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BRUCE BENNETT

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## NORMALIZATION THEOREMS FOR CERTAIN MODULAR DISCRIMINANTAL LOCI\*

Bruce Bennett

### 0. Introduction

Let  $X$  be an affine variety over an arbitrary field  $k$ ; suppose  $X$  has an isolated singular point  $x$ , with  $\kappa(x) = k$ .<sup>1</sup> We then have the ‘versal formal deformation’  $\pi : \mathfrak{X} \rightarrow S$  of  $X$  ([3], [4]). We will regard  $S = \text{Spec}(R)$ ,  $R$  a complete local  $k$ -algebra, and  $\mathfrak{X}$  is a subscheme of  $\mathbb{A}_S^m$ . The closed fibre of  $\pi$  is  $X$ , and any deformation of  $X$  over the spectrum  $T$  of a complete local  $k$ -algebra is induced by a  $k$ -morphism  $T \rightarrow S$ . Thus the family  $\mathfrak{X}/S$  contains all singularities which are ‘nearby to  $X, x$ ’.

Now for any integer  $j \geq 1$ , we have a subscheme  $N_j$  of  $\mathfrak{X}$  defined as follows:<sup>2</sup>  $N_j$  is the intersection of the components through  $x$  of the flat strata of the sheaves  $P_{\mathfrak{X}/S}^v$ ,  $v \leq j$ , and is a closed in  $\mathfrak{X}$ . In characteristic 0,  $N_j$  can be described as the locus of points in  $\mathfrak{X}$  which, when viewed in their fibre over  $S$  by  $\pi$ , have a Hilbert-Samuel function which agrees with that of  $X$  at  $x$  through at least the first  $j$  values. In characteristic  $p$  this is false; the basic reason is illustrated by the fact that the sheaf  $P_{\mathfrak{X}/S}^v$  can have high rank at a point where the fibre over  $S$  is *regular* but not *smooth*. In general, we have  $N_j \supset N_{j+1}$ ;  $N_\infty$  is called the ‘normally flat locus’.

Let  $S_j$  denote the image of  $N_j$  in  $S$ ;  $S_j$  is closed in  $S$  and  $\pi : N_j \rightarrow S_j$  is a finite morphism. Consider for example the case when  $X$  is a hypersurface in  $\mathbb{A}_k^m$ : Here the total space  $\mathfrak{X}$  is smooth over  $k$ , and  $N_1, S_1$  are the critical and discriminant loci respectively of the mapping  $\pi : \mathfrak{X} \rightarrow S$ . This latter identification is false in the non-hypersurface case.

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<sup>1</sup>  $\kappa(x)$  denotes the residue field at  $x$ .

<sup>2</sup> For details on the  $N_j$  and the deformation theory that is associated to them, the reader is referred to [5].  $P_{\mathfrak{X}/S}^v$  denotes the sheaf of relative jets of order  $v$  of  $\mathfrak{X}$  over  $S$ .

<sup>3</sup> This means that the tangent cone to  $X$  at  $x$  is a complete intersection; it implies local complete intersection in the ordinary sense. A *hypersurface* is a strict complete intersection at any point.

In [5], it is shown that when  $X$  is a ‘strict complete intersection at  $x$ ’<sup>3</sup> the  $N_j$  are smooth over  $k$ . This paper is devoted to the proof of the following theorem:

(0.1) **THEOREM:** *With the notations as above, suppose  $X$  is a strict complete intersection at  $x$ . Then for all  $j \geq 1$ , if  $\text{char.}(k) = 0$ ,  $N_j$  is the normalization of  $S_j$ , and if  $\text{char.}(k) = p$ ,  $N_j$  is a purely inseparable cover of the normalization of  $S_j$ .*

This result is known in the complex-analytic case ( $k = \mathbb{C}$ ) when  $X$  is a hypersurface ([6] II, 2.5). Note that for hypersurfaces, the result for  $j = 1$  means simply that the *critical locus* is the *normalization of the discriminant locus*.

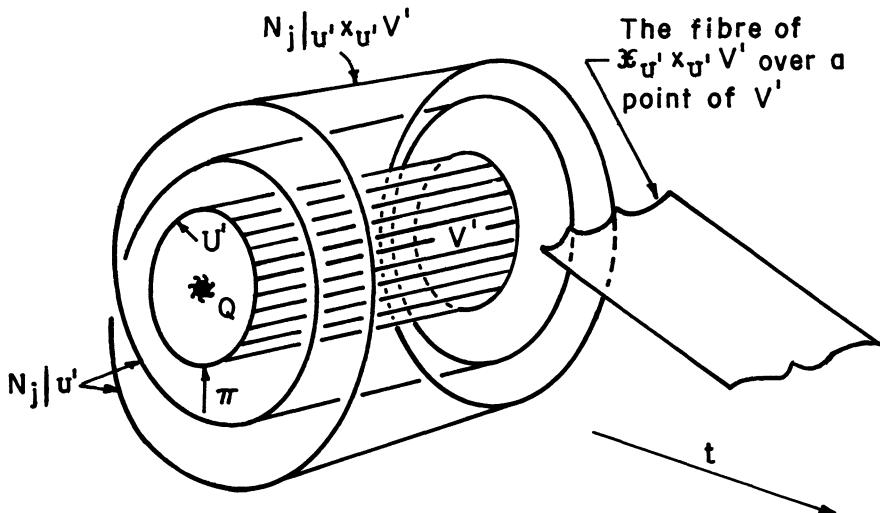
As far as generalizations of (0.1) are concerned, the situation is to the best of my knowledge completely open. We have no reason to expect that, for general  $X, x$  (even for general complete intersections), the  $N_j$  are smooth. However I would guess that they are *normal* for a large class of singularities, although I don’t know at this moment what form a general proof of this might take. Moreover, aside from the normality properties of the  $N_j$ , for general singularities it is unlikely that  $N_j \rightarrow S_j$  is ‘generically radicial’ (i.e. either birational or purely inseparable). For example, one should be able to find an isolated singularity which deforms to a variety with two distinct rigid singularities. From the arguments of Section 4 we can deduce at least that

(0.2) *If  $X$  is an arbitrary complete intersection then  $N_j \rightarrow S_j$  is generically radicial for all  $j \geq 1$ ; this is also true for  $C \rightarrow D$ , the critical and discriminant loci of  $\mathfrak{X} \rightarrow S$ .*

This basic idea of the proof of (0.2) is very simple: Suppose the fibre  $\mathfrak{X}_Q$  above a point  $Q$  in  $S_j$  contained  $n \geq 2$  distinct points of  $N_j$ . Then we can find a deformation of  $\mathfrak{X}_Q$  which leaves one of the points alone, and smooths away the others. This deformation must be contained in the family  $\mathfrak{X}/S$ , in virtue of the ‘local completeness of the versal family’ (sometimes referred to as the ‘openness of versality’; we will discuss this in more detail in Section 3). Thus, intuitively, the point  $Q$  cannot be generic in  $S_j$ .

To make this into a rigorous proof, however, requires some work. In practice, since we are working in  $\text{Spec}(R)$ , the point  $Q$  will be the generic point of a subscheme  $U$  of  $S_j$ . Suppose that generically in  $U$ , say over an open set  $U'$  of  $U$ ,  $N_j$  is an  $n$ -sheeted cover of  $U'$  (or is a purely inseparable cover of such an  $n$ -sheeted cover); thus the fibre  $\mathfrak{X}_Q$  contains one point

$P$  of  $N_j$  (set theoretically), and the separable degree of  $\kappa(P)$  over  $\kappa(Q)$  is  $n$ . Then one must first split this extension into  $n$  distinct points before smoothing away  $n-1$  of them. This job is done by a ‘norm variety’: there is a variety  $V'$  containing  $U'$  as its most singular stratum, with the following property: There is a retraction  $r : V' \rightarrow U'$  such that  $r^{-1}(Q)$  is the norm variety (Section 2) of a field extension  $\kappa(Q) \rightarrow L$ , where  $L$  splits the extension  $\kappa(P)/\kappa(Q)$ . This means geometrically that the pull-back  $N_j|_{U'} \times_{U'} V'$ , generically over  $V'$ , is just  $n$  copies of  $V'$  (or a purely inseparable cover thereof). The whole structure  $\mathfrak{X}_{U'} \times_{U'} V'$  can then be deformed, say in terms of a parameter  $t$ , to eliminate  $n-1$  of the copies.



One must be careful that the equations which work generically on  $V'$  to carry out such a deformation also make sense over all of  $V'$  (for only then can we view everything as a deformation of  $\mathfrak{X}_Q$ , and apply the completeness theorem as above). The technical solution of this problem is contained in Section 1; it amounts to showing that if a variety  $X$  over a field  $k$  has  $n$  distinct non-smooth points (with arbitrary residue fields), then there is a  $k$ -deformation of  $X$  in a parameter  $t$ , whose equations have a particularly simple form, and which smooths away  $n-1$  of the points, leaving the remaining one alone. In fact the equations can be chosen to involve only the first power of the parameter  $t$ , with polynomial coefficient ((1.2)). This is sufficient to carry out the proof of the main theorem, which is done in Section 4. Of course, we formulate everything in terms of local rings. I like to remark here that the proof of (1.2) is based in part on an idea contained in Hironaka’s  $t, r$ -index theorems ([7]), although our situation differs from his in certain respects.

In Section 5 we give three examples which illustrate the difference between our situation in characteristic 0, and characteristic  $p$ .

One feels that the significance of the norm varieties in deformation theory transcends that of a technical tool for the proof of this type of theorem. In fact, the proof itself strongly suggests that the norm varieties play the role of one type of building block of discriminant loci, and in this sense promise to give, ultimately, a good algebraic handle on their stratification and topology. Of course, the norm varieties have an intrinsic interest as abstract algebraico-geometric (or arithmetic) objects, and the basic facts in Section 2 are presented from this point of view; it is my pleasure to recall that many of the ideas here arose in conversations several years ago with B. Mazur and J.-L. Verdier.

I thank M. Schlessinger for our many discussions of questions related to the contents of this paper; also Isolde Field for the generous commitment of her time to type this manuscript.

Bruce Bennett

## 1. Some results on the ‘selective’ smoothing of points on complete intersections in arbitrary characteristic

Suppose  $X$  is a complete intersection in  $\mathbb{A}_k^m$ , say

$$X : f_1(z) = \dots = f_r(z) = 0, \quad z = (z_1, \dots, z_n).$$

Then, if  $\text{char.}(k) = 0$ , it is well known that

(\*) For generic  $b_1, \dots, b_r$ , the variety  $X_b : f_1 - b_1 = \dots = f_r - b_r = 0$  is smooth.

Equivalently,  $X_b$  is smooth over the field  $k(b)$  (or  $k((b))$ ), the field of formal-rational functions in  $b_1, \dots, b_r$ . In analysis, this follows from ‘Sard’s theorem’; in algebraic geometry it follows from one of the ‘strong forms’ of Bertini’s theorem, which asserts that a linear system without base points cannot have a moving singularity. This latter result is false in characteristic  $p$ , and indeed it is easy to find counterexamples to (\*) in this case.<sup>1</sup> For example, we can let  $X : f = b$ , where  $f = x^2 - y^3$  and  $p = 2$ . Here  $X_b$  is non-smooth at  $x = \sqrt[2]{b}, y = 0$ .

It is not hard, however, to see that

<sup>1</sup> However  $X_b$  will still be ‘regular’, as is easily seen via Zariski’s ‘mixed Jacobian’ criterion.

(1.1) For arbitrary  $k$ , for generic choice of  $a_{ij}$ ,  $b_i \in k$  ( $i = 1, \dots, r$ ,  $j = 1, \dots, m$ ) the variety

$$X_{(a, b)} : f_i + \sum_{j=1}^m a_{ij} z_j + b_i = 0 \quad (i = 1, \dots, r)$$

is smooth; in particular the family

$$f_i + t \left( \sum_{j=1}^m a_{ij} z_j + b_i \right) = 0$$

(over say  $k[[t]]$ ) deforms  $X$  to a smooth variety.

Since I can find no explicit reference for this in the literature, we give a proof here: First, consider the variety  $Z : f_i + w_i = 0$ ,  $i = 1, \dots, r$ ;  $Z$  is an  $m$ -dimensional variety in  $\mathbb{A}_k^{m+r}$ , where the coordinates are  $z_1, \dots, z_m, w_1, \dots, w_r$ . Moreover  $Z$  is smooth, since  $\partial f_i / \partial w_i = 1$ . Now we can apply a ‘weak’ Bertini theorem (see e.g. Lang [1], ) which holds in any characteristic, and which says: the generic hyperplane section of a smooth variety is smooth. Thus, if  $H_1$  denotes the generic hyperplane

$$\sum_{j=1}^m u_j^{(1)} z_j + \sum_{i=1}^r v_i^{(1)} w_i + c^{(1)} = 0,$$

$Z \cap H_1$  is smooth.

By induction, if  $H_1, \dots, H_r$  are independently generic hyperplanes, where

$$H_l : \sum_{j=1}^m u_j^{(l)} z_j + \sum_{i=1}^r v_i^{(l)} w_i + c^{(l)} = 0$$

( $l = 1, \dots, r$ ), then also  $Z \cap H_1 \cap \dots \cap H_l$  is smooth. Now since in particular the  $v_i^{(l)}$  are independently generic (so that  $\det(v_i^{(l)}) \neq 0$ ) the equations defining the  $H_l$  are equivalent, by Cramer’s Rule, to a system

$$(\dagger) \quad w_i = L_i(z), \quad i = 1, \dots, r$$

where the  $L_i$  are linear polynomials in  $z_1, \dots, z_n$  with coefficients in  $k[u_j^{(l)}, v_i^{(l)}, c^{(l)}, 1/\det(v_i^{(l)})]$ , say

$$L_i = \sum_{j=1}^m a_{ij} z_j + b_i.$$

Thus  $Z \cap H_1 \cap \dots \cap H_r$  is defined by the equations obtained by eliminating  $w$  (via  $(\dagger)$ ) in the original equations for  $Z$ , i.e. by

$$f_i + \sum_{j=1}^m a_{ij}z_j + b_i, \quad i = 1, \dots, n.$$

Since  $Z \cap H_1 \cap \dots \cap H_r$  is smooth, the proof is done.

We now consider an affine variety  $X$  over some field  $k$  (not necessarily algebraically closed) which has precisely  $n$  distinct non-smooth points  $P_1, \dots, P_n$ . As above, we suppose  $X$  is a complete intersection  $f_1 = \dots = f_r = 0$  in  $\mathbb{A}_k^m$ . We want to show that we can find a deformation  $D$  of  $X$  in  $\mathbb{A}_k^m$ , say over a one-dimensional smooth base with parameter  $t$ , whose generic fibre has only one non-smooth point, and which is formally trivial locally at  $P_1$ . In other words,  $D$  eliminates the singularities  $P_2, \dots, P_n$  but does not affect  $P_1$ . Moreover, for our later application in Section 4, we will need to find such a  $D$  whose equations have a particularly simple form; for we will have to specialize  $D$  over a certain ‘norm variety’ (Section 2), and then fit this whole structure into the moduli of some original singular variety  $X_0$ , which is our object of study. The precise statement about  $D$  is as follows:

(1.2) **THEOREM:** *Let  $k$  be arbitrary, and let  $X : f_1 = \dots = f_r = 0$  be a complete intersection in  $\mathbb{A}_k^m$ , with only isolated non-smooth points  $P_1, \dots, P_n$ . Let  $z_1, \dots, z_m$  denote the coordinate functions. Then for some polynomials*

$$g_1, \dots, g_r \in k[z] \quad (z = (z_1, \dots, z_m)),$$

*and linear polynomials  $L_1, \dots, L_r \in k[z]$ , the deformation*

$$D : f_1 + t(g_1 + L_1) = \dots = f_r + t(g_r + L_r) = 0$$

*of  $X$  over  $k[[t]]$  is trivial locally at  $P_1$ . Moreover the generic fibre of  $D$  has only one non-smooth point.*

(1.3) **REMARK:** The  $L_1, \dots, L_r$  in the statement of the theorem are the ones that work to smooth all of  $X$  as in (1.1). The point is that we can modify these with suitable  $g_i$  to have the desired effects. For our later application, all we need to know is that there exist some polynomials  $Q_1(z), \dots, Q_r(z)$  so that the deformation

$$D : (f_i + tQ_i) = 0$$

has the properties of the  $D$  in the theorem.

We now prove (1.2). To begin, let  $T_{X/k}^1$  denote the  $k$  vector space of 1st order  $k$ -deformations of  $X([2], [3])$ ; All these can be realized in  $\mathbb{A}_k^m$ .  $T_{X/k}^1$  has a natural  $\mathcal{O}_X$ -module structure; in this sense it is of finite length, supported only at the non-smooth points  $P_1, \dots, P_n$  of  $X$ . In fact, as  $\mathcal{O}_X$ -module

$$(1.3) \quad T_{X/k}^1 \cong \prod_{i=1}^n T_{X_{P_i/k}}^1$$

where  $T_{X_{P_i/k}}^1$  denotes the local contribution of  $P_i$ : it is the space of 1st order  $k$ -deformations of  $\text{Spec}(\mathcal{O}_{X, P_i})$ , or equivalently, of the formal singularity  $\hat{\mathcal{O}}_{X, P_i}$  (the hat denotes completion with respect to the maximal ideal). Let  $A = k[z]$ . We can view  $T_{X/k}^1$  as a quotient of  $\oplus^r A$ ; we think of  $A$  as the affine ring of  $\mathbb{A}_k^m$ . Similarly,  $T_{X_{P_i/k}}^1$  is a quotient of  $\oplus^r R_i$ , where  $R_i$  is the ambient local ring at  $P_i$ , or its completion. An element  $(h_1, \dots, h_r) \in \oplus^r A$  (resp.  $\oplus^r R_i$ ) corresponds to the 1st order deformation  $f_1 + th_1, \dots, f_r + th_r$  over  $k[[t]]/t^2$ . We will use the following notation: If  $(h_1, \dots, h_r)$  is in  $\oplus^r A$  or  $\oplus^r R_i$ ,  $(\bar{h}_1, \dots, \bar{h}_r)$  denotes its class in  $T_{X/k}^1$  or  $T_{X_{P_i/k}}^1$ .

Now, given our variety  $X$  as in the statement of the theorem, in view of (1.1) we can find linear functions  $L_1(z), \dots, L_r(z)$  such that the family defined by  $f_1 + tL_1, \dots, f_r + tL_r$  over say  $k[[t]]$  is generically smooth. Viewing this for the moment as a 1st order deformation (i.e. mod  $t^2$ ) it corresponds to  $(\bar{L}_1, \dots, \bar{L}_r)$  in  $T_{X/k}^1$  as above. In view of the decomposition (1.3), it is immediate that

(1.4) *There exists  $(g_r, \dots, g_r)$  in  $\oplus^r A$  such that  $(\bar{g}_1, \dots, \bar{g}_r)$  induces  $(0, \dots, 0)$  in  $T_{X_{P_i/k}}^1$  for  $i \geq 2$ , and induces  $(-L_1, \dots, -L_r)$  in  $T_{X_{P_1/k}}^1$ .*

Moreover

(1.5) *Let  $M_1$  denote the maximal ideal of the point  $P_1$  in  $\mathbb{A}_k^m$ , and let  $R_1$  be the corresponding local ring  $A_{M_1}$ . Then given an integer  $v$  we can choose  $(g_1, \dots, g_r)$  as in (1.4) above, with the additional property: each  $g_i + L_i$  is in  $M_1^v R_1$ . Namely, in the noetherian decomposition of the submodule  $\ker(\oplus^r A \rightarrow T_{X/k}^1)$  in  $\oplus^r A$ , we can just replace the  $M_1$ -primary factor  $U$  by  $M_1^v U$ , and apply the Chinese Remainder theorem in this new situation.*

Let  $I$  denote the ideal of  $X$  in  $A$ , i.e.  $I = (f_1, \dots, f_r)A$ .  $T_{X_{P_i/k}}^1$  is the quotient of  $\oplus^r R_i$  by the submodule generated by the

$$(\partial f_1/\partial z_j, \dots, \partial f_r/\partial z_j) \quad j = 1, \dots, m$$

and  $I \oplus' R_i$ ; the analogous statement with  $\oplus' A$  is true for  $T_{X/k}^1$ . Let  $s$  be an integer such that  $M_1^s R_1$  annihilates  $T_{X_{P_1/k}}^1$ , and let  $v$  be any integer  $\geq 2s$ . Then

(1.6)  $\oplus' M_1^v R_1$  is contained in the submodule of  $\oplus' R_1$  generated by  $M_1^s R_1 \partial f_1/\partial z_j \oplus \dots \oplus M_1^s R_1 \partial f_r/\partial z_j$   $j = 1, \dots, m$  and  $M_1^s I \oplus' R_1$ .

Now choose  $(g_1, \dots, g_r)$  in  $\oplus' A$  as in (1.5) with  $v \geq 2s$ . It then follows from (1.6) that

(1.7) There exist  $a_1^{(1)}, \dots, a_m^{(1)}$  in  $M_1^s R_1$ , and  $b_1^{(1)}, \dots, b_r^{(1)}$  in  $M_1^s I R_1$  such that, for each  $i = 1, \dots, r$

$$g_i + L_i = \sum_{j=1}^m a_j^{(1)} \partial f_i / \partial z_j + b_i^{(1)}.$$

I claim that these  $g_i + L_i$  work to give the theorem. We first want to show that under the conditions (1.7) the deformation  $D$  given by  $(f_1 + t(g_1 + L_1), \dots, f_r + t(g_r + L_r))$  over  $k[[t]]$  is trivial locally at  $P_1$ . For this, first note that given any  $a_1, \dots, a_m$  in  $R_1$ , we can view  $f_i(z + at)$  ( $z = (z_1, \dots, z_m)$ ,  $a = (a_1, \dots, a_m)$ ) as an element of  $R_1[[t]]$ , and in this sense

$$f_i(z + at) = f_i(z) + t \sum_{j=1}^m a_j \partial f_i / \partial z_j + \sum_{c=2}^{\infty} t^c h_{ic}$$

with  $h_{ic}$  in  $(a_1, \dots, a_m)^c R_1$ ; this is of course just formal Taylor expansion. Hence, in view of (1.7), we get:

$$(1.8) \quad f_i(z) + t(g_i(z) + L_i(z)) = f_i(z + a^{(1)}t) - \sum_{c=2}^{\infty} t^c h_{ic}(z) + tb_i^{(1)}(z),$$

where the  $a_j^{(1)}, b_i^{(1)}$  are as in (1.7), and the  $h_{ic}$  are in  $(a_1^{(1)}, \dots, a_m^{(1)})^c R_1$ . Note that since  $c \geq 2$ , the  $h_{ic}$  are themselves in  $M_1^{2s} R_1$ .

Now again by Taylor expansion, it is clear that

$$tb_i^{(1)}(z) = tb_i^{(1)}(z + a^{(1)}t) - t^2 \sum_{j=1}^m a_j^{(1)} \partial b_i / \partial z_j - t^2 B$$

with  $B \in (a_1^{(1)}, \dots, a_m^{(1)})^2 R_1[[t]] \subset M_1^{2s} R_1[[t]]$ . Moreover, since the  $b_i$  are

themselves in  $M_1^s IR_1$ ,  $a_j^{(1)} \partial b_i / \partial z_j$  is in  $M_1^{2s} R_1 + M_1 IR_1$ . Thus from (1.8) we get

$$(1.9) \quad f_i + t(g_i + L_i) = f_i(z + a^{(1)}t) + tb_i^{(1)}(z + a^{(1)}t) + \sum_{c \geq 2} h_{ic}^{(2)}t^c + \sum_{c \geq 2} b_{ic}^{(2)}t^c$$

with  $h_{ic}^{(2)}$  in  $M_1^{2s} R_1$ , and  $b_{ic}^{(2)}$  in  $M_1^s IR_1$ . The reader can check, by iteration of this process, that for each integer  $d$  there exist  $a_i^{(d)} \in M_1^s R_1$ , and  $b_i^{(d)}$  in  $M_1^s IR_1$ , such that,

$$(1.10) \quad \text{If we let } w_j = z_j + \sum_{d \geq 1} a_j^{(d)}t^d,$$

$$f_i(z) + t(g_i(z) + L_i(z)) = f_i(w) + \sum_{d \geq 1} b_i^{(d)}(w)t^d.$$

Now consider the  $k[[t]]$  homomorphism  $u : k[z][[t]] \rightarrow R_1[[t]]$  defined by  $u(z_j) = w_j$ . Note that for any  $q$  in  $k[z]$ ,  $u(q)$  is a power series in  $t$  with constant term  $q$ . It follows that if  $q$  is not in  $M_1$ ,  $u(q)$  is invertible in  $R_1[[t]]$ , so that  $u$  induces a homomorphism

$$u' : R_1[[t]] \rightarrow R_1[[t]]$$

which is an isomorphism as is easily seen. Moreover it is clear from (1.10) that

$$u'(f_i(z) + \sum_{d \geq 1} b_i^{(d)}t^d) = f_i(z) + t(g_i(z) + L_i(z)).$$

Now it is not hard to see that since the  $b_i^{(d)}$  are in  $IR_1$ , the ideal generated by the  $f_i + \sum_d b_i^{(d)}t^d$  in  $R_1[[t]]$  is the same as  $IR_1[[t]]$ . Thus if we let  $I'$  denote the ideal of  $R_1[[t]]$  generated by the  $f_i + t(g_i + L_i)$ ,  $u'$  identifies  $IR_1[[t]]$  with  $I'$ . Since  $I'$  defines the deformation  $D$ , this shows that  $D$  is trivial locally at  $P_1$ . Note that nowhere in the above argument do we encounter any difficulties due to the possibility that  $P_1$  is not a  $k$ -rational point of  $\mathbb{A}_k^m$ ; the argument does not require any reference to the explicit structure of the local ring  $R_1$  as  $k$ -algebra, to any choice of parameters, or field of representatives. Indeed for our application in Section 4,  $\kappa(P_1)$  will be a purely inseparable extension of  $k$ .

To complete the proof of Theorem (1.2), we still have to show that the deformation  $D : f_1 + t(g_1 + L_1) = \dots = f_r + t(g_r + L_r)$  above smooths away the singularities  $P_2, \dots, P_n$  of  $X$ . For this, choose one of the  $P_i$ , say  $P_2$ . We know by (1.4) that  $(g_1, \dots, g_r)$  is in the submodule of  $\bigoplus' R_2$  generated by the  $(\partial f_1 / \partial z_j, \dots, \partial f_r / \partial z_j)$ ,  $j = 1, \dots, m$  and  $I \bigoplus' R_2$ . Now again by the

Chinese Remainder Theorem, we could have chosen  $(g_1, \dots, g_r)$  so that, in addition to the property (1.5) which we needed for the local triviality at  $P_1$ , we have  $(g_1, \dots, g_r)$  in  $M_2^{2s'} \oplus^r R_2$ , where  $s'$  is an integer such that  $M_2^{s'}$  annihilates  $T_{X_{P_2/k}}^1$ . Then, as before, it follows that we even have:

$$g_i = \sum_{j=1}^m a_j \partial f_i / \partial z_j + b_i, \quad i = 1, \dots, r$$

with  $a_1, \dots, a_m$  in  $M_2^{s'} R_2$ , and  $b_i$  in  $M_2^{s'} I R_2$ . Again, by a suitable coordinate change

$$w_j = z_j + \sum_{d=1}^{\infty} a_j^{(d)} t^d$$

we can eliminate  $g_i$  from  $f_i + t(g_i + L_i)$ , replacing it with terms in positive powers of  $t$  with coefficients in  $I R_1$ . For our purpose here, however, we only want to carry out this procedure through degree  $N$  in  $t$ , where  $N \gg 0$ , say larger than  $m-r$ . In other words, there are elements  $a_j^{(d)}$  in  $R_2$  ( $j = 1, \dots, m$ ,  $d = 1, \dots, N$ ) and  $b_i^{(d)}$  in  $I R_2$  ( $i = 1, \dots, r$ ,  $d = 1, \dots, N$ ) such that

$$(1.11) \quad \text{If we let } w_j = z_j + \sum_{d=1}^N a_j^{(d)} t^d,$$

$$f_i(z) + t(g_i(z) + L_i(z)) = f_i(w) + tL_i(w) + \sum_{d=1}^N b_i^{(d)}(w) t^d + t^{N+1} F_i(w, t),$$

where  $F_i(w, t)$  is a priori in  $R_2[[t]]$ . However,  $F_i(w, t)$  is a polynomial (over  $k$ ) in  $t$ , the  $a_j^{(d)}$  and the derivatives of  $f_i$ ,  $b_i^{(d)}$ , and  $L_i$ . Since there are only a finite number of these, we can find a polynomial  $q(z)$  in  $k[z]$  so that all the terms of the equation (1.11) are in  $k[z, 1/q(z)][[t]]$ .

Now again, since the  $b_i^{(d)}$  are in  $I k[z, 1/q(z)]$ , the ideal in  $k[z, 1/q(z)][[t]]$  generated by the

$$f_i(w) + tL_i(w) + \sum_{d=1}^N b_i^{(d)}(w) t^d + t^{N+1} F_i(w, t)$$

is the same as that generated by  $f_i(w) + tL_i(w) + t^{N+1} F_i(w, t)$ . The Jacobian matrix of these last equations, viewed as functions on  $\mathbb{A}_{k(t)}^m - H$  where  $H : q(z) = 0$ , differs from that of the system  $f_i + tL_i$  only by the appearance of terms of degree  $> N$  in  $t$  (and of course the symbol  $z$  has been replaced by  $w$ ). Therefore, since the minors of this matrix all have dimension less than  $N$ , their determinants – in the case of the original system  $f_i + th_i -$

involve at most  $t^{N-1}$ ; the contribution of the  $t^{N+1}F_i(w, t)$  cannot cancel with anything coming from the original system. Hence, since the original Jacobian has maximal rank (by (1.1) (this is of course true à fortiori on  $\mathbb{A}_{k((t))}^n - H$ ), the same is true for our altered system.

We find: there exists an open set  $S = X - H$  of  $X$  containing  $P_2$  so that the deformation  $D$  contains a deformation of  $S$  in which the generic fibre is smooth. This of course means that there are no points in the generic fibre of  $D$  which specialize to  $P_2$ . A similar argument works (after suitable choice of  $(g_1, \dots, g_r)$ ) for  $P_3, \dots, P_n$ , and the proof of (1.2) is complete.

## 2. Norm varieties

Let  $L/\kappa$  be a finite separable extension, say  $d = [L : \kappa]$ . Once we choose a  $\kappa$ -basis for  $L$ , we can identify  $L$  with the set of  $\kappa$ -rational points of  $\mathbb{A}_\kappa^d$ , i.e.  $L = \kappa^d \subset \mathbb{A}_\kappa^d$ .

(2.1) *The norm function  $N : L \rightarrow \kappa$  can be expressed, in terms of the coordinate functions  $x_0, \dots, x_{d-1}$  of  $\kappa^d$ , as a form of degree  $d$  with coefficients in  $\kappa$ , say  $\varphi(x_0, \dots, x_{d-1})$ .*

This is easily seen, for example, if we suppose that  $L = \kappa(\alpha)$ , and we identify  $L$  with  $\kappa^d$  via the basis  $1, \alpha, \dots, \alpha^{d-1}$ . Let  $\sigma_i, i = 1, \dots, d$  be the distinct embeddings of  $L$  in an algebraic closure of  $\kappa$ . Then

$$(2.1.1) \quad \varphi(x) = \prod_{i=1}^d (x_0 + x_1 \sigma_i(\alpha) + \dots + x_{d-1} \sigma_i(\alpha)^{d-1}).$$

Here, the coefficient of the monomial  $x_0^{n_0} \dots x_{d-1}^{n_{d-1}}$  in

$$\varphi(x), (n_0 + \dots + n_{d-1} \leq d)$$

is that symmetric function of the  $\sigma_i(\alpha)$  where precisely  $n_j$  terms in each summand appear with degree  $j$ ; in particular the coefficients are in  $\kappa$ .  $\varphi$  is called the *norm form* of  $L$  over  $\kappa$ ; it is well defined only up to  $GL(d, \kappa)$ .

(2.2) **DEFINITION:** The norm variety of  $L$  over  $\kappa$  is the variety in  $\mathbb{A}_\kappa^d$  defined by  $\varphi(x) = 0$ , where  $\varphi$  is any norm form as above (the norm variety is also defined only up to  $GL(d, \kappa)$ ).

Note that the only  $\kappa$ -rational point of the norm variety is the origin, since 0 is the only element of  $L$  with norm 0. Henceforth we will assume:

$L/\kappa$  is Galois with  $G = G(L/\kappa)$ . We will let  $V$  denote the norm variety.

**REMARK:** Let  $V_\mu (\mu \geq 1)$  be the locus of points of  $V$  whose local rings in  $V$  have multiplicity  $\mu$ . As is well known,  $V_\mu$  is a locally closed subset of  $V$  for the Zariski topology; in fact  $\bigcup_{\mu' \geq \mu} V_{\mu'}$  is closed in  $V$ . It can be shown that  $V_\mu$  is a disjoint union of irreducible components, each of which, as an abstract  $\kappa$ -variety, is isomorphic to  $\mathbb{A}_F^{d-\mu}$ , where  $F$  is a field between  $\kappa$  and  $L$ . The number and configuration of these components is intimately related to the structure of  $G$ , but we will not pursue this here. In fact, for the purposes of this paper, we will need only the following result:

(2.3) **THEOREM:**  *$V$  is irreducible, and  $V_1$  (the regular locus of  $V$  – see the remark above) is isomorphic as an abstract  $\kappa$ -variety to a Zariski open set in  $\mathbb{A}_L^{d-1}$ . In particular the algebraic closure of  $\kappa$  in the rational function field of  $V$  is isomorphic to  $L$ .*

**PROOF:** First note that  $\varphi$  splits over  $L$  into a product of linear factors, as is evident by (2.1.1). Thus, if we let  $V_L = V \otimes_\kappa L$ ,  $V_L$  is a union of  $d$  hyperplanes in  $\mathbb{A}_L^d$ :

$$V_L = H_1 \cup \dots \cup H_d.$$

(These hyperplanes are in general position, since by (2.1.1) the determinant of their coefficients is a Vandermonde determinant.) Let  $V_{L,\mu} (\mu \geq 1)$  denote the locus of points of  $V_L$  which lie in precisely  $\mu$  of these hyperplanes, i.e. at which  $V_L$  has multiplicity  $\mu$ . Thus  $V_{L,\mu}$  is a disjoint union of irreducible components; these are the distinct  $\mu$ -fold intersections of the  $H_i$  (from which the  $\mu+1$ -fold intersections have been deleted).

Now  $V = V_L/G$ , and the projection  $p : V_L \rightarrow V$  is étale of degree  $d$ ; in particular  $V_{L,\mu} = p^{-1}(V_\mu)$  since multiplicity is preserved by étale morphisms. For  $\mu = 1$ ,  $V_{L,1}$  consists of the  $d$  hyperplanes minus their intersections, i.e.  $V_{L,1} = H'_1 \cup \dots \cup H'_d$ , where  $H'_i$  is  $H_i$  minus some linear subspaces of dimension  $d-2$ . We now observe that  $G$  acts transitively on the set  $\{H_1, \dots, H_d\}$ . In fact this set is (non-cononically) isomorphic to  $G$  in such a way that the effect of  $G$  in permuting the components of  $V_L$  looks like the action of  $G$  on itself by left translation; this also shows that no subvariety of  $V_{L,1}$  is invariant under any non-trivial element of  $G$ . It follows that  $V_1 = p(V_{L,1})$  has only one component, which is isomorphic by  $p$  to any of the  $H'_i$ .

### 3. Remarks on the completeness of the Versal family

Let  $X$  be a variety over  $k$  with isolated singularities (or more generally any geometric object with a finite dimensional local deformation theory). Let  $\pi : \mathfrak{X} \rightarrow S$  be the versal local  $k$ -deformation of  $X$ . The theorem of the completeness of the versal family states intuitively that if  $s$  is any point of  $S$ , the every deformation of the fibre  $\mathfrak{X}_s$  is contained in the family  $\mathfrak{X}/S$  in a suitable sense. This is of course true for the special point  $s_0$  of  $S$  by definition,

The theorem has been studied in various contexts by different authors. Our reference is Artin's paper [8]. He treats the theorem in a very general setting, using techniques of algebrization. In our situation, which Artin's treatment comprehends,  $S = \text{Spec}(R)$ ,  $R$  a complete local  $k$ -algebra, and we are concerned only with isolated singularities. In a certain sense Artin's approach seems the most natural, although it is possible – and quite instructive – to give a direct proof in this case without appealing to algebrization (as far as I know, such a treatment is not available in the literature).

We will be interested in the completeness of  $\mathfrak{X}/S$  at non-closed points  $s$  in  $S$ , and in stating the theorem one must be careful about the role of the residue field  $\kappa(s)$ : it turns out that the family is complete at  $s$  only ‘over  $k$ ’, i.e. certain moduli of the singularity  $\mathfrak{X}_s$  can be realized in  $\mathfrak{X}/S$  only by certain  $k$ -automorphisms of  $\kappa(s)$  acting of  $\mathfrak{X}_s$ . This phenomenon, although not so significant for our application in this paper, would seem to be of considerable independent interest, for example as a tool in the study of ‘generic singularities’.

We will state and use the completeness theorem in the following form:

(3.1) **THEOREM:** *Given  $\pi : \mathfrak{X} \rightarrow S$ , the versal  $k$ -deformation of a  $k$ -variety  $X$  with isolated singularities, where  $S = \text{Spec}(R)$ ,  $R$  local, Let  $s$  in  $S$  correspond to a prime ideal  $\mathfrak{p}$  of  $R$ . Let  $\mathfrak{C}$  denote the category of Artin local  $k$ -algebras (N.B. not  $\kappa(s)$ -algebras) with residue field  $\kappa(s)$ . Let  $A$  in  $\mathfrak{C}$ , and let  $Z$  be a flat family over  $\text{Spec}(A)$  with special fibre  $\mathfrak{X}_s$ . Then  $Z$  is induced from  $\mathfrak{X}$  by some  $k$ -homomorphism  $R_{\mathfrak{p}} \rightarrow A$ .*

Note that the theorem remains valid if we enlarge  $\mathfrak{C}$  to include all complete local  $k$ -algebras with residue field  $\kappa(s)$ .

### 4. Proof of the normalization theorems

As in the introduction, we suppose  $X$  is a  $k$ -variety with an isolated

singularity at a  $k$ -rational point  $x \in X$ . Let  $\pi : \mathfrak{X} \rightarrow S$  be the versal deformation. For each  $j \geq 1$ , we have the locus  $N_j \subset \mathfrak{X}$ , which is finite over  $S$ ; let  $S_j = \pi(N_j)$ . We suppose, moreover, that  $X$  is a strict complete intersection at  $x$ ; then we know that  $N_j$  is smooth over  $k$ , and hence normal. Therefore, to prove theorem (0.1) it suffices to show that  $\pi : N_j \rightarrow S_j$  is generically a purely inseparable morphism.

Now  $N_j$ , and therefore also  $S_j$ , is irreducible. Let  $P$  and  $Q$  denote their respective generic points, with  $\mathcal{O}_{N_j, P} = F$ ,  $\mathcal{O}_{S_j, Q} = \kappa$ ,  $\kappa \rightarrow F$  a finite extension of fields.

We first consider the case when  $\kappa \rightarrow F$  is a purely inseparable extension. Let  $A$  and  $B$  denote the affine rings of  $S_j$  and  $N_j$  respectively, and let  $\tilde{A}$  be the normalization of  $A$ . Since  $B$  is normal, we have  $A \subset \tilde{A} \subset B$ , and of course  $B$  is finite over  $\tilde{A}$ . Suppose  $B = \tilde{A}[b_1, \dots, b_n]$ ,  $b_i \in B$ . Since  $b_i \in F$ , for some integer  $e_i$ ,  $b_i^{p^{e_i}}$  is in  $\kappa$ , where  $p$  is the characteristic. But  $b_i^{p^{e_i}}$  is integral over  $\tilde{A}$  (because  $b_i$  is), so  $b_i^{p^{e_i}}$  is in  $\tilde{A}$ , since  $\tilde{A}$  is integrally closed in its field of fractions  $\kappa$ . Thus:

(4.1) *If  $\kappa \rightarrow F$  is purely inseparable,  $N_j$  is a purely inseparable extension of the normalization  $\tilde{S}_j$  of  $S_j$ , in the sense that it is obtained by extracting  $p$ th-power roots of finitely many functions on  $\tilde{S}_j$ .*

We now consider the general case, where  $\kappa \rightarrow F$  is an arbitrary finite extension. Let  $K$  denote the separable closure of  $\kappa$  in  $F$ . We want to show that  $\kappa = K$ ; this will complete the proof of the theorem, in view of (4.1).

Assume that  $\kappa \neq K$ , i.e. that  $N_j$  meets  $\mathfrak{X}_Q$  in a unique singular point  $P$  whose residue field  $F = \kappa(P)$  involves a non-trivial separable extension of  $\kappa = \kappa(Q)$ . We will show:

(4.2) *There exists a complete local domain  $C$  with residue field  $\kappa$ , and a deformation  $Z \xrightarrow{d} \text{Spec}(C)$  of  $\mathfrak{X}_Q$ , with the following properties:*

- (a) *Let  $Q'$  denote the generic point of  $\text{Spec}(C)$ . The generic fibre  $Z_{Q'}$  of  $d$  has a unique singular point  $P'$ .*
- (b) *If  $\kappa' = \kappa(Q')$  is the field of fractions of  $C$ , and if we denote  $F' = \kappa(P')$ , then  $[F' : \kappa'] = [F : K]$ .*
- (c) *For every integer  $v \geq 0$ ,*

$$rk_F(P_{Z_{Q'/\kappa'}}^v(P')) = rk_F(P_{\mathfrak{X}_{Q/\kappa}}^v(P)).$$

Assuming (4.2) is proved, we can get a contradiction (of the hypothesis that  $\kappa \neq K$ ) as follows: In view of the completeness theorem (3.1),  $Z/\text{Spec}(C)$  is induced by a morphism  $\text{Spec}(C) \xrightarrow{g} S$ , which takes the closed point of  $\text{Spec}(C)$  to  $Q$ .

$$\begin{array}{ccccccc}
 \kappa(P') = F' & P' & Z & \xrightarrow{h} & \mathfrak{X} & P & \kappa(P) = F \\
 \downarrow & & \downarrow d & \square & \downarrow \pi & \downarrow & \\
 \kappa(Q') = \kappa' & Q' & \text{Spec}(C) & \xrightarrow{g} & S & Q & \kappa(Q) = \kappa
 \end{array}$$

Then, it follows from (4.2)(c) that  $h(P') \in N_j$ , so that  $g(Q') \in S_j$ . On the other hand, it is clear that  $g(Q')$  specializes to  $Q$ ; thus, since  $Q$  is generic in  $S_j$ ,  $g(Q') = Q$ . This means that  $Z = \mathfrak{X}_Q \otimes_{\kappa} C$ , where  $C$  is viewed as a  $\kappa$ -algebra via some field of representatives; in particular  $Z_{Q'} = \mathfrak{X}_Q \otimes_{\kappa} \kappa'$ . Now every point of  $\mathfrak{X}_Q \otimes_{\kappa} \kappa'$  which projects to  $P$  in  $\mathfrak{X}_Q$  is singular. Therefore, since by (4.2)(a)  $P'$  is the unique singular point of  $Z_{Q'}$ , we must have  $[F' : \kappa'] = \dim_{\kappa'}(F \otimes_{\kappa} \kappa') = [F : \kappa]$ . This contradicts (4.2)(b), since  $[F : K] < [F : \kappa]$ .

The remainder of Section 4 will be devoted to the proof of (4.2). We begin with the following data:

(4.3) *B is a complete local  $\kappa$ -algebra with residue field  $\kappa$ , and with the property: the algebraic closure of  $\kappa$  in the field of fractions  $\Omega$  of B is isomorphic to L, where L is a normal extension of  $\kappa$  containing K.*

$$\begin{array}{ccc}
 \kappa(P) = F & & \Omega = \text{fraction field of } B \\
 & \swarrow \text{purely insep.} & \nearrow \\
 & \text{deg. } e & \\
 & \searrow & \\
 K & \xrightarrow{\quad} & L = \text{alg.cl. of } \kappa \text{ in } \Omega \\
 & \swarrow \text{deg. } n & \nearrow \text{normal} \\
 & \text{sep.} &
 \end{array}$$

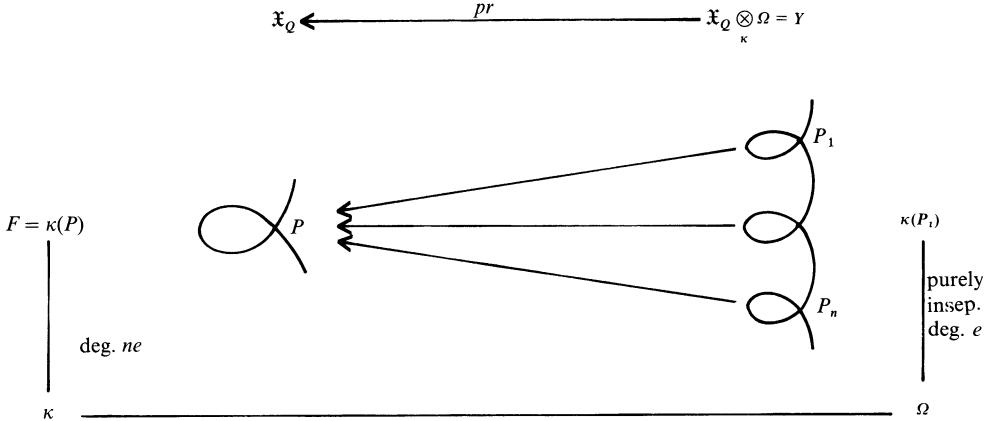
$\kappa(Q) = \kappa$

For example we can take  $B$  to be the completion of the local ring at the origin of the norm variety of  $L$  over  $\kappa$  (Section 2).

Let  $n = [K : \kappa]$ . We first notice that

(4.4)  *$\mathfrak{X}_Q \otimes_{\kappa} \Omega$  has  $n$  distinct singular points, say  $P_1, \dots, P_n$  which lie over  $P$  via the projection  $\mathfrak{X}_Q \otimes_{\kappa} \Omega \xrightarrow{\text{pr}} \mathfrak{X}_Q$ ; moreover each residue field  $\kappa(P_i)$  is a purely inseparable extension of  $\Omega$  of degree  $e = [F : K]$ ; this extension may be identified with  $F \otimes_K \Omega$ .*

**PROOF:** The fibre  $\text{pr}^{-1}(P)$  is isomorphic to  $\text{Spec}(F \otimes_{\kappa} \Omega)$ . We view  $F \otimes_{\kappa} \Omega = F \otimes_K (K \otimes_{\kappa} L) \otimes_L \Omega$ ;  $K \otimes_{\kappa} L$  is a direct product of  $n$  fields, each isomorphic to  $L$ . Hence  $F \otimes_{\kappa} \Omega \cong \prod_n F \otimes_K \Omega$ . Since  $F$  is purely inseparable over  $K$ , and since the algebraic closure  $L$  of  $K$  in  $\Omega$  is separable over  $K$ ,  $F \otimes_K \Omega$  is a field, purely inseparable of degree  $e$  over  $\Omega$  as asserted. Thus  $\text{Spec}(F \otimes_{\kappa} \Omega)$  consists of  $n$  distinct points  $P_1, \dots, P_n$  with the desired properties.



We now work with the variety  $Y = X_Q \otimes_{\kappa} \Omega$  over  $\Omega$ . As we have just seen, it has  $n$  distinct non-smooth points  $P_1, \dots, P_n$ ; each of the residue fields  $\kappa(P_i)$  is a purely inseparable extension of  $\Omega$  of degree  $e$ .

Now since our original  $X$  was a complete intersection, say in  $\mathbb{A}_k^n$ ,  $Y$  is a complete intersection in  $\mathbb{A}_{\Omega}^n$ ,

$$Y : f_1(z) = \dots = f_r(z) = 0,$$

$f_i(z) \in \Omega[z]$ ,  $z = (z_1, \dots, z_m)$ . We can therefore apply theorem (1.2) (see also Remark (1.3)) and obtain: there exist polynomials  $q_1(z), \dots, q_r(z)$  in  $\Omega[z]$  such that the family

$$D : f_1 + tq_1 = \dots = f_r + tq_r = 0$$

over  $\Omega[[t]]$  is locally trivial at  $P_1$  in  $Y$ , and has a unique non-smooth point, say  $P'$ , in its generic fibre. Let  $\kappa'$  denote the fraction field of  $\Omega[[t]]$ . Because of the local triviality of  $D$  at  $P_1$ , it is clear that

$$[\kappa(P') : \kappa'] = [\kappa(P_1) : \Omega] = e (= [F : K]).$$

Let  $C = B[[t]]$  where  $B$  is as in (4.3). Note that if  $M = \max(B)$ ,  $\Omega$  is the residue field of  $C_{MC}$ . Moreover, the residue field of  $C$  itself is  $\kappa$ . We want to find a deformation  $d : Z \rightarrow \text{Spec}(C)$  of the  $\kappa$  variety  $\mathfrak{X}_Q$  which in a suitable sense induces  $D$  above over  $C_{MC}$  (or more precisely,  $\widehat{C_{MC}} = \Omega[[t]]$ ). Note that since  $Y$  is a base extension of  $\mathfrak{X}_Q$  to  $\Omega$ , we may assume that the polynomials  $f_i(z)$  which define  $Y$  are actually in  $\kappa[z]$  and hence in  $C[z]$ . The problem then, in extending  $D$  over  $C$  as above is that the polynomials  $q_i(z)$  will have coefficients in  $\Omega$  which are definitely not in  $B$ . However, since there are only finitely many coefficients which appear in the  $q_1, \dots, q_r$ , there exists an element  $a$  in  $B$  such that  $aq_i(z)$  is in  $B[Z]$  for  $i = 1, \dots, r$ . Thus, let

$$Z : f_1 + atq_1 = \dots = f_r + atq_r = 0$$

in  $\mathbb{A}_C^m$ ;  $d : Z \rightarrow \text{Spec}(C)$  is the projection. The fibre  $Z'$  of  $Z$  over  $\text{Spec}(\widehat{C_{MC}})$  is of course defined by the same equations, but now viewed over the parameter space  $\Omega[[t]]$ . Now  $Z'$  may not be  $\Omega[[t]]$ -isomorphic to  $D$ . However it is  $\Omega$ -isomorphic, namely over the  $\Omega$ -automorphism  $t \rightarrow at$  of  $\Omega[[t]]$ . In particular  $D$ 's properties of being locally trivial at  $P_1$ , and of having a unique non-smooth point  $P'$  in the generic fibre (with  $[\kappa(P') : \kappa] = [F : K]$ ) are retained by  $Z'$ . Thus (a) and (b) of (4.2) are demonstrated for this  $d : Z \rightarrow \text{Spec}(C)$ .

For (4.2) (c), let  $\bar{Q}$  denote the point of  $\text{Spec}(C)$  corresponding to the prime ideal  $(t)C$ . Then  $Z_{\bar{Q}}$  is just  $Y = \mathfrak{X}_Q \otimes_{\kappa} \Omega$ . Now it is well known that the sheaves  $P_{Z/S}^v$  commute with base extension over  $S$ . Thus, on the one hand

$$\begin{aligned} P_{Z_{\bar{Q}}/\Omega}^v &= P_{Y/\Omega}^v = P_{\mathfrak{X}_Q/\kappa}^v \otimes_{\mathcal{O}_{\mathfrak{X}_Q}} \mathcal{O}_Y \\ P_{Z_{\bar{Q}}/\Omega}(P_1) &= P_{\mathfrak{X}_Q/\kappa}(P) \otimes_{\kappa(P)} \kappa(P_1) \\ rk_{\kappa(P)}(P_{\mathfrak{X}_Q/\kappa}^v(P)) &= rk_{\kappa(P_1)}(P_{Z_{\bar{Q}}/\Omega}^v(P_1)). \end{aligned}$$

On the other hand, since, as we have seen, the deformation  $Z'$  is locally trivial at  $P_1$ ,

$$rk_{\kappa(P_1)}(P_{Z_{\bar{Q}}/\Omega}^v(P_1)) = rk_{\kappa(P')}^v(P_{Z_{Q'}/\kappa'}^v(P'))$$

where  $Q'$  is the generic point of  $C$  and  $\kappa'$  the fraction field of  $C$ . Hence

$$rk_{\kappa(P)}(P_{\mathfrak{X}_Q/\kappa}^v(P)) = rk_{\kappa(P')}^v(P_{Z_{Q'}/\kappa'}^v(P')),$$

which is the assertion of (4.2) (c). Thus we conclude the proof of (4.2) and hence of the normalization theorem (0.1).

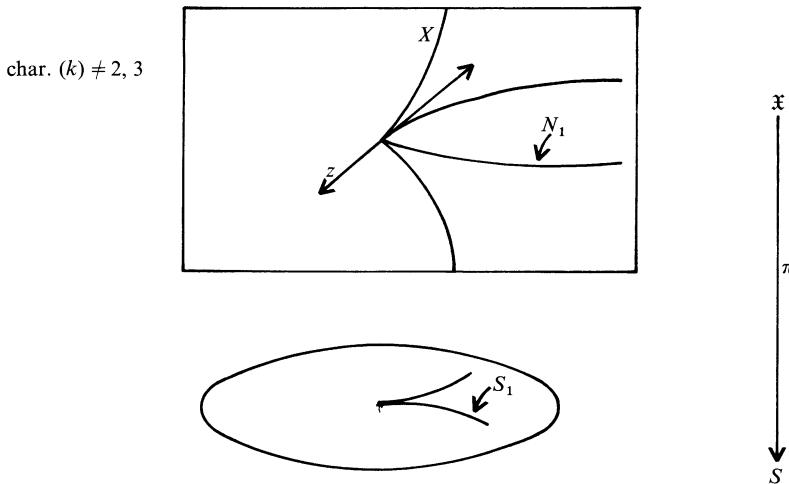
## 5. Three examples

We look at the singularity  $X : y^2 + z^3 = 0$  in  $\mathbb{A}_k^2$ ,  $x =$  the origin. We are interested in three cases: First, when the characteristic of  $k$  is either 0 or  $p \neq 3, 2$ ; secondly, when  $\text{char}(k)$  is 3, and lastly when it is 2. Let  $\pi : \mathfrak{X} \rightarrow S$ ,  $S = \text{Spec}(R)$ , denote the versal family as usual. In all cases, since the multiplicity of  $X$  at  $x$  is 2, the only stratum of interest is  $N_1$ , the critical locus of  $\pi$ ;  $S_1$  is the discriminant locus.

Now in the first two cases (i.e.  $\text{char.}(k) \neq 2$ )  $R = k[[s, t]]$ , and  $\mathfrak{X}$  is given by  $y^2 + z^3 + tz + s = 0$  in  $\mathbb{A}_S^2$ .<sup>1</sup> The critical locus  $N_1$  is given in  $\mathfrak{X}$  by the vanishing of the 1st partial derivatives with respect to  $y$  and  $z$ . Thus

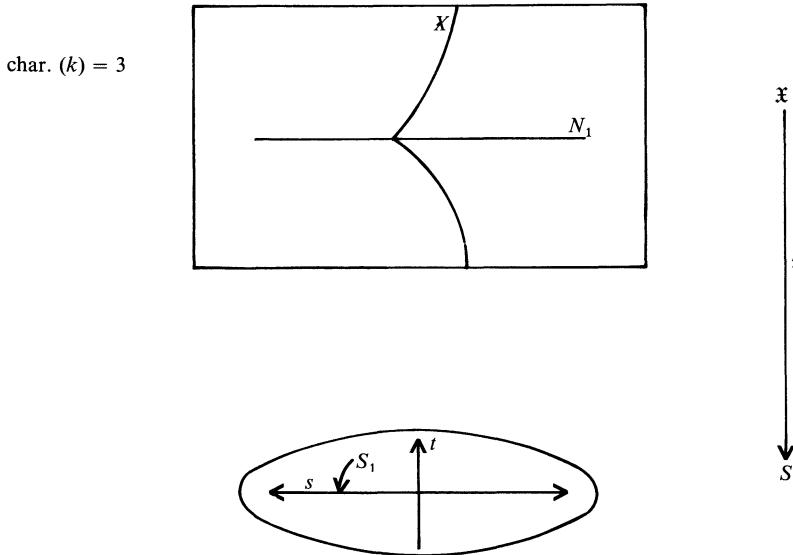
$$(*) \quad N_1 : y^2 + z^3 + tz + s = y = 3z^2 + t = 0.$$

Hence  $N_1$  is a smooth curve in  $\mathfrak{X}$  which may be parametrized by  $z$ . Now if the characteristic is not 3, and we solve for  $s$  and  $t$  in terms of  $z$ , we find that on  $N_1$ ,  $t = -3z^2$ ,  $s = 2z^3$ . Thus, if we change coordinates, replacing  $t$  by  $-t/3$ ,  $s$  by  $s/2$ , we find that  $S_1$  is the cusp given by  $s^2 - t^3$  in  $S$ , and  $N_1$  is  $\text{Spec}(k[[z]]) = \text{Spec}(k[[s/t]])$ , the normalization.



<sup>1</sup> For simplicity, for char. (2) and (3) we consider only that part of the moduli space which parametrizes terms of order  $\leq 1$  in  $y$  and  $z$ .

In case  $\text{char.}(k) = 3$ , (\*) shows that on  $N_1$ ,  $s = -z^3$ ,  $t = 0$ . Thus  $S_1$  is the  $s$ -axis in  $S$ , and  $N_1$  is the purely inseparable cover of  $S_1$  obtained by extracting a cube root of  $s$ .

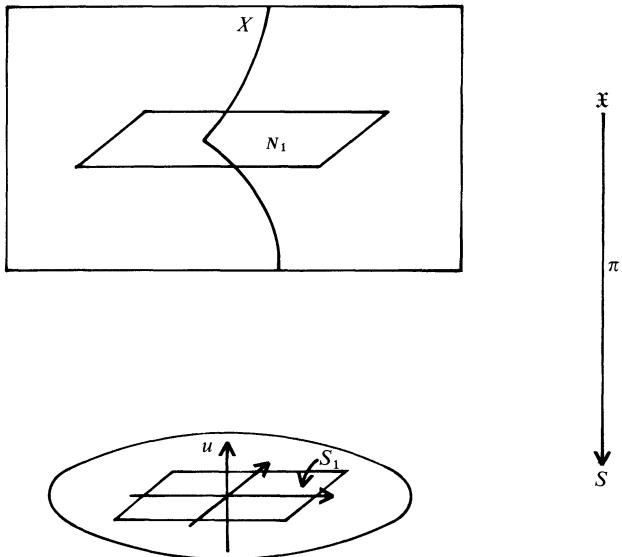


Now in characteristic 2,  $R = k[[s, t, u]]$ , and  $\mathfrak{X} : y^2 + z^3 + tz + uy + s = 0$  in  $\mathbb{A}_S^2$ .  $N_1$  is then

$$N_1 : y^2 + z^3 + tz + uy + s = u = 3z^2 + t = 0.$$

We find that  $N_1$  is a plane in  $\mathfrak{X}$  with parameters  $y, z$ . Moreover, on  $N_1$ , if we solve for  $s, t$ , and  $u$  we find:  $u = 0, t = -3z^2 (= z^2), s = y^2$ . Thus  $S_1$  is the  $s, t$ -plane in  $S$  and  $N_1 \rightarrow S_1$  is the purely inseparable cover obtained by extracting square roots of  $s$  and  $t$ .

$\text{char}(k) = 2$



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University of California  
Irvine