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TRANSFER MAPS FOR FIBRATIONS AND DUALITY

J. C. Becker and D. H. Gottlieb

1. Introduction

In this paper we will describe a transfer construction for (Hurewicz) fibrations which is a generalization of that for fiber bundles studied in [4, 5]. We suppose given a commutative triangle

$$E \xrightarrow{f} E$$

$$\bigvee_{p}^{p} \swarrow_{p}$$

$$B$$

where $p: E \rightarrow B$ is a fibration having fiber F a finite complex and base B a connected finite dimensional complex. With this data we show that there is an S-map, which we call a *transfer* map,

$$\tau(f):B^+\to E^+$$

having the property that

$$\tilde{H}^*(B^+) \xrightarrow{p^*} \tilde{H}^*(E^+) \xrightarrow{\tau(f)^*} \tilde{H}^*(B^+)$$

is multiplication by the Lefschetz number Λ of $f': F \to F$, the restriction of f to the fiber. (Although f' is not unique we allow this abuse of language since Λ is independent of the choice of f'.)

The existence of $\tau(f)$ severely restricts the projection map of the fibration. For example

$$p_*: \{X; E^+\}_q \otimes Z[\Lambda^{-1}] \to \{X; B^+\}_q \otimes Z[\Lambda^{-1}]$$

is a split epimorphism for any (pointed) finite dimensional complex X.

[2]

We will show that the boundary map $\omega: \Omega B \to F$ arising from the Puppe sequence of the fibration $p: E \to B$ is also restricted by the transfer. Precisely, we have

(1.1) THEOREM: Assume that F is connected. Then

$$\Lambda \omega_* : \{X; \Omega B\}_q \to \{X; F\}_q$$

is trivial for any finite dimensional complex X.

An independent method of extending the notion of transfer from fiber bundles to fibrations is given in [7]. The method which we describe here is intrinsic and has the advantage that many basic properties of the transfer are easily derived. A. Dold [9] has also independently defined the transfer, placing somewhat different restrictions on the projection p and fiber preserving map f.

The outline of the paper is as follows. In section 2 we give a homotopy characterization of the Lefschetz number of a map. Although an elementary fact it is the key point in defining the transfer. In section 3 we deal with some homotopy properties of ex-spaces and in section 4 with the duality theory of ex-spaces. This generalization of Spanier-Whitehead duality is purely formal except for the question of the existence of dual ex-spaces (theorem 4.2). In sections 5 thru 7 we define the transfer and establish its basic properties. In section 8 we prove theorem (1.1) mentioned above and describe some consequences of the theorem. In section 9 we consider smooth fiber bundles and in this case we give a more geometric description of the transfer.

2. The Lefschetz number

Suppose that F is a finite complex with base point and $f: F \rightarrow F$ is a base point preserving map. By the reduced Lefschetz number of f we mean

$$\tilde{\Lambda}_f = \sum (-1)^i \operatorname{tr} [f_* : \tilde{H}_i(F) \to \tilde{H}_i(F)].$$

Let $\mu: S^s \to F \land \hat{F}$ be a duality map in the sense of Spanier [15]. Then $F \land \hat{F}$ is 2s-self dual via the map

$$S^{2s} \xrightarrow{\mu \land \mu} (F \land \hat{F}) \land (F \land \hat{F}) \xrightarrow{\alpha} (F \land \hat{F}) \land (F \land \hat{F})$$

where $\alpha(x \wedge y \wedge x' \wedge y') = x' \wedge y \wedge x \wedge y'$. Denote this composite by ν and let $\hat{\mu}: F \wedge \hat{F} \rightarrow S^s$ be dual to μ relative to ν . The following lemma provides a homotopy description of the reduced Lefschetz number of f. It is the analogue for base point preserving maps of the Lefschetz fixed point theorem given by Dold [8, theorem (4.1)].

(2.1) LEMMA: The composite $S^s \xrightarrow{\mu} F \wedge \hat{F} \xrightarrow{f \wedge 1} F \wedge \hat{F}$ $\xrightarrow{\mu} S^s$ has degree $\tilde{\Lambda}_f$.

PROOF: We have the following homotopy commutative diagram

(2.2)
$$S^{2s} \xrightarrow{\iota_{\Lambda_{\mu}}} S^{s} \wedge (F \wedge \hat{F})$$

Let Q denote the rational numbers and choose a generator $\gamma \in \tilde{H}_s(S^s; Q)$. Let $\{u_p\}$ be a basis for $\tilde{H}_*(F; Q)$ and $\{v_p\}$ a basis for $\tilde{H}_*(\hat{F}; Q)$. Let $d(u_p)$ and $d(v_p)$ denote respectively the dimension of u_p and v_p . Write

and

$$\mu_{*}(\gamma) = \sum_{i,j} a_{ij}u_{i} \wedge v_{j}$$
$$\hat{\mu}_{*}(u_{i} \wedge v_{j}) = b_{ij}\gamma$$

Let $A = |a_{ij}|$, $B = |b_{ij}|$, and $D = |(-1)^{d(u,v)}\delta_{ij}|$ where δ_{ij} is the Kronecker symbol. By (2.2) we have

$$(1 \wedge \mu)_*(\gamma \wedge \gamma) = (\hat{\mu} \wedge 1)_* \nu_*(\gamma \wedge \gamma).$$

By expressing each side in terms of the basis elements $\gamma \wedge u_i \wedge v_j$ and equating coefficients, we obtain the relation $A = DAB^T A$. Since A is non-singular $AB^T = D$.

Now suppose that $f_*(u_i) = \sum_k c_{ik}u_k$. We have

$$\hat{\mu}_{*}(f \wedge 1)_{*}\mu_{*}(\gamma) = \hat{\mu}_{*}(f \wedge 1)_{*}\left(\sum_{i,j} a_{ij}u_{i} \wedge v_{j}\right)$$
$$= \hat{\mu}_{*}\left(\sum_{i,j,k} a_{ij}c_{ik}u_{k} \wedge v_{j}\right)$$
$$= \sum_{i,j,k} a_{ij}b_{kj}c_{ik}\gamma$$
$$= \sum_{i,k} (-1)^{d(u_{i})}\delta_{ik}c_{ik}\gamma$$
$$= \sum_{i} (-1)^{d(u_{i})}c_{ii}\gamma = \tilde{\Lambda}_{f}\gamma$$

This completes the proof.

[3]

3. Ex-spaces

Consider a trivial fibration $p: F \to *$ and a map $f: F \to F$, where F is a finite complex. In this case the transfer map we seek is to be of the form $\tau(f): S^s \to S^s \wedge F^+$, for large s, and is to have the property that

$$S^s \xrightarrow{\tau(f)} S^s \wedge F^+ \xrightarrow{1 \wedge p^+} S^s$$
 have degree Λ_f .

To construct $\tau(f)$ let $\mu: S^s \to F^+ \land \hat{F}$ be a duality map and take $\tau(f)$ to be composite

$$S^{s} \xrightarrow{\mu} F^{+} \land \hat{F} \xrightarrow{(1,f^{+})\land 1} F^{+} \land F^{+} \land \hat{F} \xrightarrow{1 \land \hat{\mu}} F^{+} \land S^{s} \xrightarrow{1 \land \hat{\mu}} S^{s} \land F^{+}.$$

Then it is immediate from the preceding lemma that $p\tau(f)$ has degree Λ_{f} .

In order to define $\tau(f)$ in general we intend to carry out the above construction "fiberwise". This leads naturally to the consideration of ex-spaces and duality for ex-spaces. In this section we discuss some aspects of the homotopy theory of ex-spaces and in the following section we deal with duality proper.

We shall work entirely in the category of compactly generated spaces [17]. Recall that an ex-space [13] $E = (E, B, p, \Delta)$ consists of maps $p: E \rightarrow B$ and $\Delta: B \rightarrow E$ such that $p\Delta = 1$. We assume throughout that B is a CW-complex and E has the homotopy type of a CW-complex. An *ex-map* $f: E \rightarrow E'$ is one which is both fiber and cross-section preserving, i.e. p'f = p and $f\Delta = \Delta'$. The set of ex-homotopy classes of ex-maps from E to E' is denoted by [E; E'].

An ex-space E is an *ex-fibration* if there is a lifting function

$$\Gamma: E \times_{\mathbf{B}} B^{\mathsf{I}} \to E^{\mathsf{I}}$$

with the property that $\Gamma(\Delta(b), \sigma) = \Delta \sigma$, when σ is a path in *B* beginning at *b*. We will also need the notion of a *well based* ex-space as in [13]. *E* is well based if there is a vertical retraction map $E \times I \rightarrow E \times \{0\} \cup \Delta(B) \times I$.

If $p: E \to B$ is a map we have an associated ex-space $\overline{E} = (\overline{E}, B, \overline{p}, \overline{\Delta})$ where \overline{E} is the disjoint union of E and B and \overline{p} and $\overline{\Delta}$ are the obvious maps. Observe that \overline{E} is well based, and if $p: E \to B$ is a fibration, \overline{E} is an ex-fibration.

If X is a pointed space we will also use X to denote the ex-space

 $(X \times B, B, p, \Delta)$ where p is projection on the second factor and Δ is the cross section determined by the base point.

The fiberwise reduced product of ex-spaces E and E' is denoted by $E \wedge_B E'$. Let $r: E \times_B E' \to E \wedge_B E'$ denote the identification map. Because of the exponential law in the category of compactly generated spaces, $r \times 1: (E \times_B E') \times Y \to (E \wedge_B E') \times Y$ is an identification for any space Y. From this it is easy to see that $r \times_B 1: (E \times_B E') \times_B Y \to (E \wedge_B E') \times_B Y$ is an identification for any space Y over B. With this last observation it is easy to prove the following.

(3.1) LEMMA: If E and E' are well based so is $E \wedge_B E'$. If E and E' are ex-fibrations so is $E \wedge_B E'$.

(3.2) THEOREM: (Comparison theorem): Let E and E' be exfibrations and suppose $g: E \to E'$ is such that its restriction to the fiber over $b, g_b: F_b \to F'_b$ is an n-equivalence, $b \in B$. Let X be a well based ex-space. Then $g_{\#}: [X; E] \to [X; E']$ is injective if X is n-coconnected and surjective if X is (n + 1)-coconnected.

The proof is the same as the proof given for bundles in [1; theorem 3.3]. For other versions of the comparison theorem, see Eggar [11; Theorem 3.9] and James [14; Theorem 3.2].

(3.3) COROLLARY: Suppose that E and E' are well based exfibrations and $g: E \to E'$ is such that $g_b: F_b \to F'_b$ is a homotopy equivalence, $b \in B$. Then g is an ex-homotopy equivalence.

Given $E = (E, B, p, \Delta)$ let $\Omega_B(E)$ denote the space of loops $\sigma : I \to E$ such that $\sigma(I) \subset F_b$ for some $b \in B$, and $\sigma(0) = \sigma(1) = \Delta(b)$. We have

$$\Omega(p): \Omega_B(E) \to B$$
 and $\Omega(\Delta): B \to \Omega_B(E)$

by $\Omega(p)(\sigma) = p(\sigma(0))$ and $\Omega(\Delta)(b) = \Delta(b) = \Delta(b)^*$ – the constant loop at $\Delta(b)$. If E is an ex-fibration so is $\Omega_B(E)$ as is easily checked.

There is the suspension map

(3.4)
$$\sigma:[E,E'] \to [S^1 \wedge_B E; S^1 \wedge_B E']$$

by $f \to 1 \wedge f$. By a standard argument involving the comparison theorem and the loop space $\Omega_B(S^1 \wedge E')$, we obtain the following suspension theorem (c.f. [1; Theorem 3.14] or [14; Theorem 4.3]). (3.5) THEOREM: Suppose that E' is an ex-fibration such that each fiber $F_{b'}$ is (n-1)-connected. Let E be well based. Then σ is injective if E is (2n-1)-coconnected and surjective if E is 2n-coconnected.

Let

(3.6)
$$\{E; E'\}_q = \text{LIM}_k[S^{k+q} \wedge E; S^k \wedge E']$$

with the natural abelian group structure. The cone over E is $C(E) = I \wedge_B E$ with 0 the base point of I.

Suppose that A is a subcomplex of B. Let $E_A = p^{-1}(A) \cup \Delta(B)$ regarded as an ex-space of B. Then, as in [13], we have an exact sequence

$$\cdots \rightarrow \{E \cup C(E_A); E'\}_q \rightarrow \{E; E'\}_q \rightarrow \{E_A; E'\}_q \rightarrow \cdots$$

Let E/E_A be the quotient of E obtained by identifying each fiber of E_A to its base point and let $c: E \cup C(E_A) \rightarrow E/E_A$ denote the natural map. Note that if E is well based so are E_A , E/E_A and $E \cup C(E_A)$.

(3.7) LEMMA: If E and E' are ex-fibrations and E is well based then $c^*:[E/E_A; E'] \rightarrow [E \cup C(E_A); E']$ is bijective.

A proof is given in section 10. Now if E and E' meet the requirements of the lemma we may replace $\{E \cup C(E_A); E'\}$ in the above sequence by $\{E/E_A; E'\}$ via c^* and so obtain an exact sequence

$$(3.8) \qquad \cdots \rightarrow \{E/E_A; E'\}_q \rightarrow \{E; E'\}_q \rightarrow \{E_A; E'\}_q \rightarrow \cdots$$

4. Duality

In this section we will outline Spanier-Whitehead duality theory in the category of ex-spaces. Some aspects of this theory have been dealt with by K. Tsuchida [18]. We restrict ourselves to ex-spaces which are well based ex-fibrations having base B a finite dimensional complex and each fiber homotopy equivalent to a finite complex. Briefly, we will refer to such ex-spaces as ex-fibrations.

An ex-map $\mu: S^s \times B \to E \wedge_B \hat{E}$ is a *duality map* if for each $b \in B$ the restricted map $\mu_b: S^s \to F_b \wedge \hat{F}_b$ is a duality map in the usual sense.

Given such a duality map and ex-fibrations X and Y we have

$$(4.1) D_{\mu}: \{X \land E; Y\}_q \to \{X; Y \land \hat{E}\}_{q+s}$$

defined by sending $f: S^{k+q} \wedge X \wedge E \rightarrow S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \longrightarrow S^{k+q} \wedge X \wedge S^s \xrightarrow{1 \wedge \mu}$$

$$S^{k+q} \wedge X \wedge E \wedge \hat{E} \xrightarrow{f \wedge 1} S^k \wedge Y \wedge \hat{E}$$

and

$$(4.2) D^{\mu}: \{\hat{E} \land X; Y\}_q \to \{X; E \land Y\}_{q+s}$$

by sending $f: S^{k+q} \wedge \hat{E} \wedge X \rightarrow S^k \wedge Y$ to

$$S^{k+q+s} \wedge X \xrightarrow{1 \wedge \mu \wedge 1} S^{k+q} \wedge E \wedge \hat{E} \wedge X \xrightarrow{} E \wedge S^{k+q} \wedge \hat{E} \wedge X \xrightarrow{} E \wedge S^{k} \wedge Y \longrightarrow S^{k} \wedge E \wedge Y.$$

(4.3) LEMMA: D_{μ} and D^{μ} are isomorphisms.

This follows from the corresponding fact for pointed spaces if all the ex-spaces involved are products. The proof in general is by induction over the skeleta of B using the exact sequence (3.8). The argument is standard and will be omitted.

If $\nu: S^s \times B \to X \wedge \hat{X}$ is a second duality map we have, as in the case of pointed spaces, an isomorphism

$$(4.4) D(\mu, \nu): \{E, X\}_q \to \{\hat{X}; \hat{E}\}_q$$

defined so as to make the following diagram commutative

(4.5)
$$\{E; X\}_{q} \xrightarrow{D(\mu, \nu)} \{\hat{X}; \hat{E}\}_{q}$$
$$\bigvee_{D_{\mu}} \bigvee_{D^{\nu}} \{S^{s} \times B, X \land \hat{E}\}_{q+s}$$

In particular, $f: E \to X$ is dual to $g: \hat{X} \to \hat{E}$ relative to μ and ν if and

only if the diagram

is stably homotopy commutative.

(4.7) THEOREM: If E is an ex-fibration there is an integer s, an ex-fibration \hat{E} , and a duality map $\mu: S^s \times B \to E \wedge \hat{E}$.

A proof is given in section 11.

5. Transfer

Let \mathcal{F} denote the category of fibrations $p: E \rightarrow B$ such that B is a finite dimensional complex and each fiber is homotopy equivalent to a finite complex. We consider commutative triangles



where $p: E \to B$ is in \mathcal{F} . We will construct for such a triangle and for A a subcomplex of B a *transfer* map, which is an S-map

(5.1) $\tau(f): B/A \to E/E_A.$

Here $E_A = p^{-1}(A)$.

Consider the ex-space \overline{E} , the disjoint union of E and B. Since \overline{E} is an ex-fibration in the sense of section 4, there is an ex-fibration \hat{E} and a duality map

$$\mu: S^s \times B \to \bar{E} \wedge \hat{E}.$$

Analogous to the situation for pointed spaces (see section 2), $\overline{E} \wedge \hat{E}$ is canonically 2s-self dual. Let

$$\hat{\mu}: \bar{E} \land \hat{E} \to S^s \times B$$

be dual to μ . We have

$$S^{s} \times B \xrightarrow{\mu} \bar{E} \wedge \hat{E} \xrightarrow{(1,\bar{f}) \wedge 1} \bar{E} \wedge \bar{E} \wedge \hat{E} \xrightarrow{1 \wedge \hat{\mu}} \bar{E} \wedge S^{s} \longrightarrow S^{s} \wedge \bar{E}$$

which takes $S^s \times A \cup s_0 \times B$ into $S^s \times E_A \cup s_0 \times B$.

Identifying these subspaces to a point, the above map yields

$$\tau(f): S^s \wedge B/A \to S^s \wedge E/E_A.$$

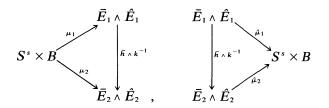
We will show now that the S-homotopy class of $\tau(f)$ is well defined. Firstly, if μ is replaced by a suspension, this has the effect of replacing $\tau(f)$ by its suspension. Suppose now that

$$(5.2) E_i \xrightarrow{I_i} E_i \\ P_i \swarrow^{p_i} \\ B \\ B$$

i = 1, 2, are given and $h: E_1 \rightarrow E_2$ is a fiber homotopy equivalence such that $hf_1 = f_2h$. Let

$$\mu_i: S^s \times B \to \overline{E}_i \wedge \hat{E}_i, \qquad i = 1, 2,$$

be duality maps and let $k: \hat{E}_2 \rightarrow \hat{E}_1$ be dual to \bar{h} . Then k is an ex-homotopy equivalence and we have commutativity relations



where the second triangle is obtained by dualizing the first. The following diagram is then commutative.

$$S^{s} \times B \xrightarrow{\mu_{1}} \bar{E}_{1} \wedge \hat{E}_{1} \xrightarrow{(1, \bar{f}_{1}) \wedge 1} \to \bar{E}_{1} \wedge \bar{E}_{1} \wedge \hat{E}_{1} \xrightarrow{1 \wedge \hat{\mu}_{1}} \bar{E}_{1} \wedge S^{s} \xrightarrow{\mu_{1}} \bar{E}_{1} \wedge S^{s} \xrightarrow{\mu_{1}} \bar{E}_{2} \wedge \bar{E}_{2} \xrightarrow{(1, \bar{f}_{2}) \wedge 1} \bar{E}_{2} \wedge \bar{E}_{2} \wedge \bar{E}_{1} \xrightarrow{1 \wedge \hat{\mu}_{2}} \bar{E}_{2} \wedge S^{s}$$

[9]

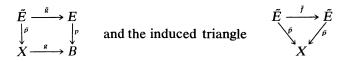
Therefore $\bar{h}\tau(f_1) = \tau(f_2)$. Taking *h* to be the identity we see that $\tau(f)$ does not depend on the choice of duality map and moreover, $\tau(f)$ depends only on the fiber homotopy class of *f*.

We also established the following functorial property

(5.3) With the data (5.2) if $h: E_1 \to E_2$ is a fiber homotopy equivalence such that hf_1 is fiber homotopic to f_2h then $h\tau(f_1) = \tau(f_2)$. Now suppose we are given



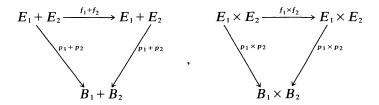
and a map $g: X \rightarrow B$. There is the pullback diagram



(5.4) We have $\tilde{g}\tau(\tilde{f}) = \tau(f)g$.

This is easily checked.

We may form the sum and product of the triangles in (5.2) obtaining



where + denotes disjoint union.

(5.5) $\tau(f_1+f_2) = \tau(f_1) \vee \tau(f_2) : (B_1/A_1) \vee (B_2/A_2) \to (E_1/E_{A_1}) \vee (E_2/E_{A_2})$

(5.6)
$$\tau(f_1 \times f_2) = \tau(f_1) \wedge \tau(f_2) : (B_1/A_1) \wedge (B_2/A_2) \to (E_1/E_{A_1}) \wedge (E_2/E_{A_2}).$$

These properties follow from standard properties of duality maps as generalized to ex-spaces.

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(5.7) For the triangle

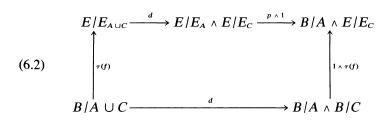


 $\tau(1): B/A \to B/A$ is the identity map.

6. Products

We consider now the multiplicative properties of the cohomology homomorphism induced by the transfer. We have a commutative diagram

where d is the diagonal map. From (5.3), (5.4), (5.6) and (5.7) we obtain, for subcomplexes A and C of B, a cummutative diagram



Let M be a ring spectrum and N an M-module as in [19]. From the commutativity of the above diagram we obtain the formulas

(6.3)
$$\tau(f)^*(p^*(x) \cup y) = x \cup \tau(f)^*(y),$$

 $x \in M^s(B/A), y \in N^t(E/E_c).$

(6.4)
$$p_{*}(\tau(f)_{*}(x) \cap y) = x \cap \tau(f)^{*}(y),$$

 $x \in N_{s}(B/A \cup C), y \in M^{t}(E/E_{c}).$

Now consider the triangle



In the diagram

the composite $(1 \land \hat{\mu})((1, f^+) \land 1)\mu$ represents $\tau(f)$. Hence, by lemma (2.1) and the commutativity of the diagram we have (identifying pt.⁺ with S°).

(6.5) $p\tau(f): S^0 \to S^0$ has degree $\tilde{\Lambda}(f^+) = \Lambda(f)$ – the Lefschetz number of f.

We can now establish the fundamental property of the transfer. Consider



with $p: E \to B$ in \mathcal{F} . Let $f_b: F_b \to F_b$ denote the restriction of f to the fiber over $b \in B$ and let Λ denote the Lefschetz number of f_b . Let $H(\ ; \Gamma)$ denote singular theory with coefficients in the abelian group Γ .

(6.6) THEOREM: If B is connected the composite

$$\tilde{H}^*(B/A;\Gamma) \xrightarrow{p^*} \tilde{H}^*(E/E_A;\Gamma) \xrightarrow{\tau(f)^*} \tilde{H}^*(B/A;\Gamma)$$

is multiplication by Λ .

PROOF: Consider the inclusion



By (6.5), for $1 \in \tilde{H}^{\circ}(B^+; Z)$

$$i_b^* \tau(f)^* p^*(1) = \tau(f_b)^* p_b^*(1) = \Lambda.$$

Since $i_b^*: \tilde{H}^{\circ}(B^+; Z) \to \tilde{H}^{\circ}(\{b\}^+; Z)$ is an isomorphism,

$$\tau(f)^{*}(1) = \tau(f)^{*}p^{*}(1) = \Lambda$$

Applying (6.3), we have for $x \in H^*(B/A; \Gamma)$,

$$\tau(f)^* p^*(x) = \tau(f)^* (p^*(x) \cup 1)$$
$$= x \cup \tau(f)^* (1) = \Lambda x.$$

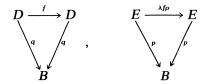
7. The retraction property

In this section we compare the transfer for a fibration with that of a retract up to homotopy.

Suppose that $p: E \to B$ and $q: D \to B$ are fibrations in \mathcal{F} and

$$D \xrightarrow{\lambda} E \xrightarrow{\rho} D$$

are fiber preserving maps such that $\rho \lambda \simeq 1$ over the identity. Then if $f: D \rightarrow D$ is a fiber preserving map we have triangles



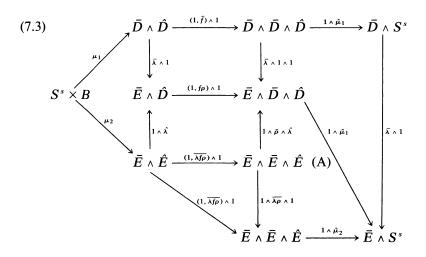
(7.1) THEOREM:
$$\lambda \tau(f) = \tau(\lambda f \rho) : B / A \to E / E_A$$
.

PROOF: Let

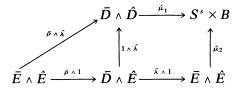
$$\mu_1: S^s \times B \to \bar{D} \wedge \hat{D}$$
$$\mu_2: S^s \times B \to \bar{E} \wedge \hat{E}$$

be duality maps. Let $\hat{\lambda}: \hat{E} \to \hat{D}$ be dual to $\bar{\lambda}: \bar{D} \to \bar{E}$ relative to μ_1 and μ_2 , so that

is commutative. Consider the diagram



The commutativity of the triangle (A) follows from the commutativity of the diagram



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where the square is the dual of (7.2). The remaining commutativity relations in (7.3) are easily checked. The theorem follows by comparing the two outside paths in (7.3) from $S^s \times B$ to $\overline{E} \wedge S^s$.

8. Proof of theorem (1.1)

We begin with an observation concerning the transfer map when the base space is a suspension. Suppose that X is a finite dimensional complex with base point x_0 and we are given



with $\tilde{p}: \tilde{E} \to S(X)$ in \mathcal{F} . Let F denote the fiber over x_0 and choose a base point $e_0 \in F$. Let

(8.1) $\Delta: S(X) \to \tilde{E}/F$

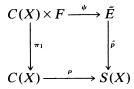
be defined by $\Delta(e^{2\pi i t} \wedge x) = \tilde{\Gamma}(e_0, \sigma(t, x))$ (1), where $\tilde{\Gamma}$ is a lifting function and $\sigma(t, x): I \to S(X)$ is the path

$$\sigma(t, x)(\lambda) = e^{2\pi i t \lambda} \wedge x.$$

Let Λ denote the Lefschetz number of $f': F \to F$, the restriction of f to F.

(8.2) LEMMA: Assume that F is connected. Then $\Lambda\Delta$ is stably homotopic to $\tau(\tilde{f}): S(X) \rightarrow \tilde{E}/F$.

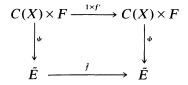
PROOF: Let $C(X) = I \wedge X$ denote the reduced cone of X (with 0 the base point of I) and consider



where ρ is the natural identification and

$$\psi(t \wedge x, y) = \tilde{\Gamma}(y, \sigma(t, x))(1).$$

Then ψ is a homotopy equivalence on each fiber and the restriction of ψ to the fiber over x_0 is the identity. It follows that



is fiber homotopy commutative. Therefore, by (5.3), (5.4) and (5.6) $\tau(\tilde{f})\rho = \psi\tau(1 \times f') = \psi(1 \wedge \tau(f')): C(X)/X \to \tilde{E}/F$, where $\tau(f'): S^{\circ} \to F^+$ is associated with the trivial triangle



Let $i_0: S^{\circ} \to F^+$ by $i_0(+1) = e_0$ and $i_0(-1) = +$. Since F is connected it is clear from the behaviour of the homomorphism in singular homology induced by $\tau(f')$ that $\tau(f') = \Lambda i_0$. Note that

(8.3)
$$C(X)/X \wedge F^{+} \xrightarrow{\psi} \tilde{E}/F$$

$$\uparrow^{1 \wedge i_{0} = j_{0}} \qquad \uparrow^{\Delta}$$

$$C(X)/X \xrightarrow{\rho} S(X)$$

is commutative. Therefore

$$\tau(\tilde{f})\rho = \psi(1 \wedge \tau(f')) = \psi(1 \wedge \Lambda i_0) = \Lambda \Delta \rho.$$

Since $\rho: C(X)/X \to S(X)$ is a homeomorphism, $\tau(\tilde{f}) = \Lambda \Delta$ and the proof is complete.

Now let

$$(8.4) E \xrightarrow{f} E \\ \searrow_{p} / p \\ B \\ B$$

be given where the fiber F of $p: E \to B$ is a finite complex and B is a complex (not necessarily finite dimensional). Choose base points $b_0 \in B$ and $e_0 \in F = p^{-1}(b_0)$ and let $\omega: \Omega B \to F$ denote the boundary map arising from the fibration $p: E \to B$.

If X is a finite dimensional complex with base point and $g: X \to \Omega B$ is a base point preserving map let

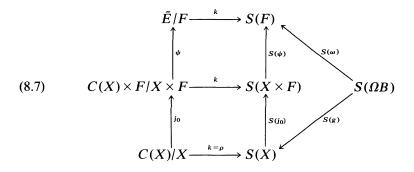
(8.5)
$$\tilde{E} \xrightarrow{f} \tilde{E} \xrightarrow{\tilde{f}} \tilde{E} \xrightarrow{\tilde{p}} \tilde{S}(X)$$

be induced by $S(X) \xrightarrow{S(g)} S(\Omega B) \xrightarrow{\epsilon} B$ where ϵ is the adjoint of the identity map. We will show now that

(8.6)
$$\begin{array}{c}
\tilde{E}/F \xrightarrow{k} S(F) \\
\uparrow^{a} & \uparrow^{s(\omega)} \\
S(X) \xrightarrow{S(g)} S(\Omega B)
\end{array}$$

is commutative, where Δ is as in (8.1) and k is from the Puppe sequence of the cofibration $F \rightarrow \tilde{E}$.

We have



where k in each case is from the appropriate Puppe sequence and j_0 is the inclusion $y \rightarrow (y, e_0)$. The commutativity of the right hand triangle is by direct calculation. The commutativity of (8.6) now follows from the commutativity of (8.3) and (8.7).

We are now in a position to prove the theorem of the introduction.

(1.1) THEOREM: Assume that F is connected. Then

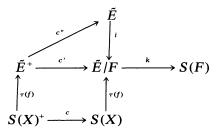
$$A\omega_*: \{X; \Omega B\}_q \to \{X; F\}_q$$

is trivial for any finite dimensional complex X.

PROOF: ΩB has the homotopy type of a *CW*-complex *Y*. If $\phi: Y \rightarrow \Omega B$ is a homotopy equivalence it is sufficient, to prove the theorem, to show that $\Lambda \omega_*(\{g\}) = 0$ if *X* is a finite dimensional subcomplex of *Y* and $g: X \rightarrow \Omega B$ is the inclusion followed by ϕ . By the commutativity of (8.6) and Lemma (8.2)

$$\Lambda \omega_*(\{g\}) = \Lambda \{k\Delta\} = \{k\tau(f)\}.$$

We have a commutative diagram



where c, c', c", and j are quotient maps. Since $\{kj\} = 0$ we have $\{k\tau(f)c\} = c*\{k\tau(f)\} = 0$. Since c* is monomorphic, $\{k\tau(f)\} = 0$ and the proof is complete.

REMARKS: (1) The map ω frequently appears in other forms, hence theorem (1.1) applies for (a) coset maps $\rho: G \to G/H$, or more generally for (b) maps $\omega: M \to X$ which factor through the evaluation map $\mathcal{H}(X) \to X$ where $\mathcal{H}(X)$ is the space of homotopy equivalences of X, or for (c) fibre inclusions of principal bundles. Theorem (1.1) states that $A\omega_* = 0$ and $A\omega^* = 0$ for all homology and cohomology theories on the category of finite dimensional complexes. This is an extension of two results of [7], wherein theorem (1.1) was proved only for singular cohomology and for homotopy groups in the stable range. See also [5].

(8.8) COROLLARY: Let $\alpha \in \pi_i(S^{2n})$. Then $[\alpha, \iota_{2n}] = 0$ implies that $2\{\alpha\} = 0$, where $\{\alpha\}$ denotes the stable homotopy element represented by

 α and $[\alpha, \iota_{2n}]$ is the Whitehead product of α with the generator of $\pi_{2n}(S^{2n})$.

PROOF: The fact that $[\alpha, \iota_{2n}] = 0$ implies there is a map $F: S^i \times S^{2n} \to S^{2n}$ such that F restricted to $* \times S^{2n}$ is the identity and F restricted to $S^i \times *$ represents α . Taking adjoints, we see that α factors through $\omega: M \to S^{2n}$ where M is the space of degree one maps on S^{2n} , and ω is the evaluation map given by evaluation at the base point. Thus (1.1) may be applied to α in view of the remark. In this case $\Lambda = \chi(S^{2n}) = 2$.

Let G be a compact connected Lie group, H a closed subgroup of G, and $\rho: G \to G/H$ the projection.

(8.9) COROLLARY: As an S-map $\chi(G|H)\rho: G \to G|H$ is trivial, where $\chi(G|H)$ is the Euler characteristic of G|H.

In particular, if N is the normalizer of a maximal torus in G then $\rho: G \to G/N$ is stably trivial since $\chi(G/N) = 1$ [6, 12]. On the other hand, it is interesting to note that $\rho_*: \pi_k(G) \to \pi_k(G/N)$ is an isomorphism for k > 2.

9. Smooth fiber bundles

In the case of a smooth fiber bundle $p: E \to B$ a more geometric description can be given for the transfer associated with a fiber preserving map. We assume that B and F are closed manifolds.

Let $\tilde{p}: E \to B \times R^s$ be a fiber preserving embedding. Its normal bundle β is inverse to the bundle α of tangents along the fiber and we have an isomorphism $\alpha \bigoplus \beta \simeq R^s$ associated with the embedding. Let

$$c: B^+ \wedge S^s \to E$$

denote the Pontryagin-Thom map of this trivialization.

The diagonal inclusion into the fiber square, $d: E \rightarrow E^2$, has normal bundle α so that we have

$$c': (E^2)^{\overset{\pi^{*}(\beta)}{\longrightarrow}} E^{\alpha \oplus \beta} = E^+ \wedge S^s$$

where $\pi_1: E^2 \to E$ is projection onto the first factor.

If $f: E \to E$ is a fiber preserving map let

$$(\widetilde{1,f}): E^{\beta} \to (E^2)^{\pi_1^*(\beta)}$$

send v_e to $(e, f(e), v_e)$.

(9.1) PROPOSITION: $\tau(f): B^+ \to E^+$ is represented by

$$B^{+} \wedge S^{s} \xrightarrow{c} E^{\beta} \xrightarrow{(1, f)} (E^{2})^{\pi^{*}_{1}(\beta)} \xrightarrow{c'} E_{s}$$

First observe that the S-map determined by c'(1, f)c is independent of the choice of embedding \tilde{p} and of the tabular neighborhood maps used in constructing c and c'.

Now, for $\tilde{p}: E \to B \times R^s$, let β denote the normal bundle, let $S(\beta)$ denote the total space of the unit sphere bundle, and let \hat{E} denote the quotient of $D(\beta)$ obtained by identifying each fiber of $S(\beta)$ to a point. We regard \hat{E} as an ex-space of B by $\hat{p}: \hat{E} \to B$ and $\hat{\Delta}: B \to \hat{E}$ where $\hat{p}([v_e]) = p(e)$ and $\hat{\Delta}(p(e)) = [v'_e]$, where $v'_e \in S(\beta)$. Then \hat{p} is the projection of a fiber bundle whose fiber over b is the Thom space $F_{b^b}^{\nu_b}$, where ν_b is the normal bundle of the embedding $F_b \to \{b\} \times R^s$.

Choose a fiber preserving tubular neighborhood $D(\beta) \subset B \times R^s$ and let $\theta: B \times S^s \to \hat{E}$ denote the associated Pontryagin-Thom map. Let

$$(9.2) \qquad \mu: B \times S^s \to \bar{E} \wedge_B \hat{E}$$

be the composite $B \times S^s \xrightarrow{\theta} \hat{E} \xrightarrow{d} \bar{E} \wedge_B \hat{E}$, where $d(v_e) = e \wedge v_e$. By Atiyah's duality theorem [1] μ is a duality map.

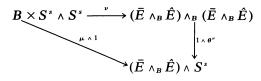
The diagonal embedding $E \to E \times_B D(\beta)$ has normal bundle $\alpha \oplus \beta = E \times R^s$. Choosing a fiber preserving tubular neighborhood $E \times D^s \subset E \times_B D(\beta)$, we obtain

(9.3)
$$\theta': \bar{E} \wedge_B \hat{E} \to \bar{E} \wedge_B S^s.$$

Let $\theta'': \overline{E} \wedge_B \widehat{E} \to B \times S^s$ denote θ' followed by the projection $\overline{E} \wedge_B S^s \to B \times S^s$.

(9.4) LEMMA:
$$\theta''$$
 is dual to μ .

PROOF: We must show that



is homotopy commutative. Since isotopic embeddings determine homotopic duality maps, the duality map determined by

$$E \xrightarrow{(\emptyset, 0)} B \times R^s \times R^s$$

is homotopic to that determined by

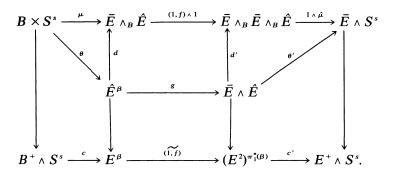
$$E \xrightarrow{(\tilde{p}, \tilde{p})} B \times R^{s} \times R^{s}.$$

The former duality map is $\mu \wedge 1$ whereas, using the factorisation

$$E \xrightarrow{d} E^2 \xrightarrow{\tilde{\rho} \times \tilde{\rho}} B \times R^s \times R^s,$$

the latter is easily seen to be homotopic to $(1 \wedge \theta'')\nu$.

Proposition (9.1) is now a consequence of the following commutative diagram.



Here $d'(e' \wedge v_e) = e \wedge e' \wedge v_e$, $g(v_e) = f(e) \wedge v_e$, and $h(e' \wedge v_e) = (e, e', v_e)$. The unlabeled arrows denote the natural identification map. The commutativity of the upper right hand triangle follows from the fact that $\hat{\mu} = \theta''$.

REMARK It follows from Proposition (9.1) and the retraction property (7.1) that the two methods of constructing the transfer which are outlined in [3], do in fact lead to the same map.

10. Proof of (3.7)

Suppose that $E = (E, B, p, \Delta)$ and $E' = (E', B', p', \Delta')$ are exspaces. If $h: E \to E'$ and $f: B \to B'$ are such that p'h = fp and $h\Delta = \Delta' f$ we will say that h is a map over f. If E' is an ex-fibration then we obtain from the special nature of the lifting function for E' the following covering homotopy property.

(10.1) Given $F: B \times I \rightarrow B'$ and a map $h: E \rightarrow E'$ over F_0 there is $H: E \times I \rightarrow E'$ such that $H_0 = h$ and H_t is a map over F_t , $0 \le t \le 1$.

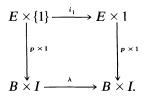
Suppose that E is an ex-space of B and $A \subset B$ is a subcomplex. As before, let $c: E \cup C(E_A) \rightarrow E/E_A$ denote the natural map. Let $\lambda: B \times I \rightarrow B \times \{1\} \cup A \times I$ be a retraction map and let $F: B \times I \rightarrow B$ denote λ followed by projection onto B.

(10.2) LEMMA: Suppose that E is an ex-fibration. There is $q: E/E_A \rightarrow E \cup C(E_A)$ over F_0 and homotopies $H: E/E_A \times I \rightarrow E/E_A$ and $K: E \cup C(E_A) \times I \rightarrow E \cup C(E_A)$ such that

(a) $H_0 = cq$, $H_1 = 1$, and H_t is over F_t , $0 \le t \le 1$.

(b) $K_0 = qc$, $K_1 = 1$, and K_t is over F_t , $0 \le t \le 1$.

PROOF: Consider



Applying (10.1) for the ex-fibration $E \times I$ there is $M: E \times I \rightarrow E \times I$ such that $M_1 = i_1$ and M_t is over λ_t , $0 \le t \le 1$. Then we actually have

$$M: E \times I \to E \times \{1\} \cup E_A \times I.$$

Let q' denote

$$E \xrightarrow{M_0} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$$

Since $M_0(E_A) \subset E_A \times \{0\}$ we have $q'(E_A) \subset \Delta(B)$. Let $q: E/E_A \to E \cup C(E_A)$ denote the collapse of q'. Then q is a map over F_0 .

To construct H, the map

$$E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A) \xrightarrow{c} E/E_A$$

has a quotient $H: E/E_A \times I \rightarrow E/E_A$ which is the desired map.

To construct K let $N: E \times I \rightarrow E \cup C(E_A)$ denote the composite

 $E \times I \xrightarrow{M} E \times \{1\} \cup E_A \times I \longrightarrow E \cup C(E_A).$

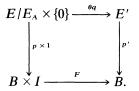
Define $K: E \cup C(E_A) \times I \rightarrow E \cup C(E_A)$ by K(e, t) = N(e, t), if $e \in E$, and K([e, s], t) = s * N(e, t), if $e \in E_A$, where s * [e', t] = [e', st] for $[e', t] \in C(E_A)$. This completes the proof.

We will now prove (3.7) which asserts that

$$c^*: [E/E_A; E'] \rightarrow [E \cup C(E_A); E']$$

is bijective provided E and E' are ex-fibrations.

To show that $c^{\#}$ is onto let $\theta: E \cup C(E_A) \rightarrow E'$ be an ex-map and consider



There is $N: E/E_A \times I \to E'$ over F such that $N_0 = \theta q$. Let $\psi = N_1: E/E_A \to E'$. Then ψ is an ex-map and we will show that $c^{\#}([\psi]) = [\theta]$. We have homotopies

$$E \cup C(E_A) \times I \xrightarrow{c \times 1} E/E_A \times I \xrightarrow{N} E$$

over F, and

$$E \cup C(E_A) \times I \xrightarrow{\kappa} E \cup C(E_A) \xrightarrow{\theta} E'$$

also over F. Then

$$(\theta K^{-1}) \circ (N(c \times 1)): E \cup C(E_A) \times I \to E'$$

is a homotopy from θ to ψc over $F^{-1} \circ F$. By a standard argument involving the covering homotopy property (10.1) we see that θ is ex-homotopic to ψc .

To show that $c^{\#}$ is one-one let $\psi_0, \psi_1: E/E_A \to E'$ be such that $\psi_0 c$ is ex-homotopic to $\psi_1 c$ by $p: E \cup C(E_A) \times I \to E'$ say. We have

$$E/E_A \times (I \times \{0\} \cup \dot{I} \times I) \xrightarrow{Q} E'$$

$$\downarrow^{p \times 1 \times 1} \qquad \qquad \downarrow^{p}$$

$$B \times I \times I \xrightarrow{\tilde{F}} B$$

where $\tilde{F}(b, t, \lambda) = F(b, \lambda)$ and Q(e, t, 0) = p(q(e), t), $Q(e, 0, \lambda) = \psi_0 H(e, \lambda)$, and $Q(e, 1, \lambda) = \psi_1 H(e, \lambda)$. There is then $R : E/E_A \times I \times I \rightarrow E'$ over \tilde{F} which extends Q. Then $S : E/E_A \times I \rightarrow E'$ by S(e, t) = R(e, t, 1) is an ex-homotopy from ψ_0 to ψ_1 .

11. Proof of (4.7)

Let $E = (E, B, p, \Delta)$ be an ex-fibration in the sense of section 4. Let $\Sigma(E)$ denote the unreduced fiberwise suspension of E and $\Sigma(p):\Sigma(E) \rightarrow B$ the projection. There is the "south pole" cross section $\delta: B \rightarrow \Sigma(E)$ given by $\delta(b) = [e, 0]$ where p(e) = b and we let $\Sigma_0(E) = (\Sigma(E), B, \Sigma(p), \delta)$. It is easy to to check that $\Sigma_0(E)$ is an ex-fibration as in section 4. The quotient map $\pi: \Sigma_{0.}(E) \rightarrow S^1 \wedge E$ is an ex-map and is a homotopy equivalence on each fiber. Hence by the comparison theorem (3.2), π is an ex-homotopy equivalence. Note that if $p: E \rightarrow B$ is a fibration with fiber a finite complex (not necessarily equipped with a cross section) we may still form $\Sigma_0(E)$ which is an ex-fibration.

Now let E be an ex-fibration. To construct an ex-fibration \hat{E} and a duality map

$$\mu: S^s \times B \to E \land \hat{E}$$

we proceed by induction over the skeleta of B. Let B have dimension n and let A denote the (n-1)-skeleton of B. Assume there is an ex-fibration D(E|A), with fiber \hat{F} say, and a duality map

(11.1)
$$\omega: S^s \times A \to E | A \wedge D(E|A).$$

Let *B* be obtained from *A* by adjoining cells via $\lambda_j : S^{n-1} \to A, j \in J$, and let $\overline{\lambda_j} : D^n \to B$ denote the characteristic map. Let $\overline{\psi_j} : F \times D^m \to \overline{\lambda_j^*}(E)$ be an ex-fiber homotopy equivalence and $\psi_j : F \times S^{n-1} \to \lambda_j^*(E)$ its restriction. We have a duality map

(11.2)
$$\eta_j: S^s \times S^{n-1} \to \lambda_j^*(E) \wedge \lambda_j^*(D(E|A))$$

induced by ω . Choose a duality map

(11.3)
$$\nu: S^s \times S^{n-1} \to (F \times S^{n-1}) \land (\hat{F} \times S^{n-1})$$

and let $\phi_i : \lambda_i^*(D(E|A)) \to \hat{F} \times S^{n-1}$ be dual to ψ_i relative to η_i and ν . Then ϕ_i is an ex-homotopy equivalence and we have a homotopy commutative triangle.

Let $\theta_i: \hat{F} \times S^{n-1} \to D(E|A)$ denote the composite

$$\hat{F} \times S^{n-1} \xrightarrow{\phi_j^{-1}} \lambda_j^*(D(E|A)) \xrightarrow{\hat{\lambda}_j} D(E|A),$$

and let X' be obtained by adjoining, for each $j \in J$, $\hat{F} \times D^n$ to D(E|A) via θ_j . We have an ex-space (X', B, p', Δ') where p' and Δ' are the obvious maps. Moreover, by results of Dold and Thom [10 (2.2) and (2.10)], $p': X' \to B$ is a quasifibration. We replace p' by a fibration in the usual way obtaining a commutative square

(11.5)
$$\begin{array}{c} X' \xrightarrow{\alpha} X \\ \downarrow p' \uparrow \downarrow \downarrow \downarrow p \\ B \xrightarrow{1} B, \end{array} \xrightarrow{\alpha} B, \end{array}$$

where $X = \{(x, \sigma) \in X' \times B^{T} | p'(x) = \sigma(0)\}, p(x, \sigma) = \sigma(1), \Delta(b) = (\Delta'(b), b^*)$ where b^* denotes the constant path at b, and $\alpha(x) = (x, p'(x)^*)$.

Now $p: X \to B$ is a fibration with each fiber $p^{-1}(b)$ of the weak homotopy type of \hat{F} . Since X (being homotopy equivalent to X') has the homotopy type of a CW-complex, it follows from [16 Proposition 0] that $p^{-1}(b)$ has the homotopy type of a *CW*-complex, hence is homotopy equivalent to \hat{F} .

Let $\hat{E} = \Sigma_0(X)$. Then \hat{E} is an ex-fibration as in section 4. We will show now that there is a duality map

$$\mu: S^{s+1} \times B \to E \wedge \hat{E}.$$

From (11.4) we obtain the following homotopy commutative diagram.

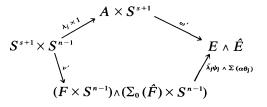
Let ω' denote the duality map

$$S^{s+1} \times A \xrightarrow{1 \wedge \omega} S^{1} \wedge E | A \wedge D(E|A) \rightarrow E | A \wedge S^{1} \wedge D(E|A)$$
$$\xrightarrow{1 \wedge \pi^{-1}} E | A \wedge \Sigma_{0}(D(E|A)) \xrightarrow{1 \wedge \Sigma(\alpha)} E | A \wedge \hat{E} | A,$$

and let ν' denote the duality map

$$S^{s+1} \times S^{n-1} \xrightarrow{1 \wedge \nu} S^{1} \wedge (F \times S^{n-1}) \wedge (\hat{F} \times S^{n-1})$$
$$\longrightarrow (F \times S^{n-1}) \wedge ((S^{1} \wedge \hat{F}) \times S^{n-1}) \xrightarrow{1 \wedge \pi^{-1}} (F \times S^{n-1}) \wedge (\Sigma_{0}(\hat{F}) \times S^{n-1})$$

We have from (11.6) a homotopy commutative diagram



It follows now that $\omega'(\lambda_j \times 1)$ has an extension over $S^{s+1} \times D^n$, for each $j \in J$, and therefore ω' has an extension $\mu: S^{s+1} \times B \to E \wedge \hat{E}$, which is clearly a duality map.

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