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W. H. Schikhof

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Introduction

Let K be any complete valued field and let G be a locally compact group. The K-vector space $BC(G \to K)$ consisting of all K-valued bounded continuous functions on G is a Banach space under the norm $f \mapsto \|f\| = \sup \{|f(x)| : x \in G\}$. A left invariant mean (l.i.m.) is a K-linear function $M : BC(G \to K) \to K$ satisfying

(1) $M(1) = 1$
(2) $\|M\| \leqq 1$ (i.e., $|M(f)| \leqq \|f\|$) for all $f \in BC(G \to K)$)
(3) $M(f_s) = M(f)$ for all $f \in BC(G \to K)$ and $s \in G$.

(Here the symbol 1 is used for the constant function one, for the unit element of K, and also for the real number 1; f_s is defined by $f_s(x) = f(sx)$ for $x \in G$). G is called K-amenable if there exists a l.i.m. on $BC(G \to K)$.

It is well known that IR-amenability in the above sense is the same as 'amenability' as it occurs in the literature: for $K = IR$ the properties (1), (2), (3) are equivalent to (1), (3), and positivity of M. (For general K we cannot use a positivity condition in the definition of a l.i.m., since an ordering is not always available in K). It is also easy to see that IR-amenability is equivalent to $\mathbb{C}$-amenability. So in order to get something new we must have that K is not isomorphic to either IR or $\mathbb{C}$, which implies that the valuation on K is non-archimedean (i.e., $|x+y| \leqq \max(|x|, |y|)$ for all $x, y \in K$). (See [2], 1.2). It turns out that the only interesting groups to consider are 0-dimensional.

As a first example, let $G = \mathbb{Z}$ (with discrete topology). The function $f : \mathbb{Z} \to K$ defined by $f(n) = n$ is bounded(!), hence in $BC(\mathbb{Z} \to K)$. If M were a l.i.m. on $BC(\mathbb{Z} \to K)$ then $1 = M(1) = M(f_1) - M(f) = 0$. So $\mathbb{Z}$ is not K-amenable. Another typical non-archimedean feature is presented by the case $G = C_p$ (group of p elements) and $K = \mathbb{Q}_p$. If f is the characteristic function of an element of C_p, and M is a l.i.m. on $BC(C_p \to \mathbb{Q}_p)$ then $M(f) = 1/p$, and $|M(f)| = |1/p| > 1$, which contradicts

(2). The reason why it goes wrong is different for both cases: $\mathbb{Z}$ is not 'torsional' and C_p is not 'p-free' (3.2 and 1.3).

It is a rather surprising fact that one can find necessary and sufficient conditions (formulated in terms of properties of G and its topology) for K-amenability. (Theorems 2.1 and 3.6).

In 'classical' analysis one often uses the fact that an $f \in BC(G \to I\!R)$ has precompact image rather than its boundedness. This leads to another non-archimedean candidate for function space namely $PC(G \to K)$, the space of all $f \in BC(G \to K)$ such that $f(G)$ is precompact. G is called weakly K-amenable if there is a l.i.m. on $PC(G \to K)$. Corollary 5.3 gives necessary and sufficient conditions for weak K-amenability in case the characteristic of the residue class field of K is non-zero.

In [5] A. C. M. van Rooij studies K-amenability for discrete abelian semigroups. Further, he proves (Theorem 7.1) that there exists a l.i.m. on $UC(G \to K)$ iff G is p-free. (Here G is an abelian zerodimensional torsional (see [5], 7) group, not necessarily locally compact; K is spherially complete; $UC(G \to K)$ is the space of the bounded uniformly continuous functions: $G \to K$). The intersection of the theory of [5] and the results of this paper (G abelian, locally compact, torsional) is rather trivial.

Note: for detailed information on facts of non-archimedean analysis needed here (for instance Ingleton's theorem: the non-archimedean form of the Hahn-Banach theorem) we refer to [2] and [4]. We use the symbols $\mathbb{Q}_p$, $\mathbb{F}_p$, $\mathbb{Q}$. They stand for the field of the p-adic numbers, the field with p elements, and the field of the rationals, respectively.

1. Non-archimedean amenability

For a topological group G and a non-archimedean complete valued field K (trivial valuation included) we define $BC(G \to K)$ to be the K-vector space of all bounded continuous functions $f : G \to K$, normed via $f \mapsto \|f\| = \sup \{|f(x)| : x \in G\}$. For $f \in BC(G \to K)$ and $s \in G$ we put $f_s(x) = f(sx)$. Then $f_s \in BC(G \to K)$. The (K-valued) characteristic function of a clopen (= closed and open) subset U of G is in $BC(G \to K)$ and we denote it by ξ_U. Many times we write 1 instead of ξ_G. (The symbol 1 will also be used for the unit element of K and for the unit element of $I\!R$). The characteristic of a field L is denoted by $\chi(L)$.

1.1 DEFINITION: *A left invariant mean (l.i.m.) on $BC(G \to K)$ is a K-linear function $M : BC(G \to K) \to K$ satisfying*

(1) $M(1) = 1$

(2) $|M(f)| \leqq \|f\|$ *for all* $f \in BC(G \to K)$

(3) $M(f_s) = M(f)$ *for all* $f \in BC(G \to K)$ *and* $s \in G$.

G is called K-amenable if there exists a l.i.m. on $BC(G \to K)$.

We shall be concerned only with locally compact groups G. Since K is totally disconnected there is a natural isomorphism

$$BC(G \to K) \to BC(G/C \to K),$$

where C is the connected component of the group identity. G/C is a totally disconnected locally compact group, hence 0-dimensional ([1], 3.5): when studying amenability of locally compact groups we may restrict ourselves to locally compact 0-dimensional groups G. Note that such groups have small open subgroups ([1], 7.7). (every neighborhood of the identity contains an open (compact) subgroup).

FROM NOW ON G IS A LOCALLY COMPACT 0-DIMENSIONAL TOPOLOGICAL GROUP, K IS A NON-ARCHIMEDEAN COMPLETE VALUED FIELD, WHOSE RESIDUE CLASS FIELD IS DENOTED BY k.

1.2 LEMMA: *Let G be K-amenable. Then*
(i) *Every open subgroup of G is K-amenable.*
(ii) *For a closed normal subgroup S, G/S is K-amenable.*

PROOF: (i) Let S be an open subgroup. For each right coset Sx, choose an element $\tilde{x} \in Sx$. The map $\sigma : x \mapsto x\tilde{x}^{-1}$ is a surjection of G onto S and $\sigma(sx) = s\sigma(x)$ for all $s \in S$, $x \in G$. If M is a l.i.m. on $BC(G \to K)$, define $N(f) = M(f \circ \sigma)(f \in BC(S \to K))$. This N is a l.i.m. on $BC(S \to K)$, which can be verified easily.

(ii) Let $\pi : G \to G/S$ be the canonical homomorphism and let M be a l.i.m. on $BC(G \to K)$. Define $N(f) = M(f \circ \pi)(f \in BC(G/S \to K))$. This N is a l.i.m. on $BC(G/S \to K)$

1.3 DEFINITION: *Let p be a prime number. We call G p-free if for every pair of open subgroups* $S_1 \supset S_2$ *the number* $[S_1 : S_2]$ *(whenever finite) is not divisible by p. By definition, every G is 0-free.*

1.4 THEOREM: *Let G be compact. Then G is K-amenable if and only if G is* $\chi(k)$*-free, and a l.i.m. on* $BC(G \to K)$ *is unique.*

PROOF: Let G be K-amenable, and let $S_1 \supset S_2$ be open subgroups. Then by Lemma 1.2.(i), S_1 is K-amenable, let M be a l.i.m. on $BC(S_1 \to K)$.

By invariance, $M(\xi_{S_2}) = [S_1 : S_2]^{-1}$, so

$$|[S_1 : S_2]|^{-1} = |M(\xi_{S_2})| \leqq \|\xi_{S_2}\| = 1.$$

Hence $|[S_1 : S_2]| = 1$ so $[S_1 : S_2]$ is not divisible by $\chi(k)$ (in case $\chi(k) \neq 0$). Conversely, if G is $\chi(k)$-free, by [3], 2.2.7 there exists a K-valued left Haar integral m on $BC(G \to K)$, for which $\|m\| = 1$. Then $M = m(\xi_G)^{-1}$. m is a l.i.m. on $BC(G \to K)$, which is unique because of [3], 2.2.3 (i).

For the locally compact case we can say the following:

1.5 Theorem: *Let G be K-amenable. Then G is $\chi(k)$-free and there exists a Haar integral m on $C_\infty(G \to K)$ $(= \{f \in BC(G \to K)$ vanishing at infinity$\})$, such that $|m(\xi_S)| = 1$ for all compact open subgroups S.*

Proof: That G is $\chi(k)$-free can be shown as in 1.4. The rest follows from [3], 2.2.7.

We refer to [3] or [2] for properties of the convolution algebra $L(G \to K)$. This non-archimedean counterpart of $L^1(G)$, as a Banach space, equals $C_\infty(G \to K)$, but it has convolution as multiplication).

2. K-amenability for non-spherically complete K

2.1 Theorem: *Let K be not spherically complete. Then G is K-amenable if and only if G is a $\chi(k)$-free compact group.*

Proof: We prove: if G is K-amenable then G is compact. (The rest follows from 1.4).

Assume that G is σ-compact. According to [2], 2.7 G is IN-compact and hence every element, including any l.i.m. M, of the dual space of $BC(G \to K)$ is tight ([2], 7.20). So there exists a compact (open) $Y \subset G$ such that $|M(f)| \leqq \max(\sup_{x\in Y}|f(x)|, \frac{1}{2}\|f\|)$ for all $f \in BC(G \to K)$. If G were not compact then there would be an $s \in G$ with $sY \cap Y = \emptyset$. Now $|M(\xi_Y)| = |M(\xi_{sY})| \leqq \frac{1}{2}$. But also $|M(\xi_Y)| = |M(1) - M(\xi_{G\setminus Y})| = 1$.

Contradiction. The general case follows from 1.2. (i) and the following lemma.

2.2 Lemma: *A non-compact G contains an open non-compact, σ-compact subgroup S.*

Proof: Choose any compact open subgroup T_0. Since G is not compact we can find $x_1 \in G\setminus T_0$. If the group T_1, generated by T_0 and $\{x_1\}$, is not compact, put $S = T_1$. Otherwise, choose $x_2 \in G \backslash T_1$ and consider the group T_2, generated by T_1 and $\{x_2\}$. If T_2 is not compact put $S = T_2$, etc. We have: either $S = T_n$ for some n, or all T_n are compact. In this last case, define $S = \bigcup_{n=1}^{\infty} T_n$.

3. K-amenability for spherically complete K

Let us denote by H the closed linear span of

$$\{f_s - f : f \in BC(G \to K),\ s \in G\}.$$

Then we have:

3.1 Theorem: *Let K be sperically complete. Then G is K-amenable if and only if* $\inf\{\|1-h\| : h \in H\} = 1$ (*notation* $1 \perp H$).

Proof: If $1 \perp H$ then define $\phi : K \cdot 1 + H \to K$ via $\phi(\lambda \cdot 1 + h) = \lambda (\lambda \in K,\ h \in H)$. Then $|\phi(\lambda \cdot 1 + h)| = |\lambda| \leqq \|\lambda \cdot 1 + h\|$ and $\phi(1) = 1$.

By Ingleton's theorem (which is also valid for trivially valued fields) we can extend ϕ to an $M \in BC(G \to K)'$ such that $|M(f)| \leqq \|f\|$ for all $f \in BC(G \to K)$. This M is a l.i.m. If $\|1-h\| < 1$ for some $h \in H$ and M were a l.i.m. on $BC(G \to K)$, then $1 > |M(1-h)| = |1 - M(h)| = 1$ (since $M = 0$ on H) which is a contradiction.

3.2 Definition: *G is called torsional if every finite subset of G is contained in a compact (open) subgroup of G.* (See also **3.7**).

3.3 Lemma: *If G is torsional and $\chi(k)$-free then G is K-amenable.*

Proof: Suppose G is not K-amenable. Then, by 3.1, there exist $f^{(1)}, \cdots, f^{(n)} \in BC(G \to K)$ and $s_1, \cdots, s_n \in G$ such that

$$\|1 - \sum (f_{s_i}^{(i)} - f^{(i)})\| < 1.$$

Let S be a compact open subgroup, containing $s_1, \cdots, s_n$. Being $\chi(k)$-free and compact S is K-amenable (1.4). But we also have

$$\|1 - \sum (g_{s_i}^{(i)} - g^{(i)})\| < 1$$

where $g^{(i)} = f^{(i)}|S \in BC(S \to K)$, from which follows via 3.1 that S is not K-amenable. Contradiction.

For a proof of the converse of lemma 3.3 we first reduce it to the case where K is trivially valued. Indeed, if G is K-amenable then G is K_0-amenable, where K_0 is the closure of the prime field. This follows from 3.1 and the fact that K_0 is always spherically complete. (K_0 is isomorphic to either $\mathbb{F}_p$, $\mathbb{Q}_p$ or $\mathbb{Q}$). It is also an easy matter to show directly that $\mathbb{Q}_p$-amenable groups are also $\mathbb{F}_p$-amenable. So we have to deal only with $\mathbb{F}_p$ and $\mathbb{Q}$ (both trivially valued).

For $x \in G$, let $\delta_x \in BC(G \to K)'$ be the evaluation map $f \mapsto f(x)$. Let $D(G)$ be the K-linear span of $\{\delta_x : x \in G\}$ and let $P(G) = \{\mu \in D(G) : \mu(1) \neq 0\}$. For

$$\mu = \sum_{i=1}^{n} \lambda_i \delta_{x_i} \in D(G)$$

and $f \in L(G \to K)$ define

$$(\mu * f)(x) = \sum_{i=1}^{n} \lambda_i f(x_i^{-1} x) \qquad (x \in G)$$

$$(f * \mu)(x) = \sum_{i=1}^{n} \lambda_i f(x x_i^{-1}) \Delta(x_i^{-1}) \qquad (x \in G)$$

$$\mu' = \sum \lambda_i \delta_{x_i^{-1}}$$

$$f'(x) = f(x^{-1}) \Delta(x^{-1}) \qquad (x \in G)$$

$$f^s(x) = f(xs^{-1}) \qquad (s, x \in G)$$

where Δ is the K-valued modular function ([3], 2.4). Clearly, both $\mu * f$ and $f * \mu$ are in $L(G \to K)$ and $(\mu * f)' = f' * \mu'$, $f_s * \mu = (f * \mu)_s$, $\mu * f^s = (\mu * f)^s$.

The space $D(G)$ becomes a K-algebra under convolution: for $f \in BC(G \to K)$, let

$$(\mu * \nu)(f) = \sum_{i,j} \lambda_i \tau_j f(x_i y_j) \qquad (\mu = \sum \lambda_i \delta_{x_i},\ \nu = \sum \tau_j \delta_{y_j}).$$

$P(G)$ is a multiplicatively closed subset of $D(G)$. We have the usual relations:

$$\left.\begin{aligned} (\mu * \nu) * f &= \mu * (\nu * f) \\ f * (\mu * \nu) &= (f * \mu) * \nu \end{aligned}\right\} (\mu, \nu \in D(G),\ f \in L(G \to K))$$

Let us denote the K-valued Haar integral on $L(G \to K)$ by m.

3.4 LEMMA: *Let G be K-amenable where K is trivially valued. For $f \in L(G \to K)$ with $m(f) = 0$ there is $\mu \in P(G)$ with $f * \mu = 0$.*

PROOF: It suffices to show that $\nu * f = 0$ for some $\nu \in P(G)$. (If $m(f) = 0$, then $m(f') = 0$. Then $\nu * f' = 0$ implies $f * \nu' = 0$). If $\mu * f \neq 0$ for all $\mu \in P(G)$, define a map $\phi \in L(G \to K)'$ by extending the map $\mu * f \mapsto \mu(1)$, defined on $D(G) * L(G \to K)$. (The definition makes sense: if $\mu * f = \nu * f$ then $(\mu - \nu) * f = 0$ so $\mu - \nu \notin P(G)$ which means $(\mu - \nu)(1) = 0$). Let M be a l.i.m. on $BC(G \to K)$ and define $\psi \in L(G \to K)'$ by

$$\psi(g) = M(x \mapsto \phi(g_x)) \qquad (g \in L(G \to K)).$$

ψ is left invariant and since $\phi(f_x) = \phi(\delta_{x^{-1}} * f) = 1$, we obtain $\psi(f) = 1$. By the uniqueness of the Haar integral, we have $\psi = cm$ for some $c \neq 0$. But $1 = \psi(f) = cm(f) = 0$. Contradiction.

3.5 LEMMA: ('Property P' of Reiter). *Let G be K-amenable, where K is trivially valued. Then for every compact set $C \subset G$ there exists a non zero $f \in L(G \to K)$ such that $f_x = f$ for all $x \in C$.*

PROOF: Choose a compact open subgroup S of G. Then C is covered by, say $Sa_1^{-1}, \cdots, Sa_n^{-1}$. Inductively, we define $\mu_1, \mu_2, \cdots, \mu_n \in P(G)$ such that

$$(\xi_S - \xi_{a_k S}) * \mu_1 * \mu_2 * \cdots * \mu_k = 0. \qquad (k = 1, \cdots, n)$$

(for any $\nu \in P(G) : m((\xi_S - \xi_{a_k S}) * \nu) = m(\xi_S - \xi_{a_k S})\nu(1) = 0$, then use 3.4). Define $f = \xi_S * \mu_1 * \cdots * \mu_n$. Then $f \neq 0$ since $m(f) = m(\xi_S) \neq 0$. Any $x \in C$ can be written as sa_i^{-1} for some $s \in S$ and i.

$$f_x = f_{sa_i^{-1}} = (\xi_S)_{sa_i^{-1}} * \mu_1 * \cdots * \mu_n = \xi_{a_i S} * \mu_1 * \cdots * \mu_n = f.$$

3.6 THEOREM: *Let K be spherically complete. Then G is K-amenable if and only if G is torsional and $\chi(k)$-free.*

PROOF: We prove: G *K-amenable* $\Rightarrow G$ is torsional. (1.5 and 3.3 take care of the rest). We assume K to have trivial valuation (that this is without loss of generality follows from the remark following 3.3). Let $C \subset G$ be compact. By 3.5 there is $f \in L(G \to K)$ such that $f \neq 0$ and $f_x = f$ for all $x \in C$. But it is easy to see that $\{x : f_x = f\}$ is an open compact subgroup S of G. Hence any compact set is contained in a compact subgroup: G is torsional.

3.7 Corollary: *For a locally compact 0-dimensional group G the following conditions are equivalent*:

(1) *G is torsional*
(2) *Every compact set is contained in a compact subgroup.*

Proof: Use the proof of 3.5 for $K = \mathbb{Q}$ (every G is 0-free).

Note: K-amenability for some non-archimedean K implies 'amenability' in the ordinary (real) sense. (G is torsional, hence inductive limit of compact (amenable) groups, so G itself is amenable).

4. Uniqueness of invariant means

We show here that, unless G is compact (see 1.4), a l.i.m. is *never* unique for K-amenable G.

4.1 Theorem: *Let G be not compact and K-amenable. Then*

(1) *There exists a l.i.m. on $BC(G \to K)$, which is an extension of the Haar integral on $C_\infty(G \to K)$.*
(2) *There exists a l.i.m. on $BC(G \to K)$, which is 0 on $C_\infty(G \to K)$.*

Proof: By 2.1 K is spherically complete, by 3.6 G is torsional and $\chi(k)$-free. Let S be any compact open subgroup of G. We show that for any $\lambda \in K$ and $h \in H$

$$\|1+\lambda\xi_S+h\| \geqq \max(1, |\lambda|)$$

First, if $\|1+\lambda\xi_S+h\|$ were < 1, then there is $h' = \sum_{i=1}^n (h_{x_i}^{(i)} - h^{(i)})$ such that

$$\|1+\lambda\xi_S+h'\| < 1$$

There is a compact open subgroup T such that $S \subset T$ and $\{x_1, \cdots, x_n\} \subset T$. Since G is not compact there is $a \in G$ with $Ta \cap T = \emptyset$. Then we may write

$$\|1+\lambda\xi_S^a+(h')^a\| < 1.$$

Restricted to T, this expression comes down to

$$\|1+\lambda\xi_{Sa\cap T}+h''\| = \|1+h''\| < 1.$$

where $h'' \in BC(T \to K)$ is of the form $\sum_{i=1}^{n}(t_{x_i}^{(i)} - t^{(i)})$ for some $t^{(i)} \in BC(T \to K)$. But this implies that T is not amenable, a contradiction.

Next, we show that $\|\xi_S + h\| \geqq 1$. (Then we are done, since

$$|\lambda| \leqq \|\lambda\xi_S + h\| \leqq \max(\|1+\lambda\xi_S+h\|, \|-1\|) = \|1+\lambda\xi_S+h\|).$$

Again, suppose

$$\|\xi_S + \sum_{i=1}^{n}(h_{x_i}^{(i)} - h^{(i)})\| < 1.$$

Let T be a compact open subgroup containing S and $\{x_1, \cdots, x_n\}$. Restricted to T the above expression yields an inequality for elements of $BC(T \to K)$:

$$\|\xi_S + \sum (t_{x_i}^{(i)} - t^{(i)})\| < 1.$$

Since T is amenable, there is a Haar integral m on $BC(T \to K)$ with $|m(\xi_S)| = 1$, $\|m\| = 1$. But

$$1 > \|\xi_S + \sum (t_{x_i}^{(i)} - t^{(i)})\| \geqq |m(\xi_S + \sum (t_{x_i}^{(i)} - t^{(i)})| = |m(\xi_S)| = 1,$$

again a contradiction.

The map

$$M : \xi \cdot 1 + \eta\xi_S + h \mapsto \xi + \eta m(\xi_S)$$

is well-defined on $K \cdot 1 + K\xi_S + H$. $M(1) = 1$ and $\|M\| \leqq 1$, and it can be extended to a l.i.m. by Ingleton's theorem. Clearly, its restriction to $C_\infty(G \to K)$ is a Haar integral. And by carrying out the same thing for the map

$$N : \xi \cdot 1 + \eta\xi_S + h \mapsto \xi$$

we find a l.i.m. that is 0 on $C_\infty(G \to K)$.

5. Invariant means on $PC(G \to K)$

Let $PC(G \to K) = \{f \in BC(G \to K) : f(G) \text{ has compact closure in } K\}$. Then $PC(G \to K)$ is a closed subspace of $BC(G \to K)$. If $f \in PC(G \to K)$ and $s \in G$ then f_s and f^s are in $PC(G \to K)$. Clearly $1 \in PC(G \to K)$.

If every closed and bounded subset of K is compact, then $PC(G \to K) = BC(G \to K)$. The latter is also true if G is compact.

5.1 Definition: *A left invariant mean on* $PC(G \to K)$ *is a* K*-linear function* $M : PC(G \to K) \to K$ *satisfying*

(1) $M(1) = 1$
(2) $|M(f)| \leqq \|f\|$ *for all* $f \in PC(G \to K)$
(3) $M(f_s) = M(f)$ *for all* $f \in PC(G \to K)$ $BC(G \to K)$ *and* $s \in G$.
G is called weakly K-amenable if there is a l.i.m. on $PC(G \to K)$.

Let Ω denote the ring of clopen subsets of G. Then $\xi_U \in PC(G \to K)$ for all $U \in \Omega$.

5.2 Theorem: *The following conditions are equivalent.*

(1) *G is weakly K-amenable*
(2) *G is weakly K_0-amenable (where K_0 is the closure of the prime field of K)*
(3) *There exists an additive set function* $\mu : \Omega \to K_0$ *with* $\mu(G) = 1$; $\mu(sA) = \mu(A)$ *and* $|\mu(A)| \leqq 1$ *for all* $s \in G$ *and* $A \in \Omega$.

Proof: We prove: (1) → (2) → (3) → (1). If M is a l.i.m. on $PC(G \to K)$, take $\phi : K \to K_0$ with $\phi(1) = 1$, $|\phi(x)| \leqq |x|$ for all $x \in K$, ϕ is K_0-linear. (Such ϕ exists since K_0 is spherically complete). Define

$$N : PC(G \to K_0) \to K_0$$

via $N(f) = \phi(M(f))$. This N is a l.i.m. on $PC(G \to K_0)$. (2) → (3) is almost trivial (if M is a l.i.m. on $PC(G \to K_0)$, put $\mu(A) = M(\xi_A)$ for $A \in \Omega$). (3) → (1): If $f \in PC(G \to K)$ has the form $\sum_{i=1}^n \lambda_i \xi_{U_i}$ where $U_i \in \Omega$ are disjoint, define $M(f) = \sum \lambda_i \mu(U_i)$. This way M is well-defined on the set $\mathscr{T}$ of 'simple functions' and has the properties (1), (2), (3) of 5.1. For $f \in PC(G \to K)$ and $\varepsilon > 0$ define $x \sim y$ if $|f(x) - f(y)| < \varepsilon$ $(x, y \in G)$.
Let $U_1, U_2, \cdots, U_n$ be the (clopen) equivalence classes. (Since $\overline{f(G)}$ is compact the number of equivalence classes is finite). Choose $a_i \in U_i$ for each i. Then $g = \sum f(a_i)\xi_{U_i} \in \mathscr{T}$ and $\|g - f\| < \varepsilon$. Thus $\mathscr{T}$ is dense in $PC(G \to K)$ and the continuous extension of M is a l.i.m. on $PC(G \to K)$.

5.3 Corollary: *Let* $\chi(k) \neq 0$. *Then the following conditions are equivalent.*

(1) *G is weakly K-amenable.*
(2) *G is torsional and $\chi(k)$-free.*
If K is spherically complete, then G is weakly K-amenable if and only if G is K-amenable.

PROOF: (1)$\to$(2): by 5.2. G is weakly K_0-amenable. Since $\chi(k) \neq 0$ we have either $K_0 = \mathbb{F}_p$ or $K_0 = \mathbb{Q}_p$, in both cases $PC(G \to K_0) = BC(G \to K_0)$ and K_0 is spherically complete. Now use 3.6. (2)$\to$(1): by 3.6 G is K_0-amenable, hence weakly K_0-amenable. Now use 5.2. The second part is obvious (use (1)$\to$(2) and 3.6).

The situation is radically different if $\chi(k) = 0$ (note that in general $PC(G \to \mathbb{Q}) \neq BC(G \to \mathbb{Q})$).

Let us call G IR-amenable if there exists a left invariant mean on $BC(G \to IR)$ (the 'classical' definition of amenability). We have:

5.4 THEOREM: *If G is IR-amenable and $\chi(k) = 0$ then G is weakly* K-amenable.

PROOF: By 5.2 it suffices to show that there exists a l.i.m. on $PC(G \to \mathbb{Q})$, where $\mathbb{Q}$ has the trivial valuation. Compact subsets of $\mathbb{Q}$ are finite so every $f \in PC(G \to \mathbb{Q})$ is a simple function and we have an embedding $PC(G \to \mathbb{Q}) \to BC(G \to IR)$. Construct a $\mathbb{Q}$-linear $\phi : IR \to \mathbb{Q}$ with $\phi(1) = 1$. If M is a l.i.m. on $BC(G \to IR)$ define $N(f) = \phi(M(f))$ $(f \in PC(G \to \mathbb{Q})$. This N is a l.i.m. on $PC(G \to \mathbb{Q})$.

It is still an open question whether the converse of 5.4 holds. As an example we show that the discrete free group on two generators F_2, the classical example of a non-IR-amenable group, is also not weakly K-amenable.

5.5 LEMMA: *Let F_2 have generators a, b and let $h : F_2 \to K$ (here K may be any additive group). Then there exist $f, g : F_2 \to K$ such that*
(1) $f - f_a + g - g_b = h$
(2) $f(F_2) \subset h(F_2) \cup \{0\}$; $g(F_2) \subset h(F_2) \cup \{0\}$.

PROOF: Define $f(e) = f(a) = 0$; $g(e) = h(e)$, $g(b) = 0$. Then

$$(*) \qquad f(x) - f(ax) + g(x) - g(bx) = h(x)$$

holds for $x = e$ (all x with length $\leqq 0$). Suppose we have defined already $f(x)$, $g(x)$ for all x with length $\leqq n-1$ and $f(y)$ for all y with length n of the form $y = a\cdots$ and $g(z)$ for all z of length n of the form $b\cdots$ such that

($*$) holds for all words with length $\leqq n-1$. Then we extend f and g as follows:

(1) If x has length n:

$$f(x) = h(x) \quad \text{if} \quad x = b^{\pm 1}\cdots \quad \text{and} \quad f(x) = f(ax) \quad \text{if} \quad x = a^{-1}\cdots$$
$$g(x) = h(x) \quad \text{if} \quad x = a^{\pm 1}\cdots \quad \text{and} \quad g(x) = g(bx) \quad \text{if} \quad x = b^{-1}\cdots$$

(2) If x has length $n+1$:

$$\begin{aligned} f(x) &= f(a^{-1}x) && \text{if } x = aa\cdots \\ f(x) &= 0 && \text{if } x = ab^{\pm 1}\cdots \\ g(x) &= g(b^{-1}x) && \text{if } x = bb\cdots \\ g(x) &= 0 && \text{if } x = ba^{\pm 1}\cdots \end{aligned}$$

This way we now have defined $f(x)$, $g(x)$ for all x with length $\leqq n$, $f(y)$ for all y with length $n+1$ of the form $y = a\cdots$, $g(z)$ for all z with length $n+1$ of the form $z = b\cdots$.

It is easy to check that now ($*$) holds for all x with length $\leqq n$. Inspection of the above inductive definition of f and g learns us right away that also (2) holds.

5.6 COROLLARY: *F_2 is not weakly K-amenable. In fact, every left invariant linear function on $PC(F_2 \to K)$ is the zero map.*

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W. H. Schikhof
Mathematisch Instituut, K.U.
Tournooiveld, Nijmegen