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ON $\lambda(\phi, P)$-NUCLEARITY

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Introduction

We consider sequence spaces $\lambda(\phi, P, \infty)$ and $\lambda(\phi, P, k)$ and define $\lambda(\phi, P, \infty)$-nuclear resp. $\lambda(\phi, P, \mathbb{N})$-nuclear spaces in order to unify the concepts of $\lambda(P)$-nuclearity, $\Lambda_{\mathbb{N}}(\alpha)$-nuclearity, and ϕ-nuclearity considered in [7, 8, 9]; we are especially interested in obtaining universal generators for the varieties of $\lambda(\phi, P, \infty)$-nuclear and $\lambda(\phi, P, \mathbb{N})$-nuclear spaces. Section 1 of the paper contains various definitions and some remarks on the operator ideal of $\lambda(P, \infty)$-nuclear maps, P a stable, nuclear G_∞-set; in section 2 we consider $\lambda(\phi, P, \infty)$-nuclear spaces and show that $\lambda(\phi, P, \infty)$-nuclearity is the same as $\lambda(Q, \infty)$-nuclearity, Q a suitably chosen G_∞-set. In section 3 we introduce the concept of $\lambda(\phi, P, \mathbb{N})$-nuclearity and extend a result in [8] by showing that $\lambda(Q, \infty)$ – Q suitably chosen G_∞-set – is a universal generator for the variety of $\lambda(\phi, P, \mathbb{N})$-nuclear spaces whenever P is a countable, monotone, stable, nuclear G_∞-set.

1. Definitions, notations, and some remarks on $\lambda(P, \infty)$-nuclear maps

For terminology and notations not explained here we refer to Köthe [3], Pietsch [4], Dubinsky and Ramanujan [1], and Terzioglu [12].

Let X and Y be Banach spaces, λ a normal sequence space, and $\lambda^\times$

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its Köthe-dual. A continuous linear map $T \in L(X, Y)$ is said to be

(i) *λ-nuclear* (written $T \in N_\lambda(X, Y)$) if there exists a representation

$$Tx = \sum_{n=0}^{\infty} \gamma_n \langle x, a_n \rangle y_n \quad \text{for } x \in X$$

with $\{\gamma_n\}_n \in \lambda$, $a_n \in X'$, $\|a_n\| \leq 1$, and $y_n \in Y$, $\{\langle y_n, b\rangle\}_n \in \lambda^\times$ for each $b \in Y'$;

(ii) *pseudo-λ-nuclear* or *$\tilde{\lambda}$-nuclear* (written $T \in \tilde{N}_\lambda(X, Y)$) if T has a representation

$$Tx = \sum_{n=0}^{\infty} \gamma_n \langle x, a_n \rangle y_n \quad \text{for } x \in X$$

with $\{\gamma_n\}_n \in \lambda$, $a_n \in X'$, $\|a_n\| \leq 1$, $y_n \in Y$, and $\|y_n\| \leq 1$;

(iii) *of type* λ if $\{s_n^{\text{app}}(T)\}_n \in \lambda$ where $s_n^{\text{app}}(T)$ denotes the n-th approximation number of T.

For a locally convex Hausdorff space (l.c.s.) E, $\mathcal{U}(E)$ will denote a neighbourhood base of 0 of absolutely convex, closed sets; E_U will denote the completion of the normed space $E/p_U^{-1}(0)$ $U \in \mathcal{U}(E)$; $\delta_n(V, U)$ denotes the n-th Kolmogorov-diameter of $V \in \mathcal{U}(E)$ with respect to $U \in \mathcal{U}(E)$; $\Delta(E)$ denotes the Δ-diametral dimension of E, viz., the sequences $\{\gamma_n\}_n$ such that given $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ with $\gamma_n \delta_n(V, U) \to 0$.

Let $\lambda(P)$ be a Köthe space with its generating Köthe set P. The Köthe set P is called a *power set of infinite type* if it satisfies the following additional conditions:

(i) for each $a \in P$, $0 < a_n \leq a_{n+1}$, $n \in \mathbb{N}$;

(ii) for each $a \in P$, there exists a $b \in P$ with $a_n^2 \leq b_n$, $n \in \mathbb{N}$.

The corresponding space $\lambda(P, \infty)$ is called a *smooth sequence space of infinite type* or a *G_∞-space*. For an example of a G_∞-space which is not a power series space of infinite type see [1; theorem 2.25].

Throughout, $\lambda(P, \infty)$ is assumed to be a G_∞-space. The nuclearity and related concepts of such spaces are discussed in [1, 12, 13]; we only need the following result.

1.1 LEMMA: *$\lambda(P, \infty)$ is nuclear if and only if there exists a sequence $\{p_n\}_n \in P$ such that $\{1/p_n\}_n \in l_1$.*

We shall frequently say "P is a nuclear G_∞-set" to mean that the corresponding $\lambda(P, \infty)$ is a nuclear G_∞-space; P is said to be a countable, monotone, nuclear G_∞-set if $P = \{\{p_n^i\}_n : i = 1, 2, \ldots\}$ with

$p_n^i \le p_n^{i+1}$ for each $i, n \in \mathbb{N}$ and P is a nuclear G_∞-set; note that P is a G_∞-set already implies $0 < p_n^i \le p_{n+1}^i$.

Let $\lambda(P, \infty)$ be nuclear; then a l.c.s. E is said to be *$\lambda(P, \infty)$-nuclear* if for each $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ such that V is absorbed by U and the canonical map $K(V, U)$ on E_V to E_U is a $\lambda(P, \infty)$-nuclear map. In [1] it is shown that a l.c.s. E is $\lambda(P, \infty)$-nuclear if and only if for each $U \in \mathcal{U}(E)$ there exists $V \in \mathcal{U}(E)$ such that $\{\delta_n(V, U)\}_n \in \lambda(P, \infty)$. We shall denote the class of all $\lambda(P, \infty)$-nuclear spaces by $\mathcal{N}_{\lambda(P,\infty)}$.

A l.c.s. E is said to be *stable* if $E \times E$ is isomorphic to E. It is known that a nuclear G_∞-space is stable if and only if for each $p \in P$ there exists a $q \in P$ such that $\{p_{2n}/q_n\}_n \in l_\infty$ [14]. We say "P is a stable G_∞-set" to mean that $\lambda(P, \infty)$ is a stable G_∞-space. Stability plays an important role in the study of various permanence properties of $\lambda(P, \infty)$-nuclear spaces; the following result can be found in [7; Proposition 4.5].

1.2 Proposition: *Let P be a countable, monotone, nuclear G_∞-set, $P = \{\{p_n^i\}_n : i = 1, 2, \ldots\}$. Then the following statements are equivalent:*

(i) *For each $j \in \mathbb{N}$ there exists a $\gamma(j) \in \mathbb{N}$ such that $\{p_{2n}^j / p_n^{\gamma(j)}\}_n \in l_\infty$.*

(ii) *If $\xi^k \in \lambda(P, \infty)$ for each $k \in \mathbb{N}$ and $\beta: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection defined by $\beta^{-1}(k, m) := 2^{k-1}(2m - 1)$ then there exist $t_k > 0$, $k \in \mathbb{N}$, such that the sequence $\{t_{\beta_1(n)} \xi_{\beta_2(n)}^{\beta_1(n)}\}_n \in \lambda(P, \infty)$ where $\beta(n) = (\beta_1(n), \beta_2(n))$.*

(iii) *If $\xi, \eta \in \lambda(P, \infty)$ and $\zeta = \xi * \eta := (\xi_1, \eta_1, \xi_2, \eta_2, \ldots)$ then there exists a bijection $\pi: \mathbb{N} \to \mathbb{N}$ such that $\{\zeta_{\pi(n)}\}_n \in \lambda(P, \infty)$.*

(iv) *$\mathcal{N}_{\lambda(P,\infty)}$ is closed under the operation of forming countable direct sums.*

(v) *$\mathcal{N}_{\lambda(P,\infty)}$ is closed under the operation of forming finite Cartesian products.*

(vi) *$\mathcal{N}_{\lambda(P,\infty)}$ is closed under the operation of forming arbitrary Cartesian products.*

(vii) *The sum of two $\lambda(P, \infty)$-nuclear maps is a $\lambda(P, \infty)$-nuclear map.*

(viii) *$\lambda(P, \infty)$ is stable.*

An operator ideal A is said to be

(i) *surjective* if for each closed subspace N of a Banach space X and each $T \in L(X/N, Y)$, Y a Banach space, $TQ_N^X \in A(X, Y)$ implies $T \in A(X/N, Y)$, $Q_N^X : X \to X/N$ denotes the canonical map onto X/N;

(ii) *injective* if for each closed subspace M of a Banach space Y and each $T \in L(X, M)$, X a Banach space, $J_M^X T \in A(X, Y)$ implies $T \in A(X, Y)$, $J_M^X : M \to Y$ denotes the injection.

The following result shows that the operator ideal of $\lambda(P, \infty)$-nuclear maps – P a stable, countable, monotone, nuclear G_∞-set – is injective and surjective.

1.3 Proposition: *Let P be a stable, countable, monotone, nuclear G_∞-set, X and Y be Banach spaces with closed unit balls U_X and U_Y. Then the following statements are equivalent.*

(i) *$T \in L(X, Y)$ is $\lambda(P, \infty)$-nuclear.*

(ii) *$T \in L(X, Y)$ is of type $\lambda(P, \infty)$.*

(iii) *$\{s_n^{\text{gel}}(T)\}_n \in \lambda(P, \infty)$ where $s_n^{\text{gel}}(T) := \inf\{\|TJ_M^X\| : \text{codim } M < n, M$ closed subspace of $X\}$ denotes the n-th Gelfand-number.*

(iv) *$\{s_n^{\text{kol}}(T)\}_n \in \lambda(P, \infty)$ where $s_n^{\text{kol}}(T) := \delta_n(TU_X, U_Y)$.*

Proof: (i) $\Leftrightarrow$ (ii) is shown in [1] and [7]. (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are consequences of the facts that $s_n^{\text{gel}}(T) \le s_n^{\text{app}}(T)$ and $s_n^{\text{kol}}(T) \le s_n^{\text{app}}(T)$ for each $n \in \mathbb{N}$ [5]. (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii) are consequences of the facts that $s_n^{\text{app}}(T) \le (n+1)\, s_n^{\text{gel}}(T)$ and $s_n^{\text{app}}(T) \le (n+1) s_n^{\text{kol}}(T)$ for each $n \in \mathbb{N}$ [5] and the well known fact that $\{(n+1)\xi_n\}_n \in \lambda(P, \infty)$ for $\{\xi_n\}_n \in \lambda(P, \infty)$.

1.4 Corollary: *Let P be a stable, countable, monotone, nuclear G_∞-set. Then the operator ideal of $\lambda(P, \infty)$-nuclear maps is injective and surjective.*

Proof: Let X and Y be Banach spaces, M a closed subspace of Y, and X/N a quotient space of X. Then $s_n^{\text{gel}}(T) = s_n^{\text{gel}}(J_M^Y T)$ for all $T \in L(X, M)$ and $s_n^{\text{kol}}(SQ_N^X) = s_n^{\text{kol}}(S)$ for all $S \in L(X/N, Y)$ [5]. Now the proof easily follows from Proposition 1.3.

1.5 Corollary: *Let P be a stable, countable, monotone, nuclear G_∞-set, X and Y be Banach spaces such that each map $T \in L(X, Y)$ is $\lambda(P, \infty)$-nuclear. Then X or Y must be finite-dimensional.*

Proof: By Proposition 1.3 every operator $T \in L(X, Y)$ is of type $\lambda(P, \infty)$, hence of type l_1. By a result of Pietsch [6], X or Y must be finite-dimensional.

REMARK: With methods used in [11], it can be also shown that X or Y is finite-dimensional if each nuclear map $T \in L(X, Y)$ is $\lambda(P, \infty)$-nuclear, P as in Corollary 1.5.

2. On $\lambda(\phi, P, \infty)$-Nuclearity

Throughout, let Φ denote the set of all functions ϕ: $[0, \infty] \to [0, \infty]$ which are continuous, strictly increasing, subadditive with $\phi(0) = 0$ and $\phi(1) = 1$, and satisfy the additional condition (+) there exist constants $M \geq 1$ and $t_\phi \in [0, \infty]$ such that $\phi(\sqrt{t}) \leq \sqrt{\phi(Mt)}$ for $t \in [0, t_\phi)$.

Note that all examples given in [10;2.6] do have the additional property (+). So far we have been unable to find an example of a continuous, strictly increasing, subadditive function ϕ: $[0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ which does not fulfill condition (+).

2.1 DEFINITION: Let $\lambda(P, \infty)$ be nuclear. For $\phi \in \Phi$ define the sequence space $\lambda(\phi, P, \infty)$ by

$$\lambda(\phi, P, \infty) := \{\{\xi_n\}_n : \{\phi(|\xi_n|)\}_n \in \lambda(P, \infty)\}.$$

A l.c.s. E is called *$\lambda(\phi, P, \infty)$-nuclear* if for each $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ such that $\{\delta_n(V, U)\}_n \in \lambda(\phi, P, \infty)$.

For $\phi = \mathrm{id}$ this definition agrees with the definition of a $\lambda(P, \infty)$-nuclear space, so we write $\lambda(P, \infty)$-nuclear instead of $\lambda(\mathrm{id}, P, \infty)$-nuclear.

We now show that the concept of $\lambda(\phi, P, \infty)$-nuclearity is exactly the same as the concept of $\lambda(Q, \infty)$-nuclearity where Q is suitably chosen depending on ϕ.

2.2 LEMMA: *Let P a countable, monotone, nuclear G_∞-set, $P = \{\{p_n^i\}_n : i = 1, 2, \ldots\}$. Take $\phi \in \Phi$; define q_n^i by $1/q_n^i := \phi^{-1}(1/p_n^i)$ n, $i \in \mathbb{N}$. Then $Q := \{\{q_n^i\}_n : i = 1, 2, \ldots\}$ is a countable, monotone, nuclear G_∞-set.*

PROOF: (i) It is obvious that Q is countable and monotone and that for each n, $i \in \mathbb{N} q_n^i > 0$ and $q_n^i \leq q_{n+1}^i$.

(ii) Given q^k, $q^l \in Q$, we have to show the existence of a $q^r \in Q$ such that $q^k q^l < q^r$, i.e. $q_n^k q_n^l \leq M q_n^r$ for a constant $M > 0$ and each $n \in \mathbb{N}$. Since P is a Köthe set we find $p^m \in P$ and $M_1 > 0$ such that $p_n^k \leq M_1 p_n^m$ and $p_n^l \leq M_1 p_n^m$ for each $n \in \mathbb{N}$. Since $\lambda(P, \infty)$ is nuclear, there exists a $p^s \in P$ with $\{p_n^m/p_n^s\}_n \in l_1$. This can be seen as follows. By Lemma 1.1 there exists a $p^j \in P$ with $\{1/p_n^j\}_n \in l_1$; we choose $p^h \in P$ with $p_n^j \leq M_2 p_n^h$ and $p_n^m \leq M_2 p_n^h$ for $n \in \mathbb{N}$, we also choose $p^s \in P$ with $(p_n^h)^2 \leq p_n^s$ for all $n \in \mathbb{N}$. (This can be done by condition (ii) of a G_∞-set.) From the inequalities

$$p_n^m/p_n^s \leq p_n^m/(p_n^h)^2 \leq M_2/p_n^h \leq M_2^2/p_n^j$$

we get $\{p_n^m/p_n^s\}_n \in l_1$. We also find an integer n_0 such that $p_n^k/p_n^s \leq 1$, $p_n^l/p_n^s \leq 1$, and $1/p_n^s < t_\phi$ for each $n \geq n_0$ (here t_ϕ is determined by condition (+)). Because of condition (+) for the function ϕ we then have

$$M_\phi^{-1}(1/p_n^k)\phi^{-1}(1/p_n^l) \geq M\phi^{-1}(1/p_n^s)\phi^{-1}(1/p_n^s) \geq \phi^{-1}(1/p_n^s p_n^s)$$

for each $n \geq n_0$. Again because of the nuclearity of $\lambda(P, \infty)$ and the second additional condition for a G_∞-set we find $p^r \in P$, $n_1 \in \mathbb{N}$, $n_1 \geq n_0$, such that $p_n^s p_n^s \leq p_n^r$ for each $n \geq n_1$. Therefore there exists $K > 0$ such that $\phi^{-1}(1/p_n^r) \leq K\phi^{-1}(1/p_n^k)\phi^{-1}(1/p_n^l)$ for each $n \in \mathbb{N}$, i.e. $q^k q^l < q^r$.

(iii) To prove the nuclearity of $\lambda(Q, \infty)$ we use Lemma 1.1 and show the existence of a sequence $\{q_n^k\}_n \in Q$ such that $\{1/q_n^k\}_n \in l_1$. Since $\lambda(P, \infty)$ is nuclear we can find $p^k \in P$ such that $\{1/p_n^k\}_n \in l_1$. Since ϕ is subadditive we have $1/q_n^k = \phi^{-1}\ (1/p_n^k) \leq M/p_n^k$ for each $n \in \mathbb{N}$, and $\lambda(Q, \infty)$ is nuclear.

If $\lambda(P, \infty)$ is a power series space of infinite type then $\lambda(Q, \infty)$ is in general not a power series space as the following example shows.

Example: Define $p^i \in P$ by $p_n^i := n^i$, n, $i \in \mathbb{N}$. Let $\phi \in \Phi$ be the function $\phi_{\log}$ as given in [10], i.e.

$$\phi_{\log}(t) := \begin{cases} 0 & \text{for } t = 0 \\ -\alpha/\log t & \text{for } t \in (0, t_0],\ t_0 \text{ sufficiently small} \\ \beta t + \gamma & \text{for } t \geq t_0 \end{cases}$$

where α, β, $\gamma \geq 0$ are chosen in such a way that $\phi_{\log}$ is continuous with $\phi(1) = 1$. Then $\phi_{\log} \in \Phi$ and $\phi_{\log}^{-1}(t) = \exp(-\alpha/t)$ for $t \in (0, t_0]$. Therefore $q_n^i = \exp(\alpha n^i)$ for $n \geq n_0$, $i \in \mathbb{N}$, and $\lambda(Q, \infty)$ is not a power series space [1].

The following lemma is well known and can be found in [4].

2.3 Lemma: *Let P be a countable, monotone, nuclear G_∞-set. Then $x = \{\xi_n\}_n \in \lambda(P, \infty)$ if and only if $\{\xi_n p_n^i\}_n \in l_\infty$ for each $i \in \mathbb{N}$. The* *topologies determined by the semi-norms* $\pi_i(x) := \Sigma_n |\xi_n| p_n^i$ *resp.* $\sigma_i(x) := \sup |\xi_n| p_n^i$, $i \in \mathbb{N}$, *are equivalent.*

2.4 PROPOSITION: *Let P be a countable, nuclear G_∞-set, $\phi \in \Phi$. Then the following statements are true.*

(i) $\lambda(\phi, P, \infty) = \lambda(Q, \infty)$

(ii) *On* $\lambda(\phi, P, \infty)$ *the two topologies given by the F-norms* $\tilde{\pi}_i(x) := \Sigma_n \phi(|\xi_n|) p_n^i$ *resp.* $\tilde{\sigma}_i(x) := \sup \phi(|\xi_n|) p_n^i$, $i \in \mathbb{N}$, *are equivalent.*

(iii) $\lambda(\phi, P, \infty)$ *with the above topologies is topologically equivalent to* $\lambda(Q, \infty)$.

PROOF: (ii) For $x = \{\xi_n\}_n \in \lambda(\phi, P, \infty)$ we always have $\tilde{\sigma}_i(x) \leq \tilde{\pi}_i(x)$, $i \in \mathbb{N}$. On the other hand for $i \in \mathbb{N}$ there exists $p^r \in P$ and $p^l \in P$ such that $\{1/p_n^r\} \in l_1$ and $p^i p^r < p^l$; therefore

$$\begin{aligned}\tilde{\pi}_i(x) &= \sum_n \phi(|\xi_n|) p_n^i = \sum_n \phi(|\xi_n|) p_n^i p_n^r / p_n^r \leq M \sum_n \phi(|\xi_n|) p_n^l / p_n^r \\ &\leq M_0 \tilde{\sigma}_l(x) \quad \text{for } x \in \lambda(\phi, P, \infty).\end{aligned}$$

To prove (i) and (iii) notice that because of Lemma 2.3 $x = \{\xi_n\}_n \in \lambda(\phi, P, \infty) \Leftrightarrow \phi(|\xi_n|) p_n^k \leq 1$ for $k \in \mathbb{N}$ and $n \geq n_k \Leftrightarrow |\xi|_n \leq \phi^{-1}(1/p_n^k) = 1/q_n^k$ for $k \in \mathbb{N}$ and $n \geq n_k \Leftrightarrow x \in \lambda(Q, \infty)$.

2.5 COROLLARY: *Let P be a countable, nuclear G_∞-set. $\phi \in \Phi$. Then* $[\lambda(Q, \infty)]'_b = [\lambda(\phi, P, \infty)]'_b$.

As an immediate consequence of Proposition 4.6 in [7] we finally get

2.6 PROPOSITION: *Let P be a stable, countable, monotone, nuclear G_∞-set; $\phi \in \Phi$. Let E be a l.c.s. Then the following statements are equivalent.*

(i) *E is $\lambda(\phi, P, \infty)$-nuclear.*

(ii) *E is $\lambda(Q, \infty)$-nuclear.*

(iii) *E is isomorphic to a subspace of a suitable I-fold product* $[\lambda(Q, \infty)'_b]^I$.

3. On $\lambda(\phi, P, \mathbb{N})$-nuclearity

In this section we study the concepts of $\lambda(\phi, P, \mathbb{N})$-nuclearity, $\phi \in \Phi$, and $\lambda(P, \mathbb{N})$-nuclearity, P a countable, monotone G_∞-set. As in the case of $\lambda(\phi, P, \infty)$-nuclearity we obtain that both concepts are closely related. The main result in this section is that whenever P is a countable, monotone, stable, nuclear G_∞-set $\lambda(P,\infty)$ is a universal generator for the variety of $\lambda(P,\mathbb{N})$-nuclear spaces. For $\lambda(P,\infty) = \Lambda_\infty(\alpha)$, α a stable exponent sequence, this result has been established in [8].

Let $P = \{\{p_n^i\}_n \colon i = 1, 2, 3, \ldots\}$ be a countable, monotone G_∞-set. Fix $k \in \mathbb{N}$. We define the sequence space $\lambda(P, k)$ by

$$\lambda(P,k) := \{\{\xi_n\}_n \colon \sum_n |\xi_n| p_n^k < \infty\}.$$

It is obvious that $\lambda(P, k+1) \subset \lambda(P,k)$ and $\lambda(P,\infty) = \bigcap_k \lambda(P,k)$. For $\phi \in \Phi$ define $\lambda(\phi, P, k)$ by $\lambda(\phi,P,k) := \{\{\xi_n\}_n \colon \{\phi(|\xi_n|)\}_n \in \lambda(P,k)\}$.

3.1 Definition: Let P be a countable, monotone, nuclear G_∞-set, $\phi \in \Phi$. A l.c.s. E is said to be

(i) $\tilde{\lambda}(\phi, P, k)$*-nuclear* if for each $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ such that $\{\delta_n(V, U)\}_n \in \lambda(\phi, P, k)$;

(ii) $\lambda(\phi, P, \mathbb{N})$*-nuclear* if E is $\tilde{\lambda}(\phi, P, k)$-nuclear for each $k \geq k_0$, where $\{1/p_n^{k_0}\}_n \in l_1$ because of the nuclearity of $\lambda(P,\infty)$.

For $\phi = \mathrm{id}$ we write $\tilde{\lambda}(P,k)$-nuclear resp. $\lambda(P,\mathbb{N})$-nuclear instead of $\tilde{\lambda}(\mathrm{id}, P, K)$-nuclear resp. $\lambda(\mathrm{id}, P, \mathbb{N})$-nuclear.

If α is an exponent sequence and if $P = \{\{k^{\alpha_n}\}_n \colon k \in \mathbb{N}\}$, then a l.c.s. E is $\lambda(P,\mathbb{N})$-nuclear iff E is $\Lambda_{\mathbb{N}}(\alpha)$-nuclear, i.e. $\tilde{\Lambda}_k(\alpha)$-nuclear for each $k > 1$. So we indeed generalize the concept of $\Lambda_{\mathbb{N}}(\alpha)$-nuclearity considered in [8].

As in section 2 we now show that the concepts of $\lambda(\phi, P, \mathbb{N})$-nuclearity and $\lambda(Q,\mathbb{N})$-nuclearity are the same if Q is suitably chosen.

3.2 Proposition: *Let P be a countable, monotone, nuclear G_∞-set, $\phi \in \Phi$. Define $\{q_n^i\}_n$ by $1/q_n^i := \phi^{-1}(1/p_n^i)$ and $Q := \{\{q_n^i\}_n\}$. Let E be a l.c.s. Then the following statements are equivalent.*

(i) *E is $\lambda(\phi, P, \mathbb{N})$-nuclear.*

(ii) *E is $\lambda(Q,\mathbb{N})$-nuclear.*

Proof: Note that E is $\lambda(\phi, P, \mathbb{N})$-nuclear (resp. $\lambda(Q,\mathbb{N})$-nuclear) will follow if E is shown to be $\tilde{\lambda}(\phi, P, k)$-nuclear (resp. $\tilde{\lambda}(Q,k)$-nuclear) for each $k \geq k_0$, k_0 as in 3.1. Therefore we will show (1) given $k \geq k_0$ there exists $l \in \mathbb{N}$ such that $\lambda(\phi, P, l+1) \subset \lambda(Q,k)$; (2) given $r \geq r_0$ there exists $s \in \mathbb{N}$ such that $\lambda(Q, s+1) \subset \lambda(\phi, P, r)$. Given k, by nuclearity of $\lambda(Q,\infty)$ (Lemma 2.2) there exists $l \in \mathbb{N}$ such that

$\{q_n^k/q_n^l\}_n \in l_1$. If $x = \{\xi_n\}_n \in \lambda(\phi, P, l+1)$ then $\phi(|\xi_n|)p_n^l \le 1$ for $n \ge n_l$ and therefore $\Sigma_n |\xi_n| q_n^k = \Sigma_n |\xi_n| q_n^k q_n^l / q_n^l < \infty$. On the other hand given r, by nuclearity of $\lambda(P, \infty)$, find $s \in \mathbb{N}$ such that $\{p_n^r/p_n^s\}_n \in l_1$. So if $x = \{\xi_n\}_n \in \lambda(Q, s+1)$ we have $|\xi_n| \le 1/q_n^s$ for $n \ge n_s$ and therefore $\Sigma_n \phi(|\xi_n|)p_n^r = \Sigma_n \phi(|\xi_n|)p_n^s p_n^r/p_n^s < \infty$; this shows (1) and (2).

To prove (i) $\Rightarrow$ (ii) fix $k \ge k_0$, find $l \in \mathbb{N}$ such that $\lambda(\phi, P, l+1) \subset \lambda(Q, k)$. Since E is $\lambda(\phi, P, \mathbb{N})$-nuclear, E is $\tilde{\lambda}(\phi, P, l+1)$-nuclear and therefore $\lambda(Q, \mathbb{N})$-nuclear. The implication (ii) $\Rightarrow$ (i) can be proved in the same way.

REMARK: It is easily seen that for $\phi \in \Phi$ $\lambda(\phi, P, \mathbb{N})$-nuclearity implies ϕ-nuclearity [9]. But the reverse of this implication is not true in general as the following example shows.

Take $\phi = \phi_{\log}$; consider the power series space $\Lambda_\infty(\alpha)$ with $\alpha_n := (n+1)^2$. We show that $\Lambda_\infty(\alpha)$ is $\phi_{\log}$-nuclear but not $\lambda(\phi_{\log}, \mathrm{R}, \mathbb{N})$-nuclear if $R := \{\{(n+1)^k\}_n : k = 1, 2, \ldots\}$. The $\phi_{\log}$-nuclearity of $\Lambda_\infty(\alpha)$ immediately follows from Korollar 3.6 in [9]. But $\Lambda_\infty(\alpha)$ is $\lambda(\phi_{\log}, R, \mathbb{N})$-nuclear if and only if there exists $M > 1$ such that $\{\phi_{\log}(1/M^{\alpha_n})\}_n \in \lambda(R, k)$ for each k [8]. Therefore $\Lambda_\infty(\alpha)$ is $\lambda(\phi_{\log}, R, \mathbb{N})$-nuclear if and only if for each $k \ge k_0$ the sum $\Sigma_n (n+1)^k/\alpha_n \log M = \Sigma_n (n+1)^{k-2}/\log M$ is finite which obviously is not true.

In order to answer the question what the model of a universal $\lambda(\phi, P, \mathbb{N})$-nuclear space is, it is enough to describe the model of a universal $\lambda(Q, \mathbb{N})$-nuclear space; so from now on P is always supposed to be a countable, monotone, nuclear G-set.

Since $\lambda(P, \mathbb{N})$-nuclearity implies nuclearity one easily obtains following permanence properties.

3.3 PROPOSITION: *Subspaces, quotient spaces by closed subspaces, and completions of $\lambda(P, \mathbb{N})$-nuclear spaces are $\lambda(P, \mathbb{N})$-nuclear.*

3.4 LEMMA: *For each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that $\{(n+1)\xi_n\}_n \in \lambda(P, k)$ whenever $\{\xi_n\}_n \in \lambda(P, r+1)$.*

PROOF: Fix $k \in \mathbb{N}$. Since $\lambda(P, \infty)$ is nuclear there exists $l \in \mathbb{N}$ such that $\{p_n^k/p_n^l\}_n \in l_1$. Therefore for each $n \in \mathbb{N}$ $(p_0^k/p_0^l) + \cdots + (p_n^k/p_n^l) \le M$ and $n + 1 \le p_n^l M/p_0^k =: M_1 p_n^l$. This implies

$$\sum_n (n+1)\,|\xi_n| p_n^k \le M_1 \sum_n |\xi_n| p_n^k p_n^l \le M_2 \sum_n |\xi_n| p_n^r < \infty.$$

3.5 LEMMA: *Let H_1, H_2 be Hilbert spaces and $T \in L(H_1, H_2)$. Then*
(i) *given k, T is $\tilde{\lambda}(P, k)$-nuclear if T is of type $\lambda(P, k)$;*

(ii) *given* k, *there exists* r *so that* T *is of type* $\lambda(P,k)$ *if* T *is* $\tilde{\lambda}(P,r+1)$-*nuclear.*

PROOF: (i): We recall that for a compact operator $T\in L(H_1,H_2)$ T has a representation $Tx=\Sigma_n\lambda_n(x,e_n)f_n$ for suitable orthonormal sequences $\{e_n\}$, $\{f_n\}$ in H_1 resp. H_2; also $\lambda_n=\delta_n(TU_{H_1},U_{H_2})=s_n^{\text{app}}(T)$.

(ii) Fix k. Lemma 3.4 guarantees the existence of $r\in\mathbb{N}$ such that $\{(n+1)\xi_n\}_n\in\lambda(P,k)$ if $\{\xi_n\}_n\in\lambda(P,r+1)$. Now let T be $\tilde{\lambda}(P,r+1)$-nuclear, then T has a representation $Tx=\Sigma_n\gamma_n\langle x,a_n\rangle y_n$, $\{\gamma_n\}_n\in\lambda(P,r+1)$, $\|a_n\|=\|y_n\|=1$, $a_n\in H_1'$, $y_n\in H_2'$. Since $s_n^{\text{app}}(T)\le\Sigma_{i=n}^{\infty}\gamma_i$, we have

$$\sum_{n=0}^{\infty}s_n^{\text{app}}(T)p_n^k\le\sum_{n=0}^{\infty}p_n^k\sum_{i=n}^{\infty}\gamma_i=\sum_{i=0}^{\infty}\sum_{n=0}^{i}\gamma_ip_n^k$$
$$\le\sum_{i=0}^{\infty}(i+1)\gamma_ip_i^k\le M\sum_{i=0}^{\infty}\gamma_ip_i^r<\infty.$$

3.6 PROPOSITION: *Let* P *be a countable, monotone, stable, nuclear* G_∞-*set. Then countable direct sums of* $\lambda(P,\mathbb{N})$-*nuclear spaces are* $\lambda(P,\mathbb{N})$-*nuclear.*

PROOF: We only indicate a partial proof and refer the reader to [1, Theorem 2.8] for the rest of the proof.

In $E=\bigoplus_{i=1}^{\infty}E_i$ a typical fundamental system of neighbourhoods of 0 is of the form $U=\Gamma((U_i)_i)$ where each U_k is an absolutely convex, closed, and absorbing neighbourhood of 0 in E_k and Γ represents the closed convex hull of the union. Stability of $\lambda(P,\infty)$ now gives a single valued, increasing map γ: $\mathbb{N}\to\mathbb{N}$ such that $\gamma(j)>j$ and $p_{2n}^j=0(p_n^{\gamma(j)})$. Then

(1) $p^k_{2^{j-1}(2m-1)}\le p^k_{2^jm}\le M_{k,j}p_m^{\gamma^j(k)}$ for each j, k, m where $\gamma^j:=\gamma\circ\gamma\circ\cdots\circ\gamma$ (j-times)

(2) $p^k_{2^{j-1}(2m-1)}\le p^j_{2^{j-1}(2m-1)}\le M_jp_m^{\gamma^j(j)}$ for each $k\le j$.

Define β: $\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ by $\beta^{-1}(j,m):=2^{j-1}(2m-1)$; β is a bijection and $\beta(n)=(\beta_1(n),\beta_2(n))$.

We have to show that $E=\bigoplus_iE_i$ is $\tilde{\lambda}(P,k)$-nuclear for each $k\ge k_0$. Fix k, assume $i<k$. Using the fact that E_i is $\lambda(P,\mathbb{N})$-nuclear and therefore nuclear for each neighbourhood $U_i\in\mathcal{U}(E_i)$ we find a neighbourhood $W_i\in\mathcal{U}(E_i)$ so that the canonical map K_i: $(E_i)_{W_i}\to(E_i)_{U_i}$ is represented by

$$K_i(x_i)=\sum_{m=0}^{\infty}\xi_m^i\langle x,a_m^i\rangle y_m^i$$

where $\{\xi_m^i\}_m\in\lambda(P,\gamma^i(k)+1)$, $\|a_m^i\|\le1$, $\|y_m^i\|\le1$; then we have $\Sigma_{m=0}^{\infty}\xi_m^ip_m^{\gamma^i(k)}<\infty$. Now pick $t_i>0$ so that $\Sigma_{m=0}^{\infty}t_i\xi_m^ip_m^{\gamma^i(k)}\le1/2^iM_{k,i}$. For

$i \geq k$ we get a representation

$$K_i(x_i) = \sum_{m=0}^{\infty} \xi_m^i \langle x, a_m^i \rangle y_m^i$$

where $\{\xi_m^i\}_m \in \lambda(P, \gamma^i(i)+1)$, $\|a_m^i\| \leq 1$, $\|y_m^i\| \leq 1$; then we have $\Sigma_{m=0}^{\infty} \xi_m^i p_m^{\gamma^i(i)} < \infty$. Pick $t_i > 0$ so that $\Sigma_{m=0}^{\infty} t_i \xi_m^i p_m^{\gamma^i(i)} \leq 1/2^i M_i$. We then get

$$\begin{aligned}\sum_{n=1}^{\infty} t_{\beta_1(n)} |\xi_{\beta_2(n)}^{\beta_1(n)}| p_n^k &= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| p^k_{\beta^{-1}(j,m)} \\ &= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| p^k_{2^{j-1}(2m-1)} \\ &= \sum_{j=1}^{k-1} \sum_{m=1}^{\infty} t_j |\xi_m^j| p^k_{2^{j-1}(2m-1)} + \sum_{j=k}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| p^k_{2^{j-1}(2m-1)} \\ &\leq \sum_{j=1}^{k-1} \sum_{m=1}^{\infty} t_j |\xi_m^j| M_{k,j} p_m^{\gamma^j(k)} + \sum_{j=k}^{\infty} \sum_{m=1}^{\infty} t_j |\xi_m^j| M_j p_m^{\gamma^j(j)} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} < \infty.\end{aligned}$$

Thus we have shown that $\{t_{\beta(n)} \xi_{\beta(n)}^{\beta(n)}\} \in \lambda(P, k)$. As in the proof of Theorem 2.8 in [1], we now can show that E is $\tilde{\lambda}(P, k)$-nuclear and since this can be done for each $k \geq k_0$, E is $\lambda(P, \mathbb{N})$-nuclear.

3.7 COROLLARY: *Arbitrary products of $\lambda(P, \mathbb{N})$-nuclear spaces again are $\lambda(P, \mathbb{N})$-nuclear whenever P is a countable, monotone, stable, nuclear G_∞-set.*

3.8 PROPOSITION: *Let P be a countable, monotone, nuclear G_∞-set. Then $\lambda(P, \infty)$ is $\lambda(P, \mathbb{N})$-nuclear.*

PROOF: Since $\lambda(P, \infty)$ is a nuclear G_∞-space, there exists a $p^i \in P$ such that $\{1/p_n^i\} \in l_1$. Given k, $r \in \mathbb{N}$, there exists a $p^t \in P$ such that $p^r p^k < p^t$; we also find a $p^s \in P$ so that $p^t p^i < p^s$. We then have

$$\sum_n p_n^k p_n^r / p_n^s \leq M \sum_n p_n^t / p_n^s = M \sum_n (p_n^t p_n^i)/(p_n^s p_n^i) \leq M_0 \sum_n 1/p_n^i < \infty,$$

so $\lambda(P, \infty)$ is $\tilde{\lambda}(P, k)$-nuclear for each k.

Our next result shows that $\lambda(P, \infty)$ is a universal generator for the variety of $\lambda(P, \mathbb{N})$-nuclear spaces if P is stable.

3.9 PROPOSITION: *Let P be a countable, monotone, stable, nuclear G_∞-set; then each $\lambda(P, \mathbb{N})$-nuclear space E is isomorphic to a subspace of $[\lambda(P, \infty)]^I$ for a suitable I.*

PROOF: Let $k \in \mathbb{N}$ be fixed. By stability let γ: $\mathbb{N} \to \mathbb{N}$ be a single valued, increasing function such that $\gamma(j) > j$ and $p^j_{2n} = 0(p^{\gamma(j)}_n)$. Write $\bar{k} := \gamma^k(k)$. Since E is $\lambda(P, \mathbb{N})$-nuclear we have $p^{\bar{k}} \in \Delta(E)$, and there exists an absolutely convex, closed, and absorbing neighbourhood $U \in \mathcal{U}(E)$ so that E'_{U^0} (U^0 is the polar of U) is a Hilbert space; now by Proposition IV.1 of [12] there exists an orthonormal basis $\{e^k_n\rangle_n$ in E'_{U^0} so that the set

$$A_k := \left\{ \sum_{n=0}^{\infty} \xi_n p^{\bar{k}}_n e^k_n : \sum_n |\xi_n|^2 \leq 1 \right\}$$

is equicontinuous in E'. Rearrange the set $\{e^k_n: k, n = 1, 2, \ldots\}$ into a single sequence by using the bijection map β: $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with $\beta^{-1}(k, n) := 2^{k-1}(2n-1)$; use the Gram-Schmidt process to obtain a new orthonormal basis $\{e_m\rangle$ for E'_{U^0}. We then have $e_m = \Sigma_{n \geq m/2^k} (e_m, e^k_n) e^k_n$ and therefore

$$\sum_{n \geq m/2^k} |(e_m, e^k_n)|^2/(p^{\bar{k}}_n)^2 \leq \sum_{n \geq m/2^k} |(e_m, e^k_n)|^2/(p^{\bar{k}}_{m/2^k})^2.$$

Since $p^k_{2^k m} \leq M_k p^{\gamma^k(k)}_m = M_k p^{\bar{k}}_m$ and since without loss of generality we may assume $M_k = 1$, we get

$$\sum_{n \geq m/2^k} |(e_m, e^k_n)|^2/(p^{\bar{k}}_{m/2^k})^2 \leq \sum_{n \geq m/2^k} |(e_m, e^k_n)|^2/(p^k_m)^2 \leq 1/(p^k_m)^2$$

and therefore $p^k_m e_k \in A_k$.

So we have shown that there exists an orthonormal basis $\{e_m\}$ of E'_{U_0} such that $\{p^k_m e_m: m = 1, 2, \ldots\}$ is equicontinuous in E'_{U_0}, for each fixed k.

Let $\mathcal{U} = (U_i: i \in I)$ be a base of neighbourhoods of 0 in E so that each U_i is absolutely convex, closed, and absorbing and $E'_{U^0_i}$ is a Hilbert space. For each $i \in I$ we can get an orthonormal basis $\{e^i_m$: $m = 1, 2, \ldots\}$ of $E'_{U^0_i}$ such that the sets $B_{i,k} := \{p^k_m e^i_m: m = 1, 2, \ldots\}$ are equicontinuous for each fixed k; for each $i \in I$, define the map T_i: $E \to \lambda(P, \infty)$ by $T_i x := \{\langle x, e^i_m \rangle\}_m$; T_i goes into $\lambda(P, \infty)$ and is continuous. Define T: $E \to [\lambda(P, \infty)]^I$ by $Tx := \{T_i x\}_i$. Then T is continuous and one-to-one. With obvious changes, the rest of the proof is exactly the same as of Proposition 3.4 in [8].

In [2], Fenske and Schock consider the class Ω of all l.c.s. E such that the set ω of all strictly positive, non-decreasing sequences of reals is contained in $\Delta(E)$. They prove Ω to be a stability class of nuclear spaces and therefore closed under the operations of forming completions, subspaces, quotients by closed subspaces, arbitrary products, countable direct sums, tensor products, and isomorphic