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**A SHORT PROOF OF DVORETZKY'S THEOREM
 ON ALMOST SPHERICAL SECTIONS OF CONVEX BODIES**

T. Figiel

In this note we present a short proof of the following important result of A. Dvoretzky [1].

THEOREM: *For every integer $k \geq 2$ and every $\epsilon > 0$, there exists an $N = N(k, \epsilon)$ such that every normed space (X, p) with $\dim X \geq N$ contains a k -dimensional subspace that is ϵ -Euclidean.*

Let us recall that a normed space (X, p) is said to be ϵ -Euclidean if there exist an inner-product norm, say $|\cdot|$, and a constant C such that

$$C(1 - \epsilon)|x| \leq p(x) \leq C|x|, \quad \text{for } x \in X.$$

We shall use another measure of the "distance" between p and $|\cdot|$, which will be denoted here by $v(X, p, |\cdot|)$. The theorem will follow, once we have established that:

A. There exists a sequence (c_n) tending to 0 such that, for any n -dimensional normed space (X, p) , there exists an inner-product norm $|\cdot|$ on X with $v(X, p, |\cdot|) \leq c_n$.

B. For any $(X, p, |\cdot|)$ and any integer k with $1 < k < \dim X$, there exists a subspace E of X such that $\dim E = k$ and $v(E, p|_E, |\cdot|_E) \leq v(X, p, |\cdot|)$.

C. For any $k, \epsilon > 0$, there exists a $\delta > 0$ such that, if $\dim E = k$ and $v(E, p, |\cdot|) < \delta$, then (E, p) is ϵ -Euclidean.

Let us introduce some notation. Given a normed real or complex space (X, p) , $2 \leq \dim X < \infty$, with a Euclidean norm $|\cdot|$, we set

$$S_X = \{x \in X : |x| = 1\},$$

$$x^\perp = \{y \in X : |x + y| = |x - y|\}, \quad \text{for } x \in X,$$

$$\Sigma_X = \{(x, y) \in S_X \times S_X : y \in x^\perp\},$$

λ_x (resp. σ_x) = the normalized $|\cdot|$ -rotation
invariant Borel measure on S_x
(resp. Σ_x),

$$v(x, p, |\cdot|) = \int_{\Sigma_x} (Dp(x)(y))^2 p(x)^{-2} d\sigma_x(x, y).$$

The last formula makes sense because the convex function p is differentiable almost everywhere on S_x and Dp is a measurable and bounded function of x .

Our proof of Property A uses the well known Dvoretzky–Rogers lemma, which can be stated as follows (cf. [1]).

(D–R) For every normed space (X, p) with $\dim X = n$, there exists an integer $m > \frac{1}{2}\sqrt{n} - 1$ and linear operators $T: l_2^n \rightarrow X$, $U: X \rightarrow l_\infty^m$ such that $\|T\| = 1$, $\|U\| \leq 2$, T is one-to-one, and $UT((x_1, \dots, x_n)) = (x_1, \dots, x_m)$ for $(x_1, \dots, x_n) \in l_2^n$.

We define the Euclidean norm on X letting $|x| = \|T^{-1}(x)\|_{l_2^n}$. If p is differentiable at an $x \in X$, and $y \in X$, then $|Dp(x)(y)| \leq p(y) \leq |y|$, therefore we have

$$\begin{aligned} q(x) &\stackrel{\text{def}}{=} \int_{S_x \cap x^\perp} |Dp(x)(y)|^2 d\lambda_{x^\perp}(y) \\ &= \frac{1}{n-1} \sup \{ |Dp(x)(y)|^2 : y \in S_x \cap x^\perp \} \leq \frac{1}{n-1}. \end{aligned}$$

Writing $v(X, p, |\cdot|)$ as an iterated integral, and using (D–R), we get

$$\begin{aligned} v(X, p, |\cdot|) &= \int_{S_x} q(x) p(x)^{-2} d\lambda_x(x) \\ &\leq \frac{1}{n-1} \int_{S_x} \|U\|^2 \|Ux\|^{-2} d\lambda_x(x) = \frac{\|U\|^2}{n-1} \int_{S_{l_2^n}} \|UTz\|^{-2} d\lambda_{l_2^n}(z) \\ &\leq \frac{4}{n-1} \int_{S_{l_2^n}} \left(\max_{1 \leq i < \sqrt{n}/2} |x_i| \right)^{-2} d\lambda_{l_2^n}(z) \stackrel{\text{def}}{=} c_n. \end{aligned}$$

It is known (cf. [4]) that $\lim_{n \rightarrow \infty} c_n = 0$, we shall prove a more general fact in a lemma below.

Property B follows easily from a well known formula, in which γ denotes the normalized rotation invariant measure on the Grassmann manifold Γ of all k -dimensional linear subspaces E of X ,

$$\int_{\Sigma_x} f(x, y) d\sigma_x(x, y) = \int_{\Gamma} d\gamma(E) \int_{\Sigma_x} f(x, y) d\sigma_E(x, y),$$

after substituting $f(x, y) = (Dp(x)(y))^2 p(x)^{-2}$. The formula is valid for any function f that is σ_x -integrable on Σ_x , since the right hand side also defines a normalized invariant integral on Σ_x .

Finally, if C were false, then there would exist numbers k, ϵ and a sequence $(p_n), n = 1, 2, \dots$, of norms on $E = l_2^k$ such that $v(E, p_n, |\cdot|) < 1/n$ and p_n fails to be ϵ -Euclidean for $n = 1, 2, \dots$. Let $S = \{x \in E : |x| = 1\}$. We may assume that $\sup_{x \in S} p_n(x) = 1$ for all n , and hence $\inf_{x \in S} p_n(x) \leq 1 - \epsilon$. By passing to a subsequence (Ascoli's theorem) we may also assume that $p_0(x) = \lim_{n \rightarrow \infty} p_n(x)$ exists for $x \in E$. Clearly

$$\sup_{x \in S} p_0(x) = 1 > 1 - \epsilon \geq \inf_{x \in S} p_0(x).$$

Let

$$A = \{x \in S : Dp_n(x) \text{ exists for } n = 0, 1, 2, \dots\},$$

$$B = \{x \in A : Dp_0(x)(y) = 0 \text{ for } y \in x^\perp\}.$$

Since the p_n 's are convex functions, we have $\lambda_E(A) = 1$, and

$$\lim_{n \rightarrow \infty} Dp_n(x) = Dp_0(x), \text{ for } x \in A.$$

Hence, by Fatou's lemma,

$$\int_{\Sigma_x} Dp_0(x)(y))^2 d\sigma_E(x, y) \leq \underline{\lim} \int (Dp_n(x)(y))^2 d\sigma_E(x, y)$$

$$\leq \underline{\lim} v(E, p_n, |\cdot|) = 0.$$

It follows that $\lambda(A \setminus B) = 0$. Therefore any two points $x_1, x_2 \in S$ can be connected in S by a rectifiable curve $g(t), a \leq t \leq b$, whose almost all points are in B . Consequently,

$$p_0(x_2) - p_0(x_1) = \int_a^b Dp_0(g(t))(g'(t)) dt = \int_a^b 0 dt = 0,$$

i.e. p_0 is constant on S . This contradiction completes the proof of C .

For the sake of completeness we include the following lemma (the probabilistic argument has been indicated by D. L. Burkholder; the approach used in [2] can also be adapted).

LEMMA: Let $m(n)$ be a sequence of positive integers, such that $m(n) \leq n$ and $\lim_{n \rightarrow \infty} m(n) = \infty$, and let

$$\alpha(n) = \frac{1}{n} \int_S \left(\max_{1 \leq i \leq m(n)} |x_i| \right)^{-2} d\lambda(x).$$

where λ is the normalized invariant measure on the unit sphere S of l_2^n . Then $\lim_{n \rightarrow \infty} \alpha(n) = 0$.

PROOF: Let X_1, X_2, \dots be a sequence of independent normalized Gaussian random variables on a probability space (Ω, Σ, P) . If one considers the complex spaces l_2^n , then also the X_i 's should be complex-valued.

Fix n , for the time being, and let $Y_i(\omega) = X_i(\omega)/(\sum_{i=1}^n |X_i(\omega)|^2)^{1/2}$. The map $\omega \rightarrow (Y_1(\omega), \dots, Y_n(\omega))$ transports the measure P onto a normalized rotation invariant measure on S . Thus we have

$$\begin{aligned} \alpha(n) &= \frac{1}{n} \int_{\Omega} \left(\max_{1 \leq i \leq m(n)} |Y_i| \right)^{-2} dP \\ &= \frac{1}{n} \int_{\Omega} \sum_{i=1}^n |X_i|^2 \left(\max_{i \leq m(n)} |X_i| \right)^{-2} dP \\ &= \frac{m(n)}{n} \int_{\Omega} |X_1|^2 \left(\max_{i \leq m(n)} |X_i| \right)^{-2} dP + \frac{n - m(n)}{n} \int_{\Omega} \left(\max_{i \leq m(n)} |X_i| \right)^{-2} dP. \end{aligned}$$

Now we let n tend to infinity. Then the integrals on the right-hand side tend to zero, by the dominated convergence theorem, because they are finite (the second one, if $m(n) \geq 4$) and the X_i 's are unbounded almost surely. This completes the proof.

REMARKS: Our introducing of the quantity $v(X, p, |\cdot|)$ has been suggested by Szankowski's [6]. The lemma was motivated by Lemma 9 in [4]. In fact, our Property A is essentially equivalent to Lemma 10 in [4].

It is not difficult to prove quantitative versions of properties A and C, but our estimate for $N(k, \epsilon)$ is not as good as those given in [5] and [6].

Finally, we should mention that there exists a "combinatorial" proof of Dvoretzky's theorem, at least in the real case, (cf. [7] and [3]).

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