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# CURVATURE ON HOLOMORPHIC PLANE CURVES II

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## Introduction

Let $C: f(y, z) = 0$ be a holomorphic curve with an ordinary double point at Pand let $C_t: f + t = 0$. Let $B$ be an open ball centered at $P$ which is so small that $B \cap C = C_1 \cup C_2$ where $C_1$ and $C_2$ are non-singular connected holomorphic curves and $C_1 \cap C_2 = \{P\}$. For sufficiently small $t$, $C_t \cap B$ is nonsingular.

As in Part I [3] we will assume that all holomorphic curves are endowed with the metric induced by the Fubini-Study metric on $\mathbb{C}^2$. The Fubini-Study metric on $\mathbb{C}^2$ is given by

$$\frac{2}{1+|y|^2+|z|^2}((1+|z|^2)dyd\bar{y} - \bar{y}zdyd\bar{z} - y\bar{z}dzd\bar{y} + (1+|y|^2)dzd\bar{z})$$

where $y$ and $z$ denote the usual coordinates on $\mathbb{C}^2$.

Here $C_1$ and $C_2$ are Riemannian surfaces as well as Riemann surfaces. In Part I we obtained a formula for the Gaussian curvature and studied the Gaussian curvature on $C_t \cap B$ in the more general case that $C$ had an ordinary singularity at $P$. In Part II we will study the closed geodesics on $C_t \cap B$ and the lines of constant Gaussian curvature on $C_t \cap B$, when $C$ has an ordinary double point at $P$.

From Part I [3] we recall

THEOREM 1: *Let $C \subset \mathbb{C}^2$ be a holomorphic curve defined by $f(y, z) = 0$. The Gaussian curvature at a nonsingular point of $\mathbb{C}$ is given by*

$$K(y, z) = 2 - \frac{(1+|y|^2+|z|^2)^3|H(f)|^2}{(|f_y|^2+|f_z|^2+|yf_y+zf_z|^2)^3} \tag{1}$$

*where*

$$H(f) = 2f_{yz}f_yf_z - f_{yy}f_z^2 - f_{zz}f_y^2 \tag{2}$$

*($H(f)$ is the affine hessian of $f$.)*

We now state the first main result in Part II. For convenience in referral, we number the theorems consecutively throughout Parts I and II.

THEOREM 3: *Let $D$ be a small solid cone with vertex at $P$ and with the line tangent to $C_1$ at $P$ as its axis. Let $D_2$ be the image of $D_1$ under the unitary transformation of $\mathbb{C}^2$ that fixes $P$ and interchanges the tangents to $C_1$ and $C_2$. Let $\tilde{C}_t = C_t \cap B - D_1 - D_2$. Given $M > 0$ there exists $\epsilon > 0$ such that if $-\epsilon < t < \epsilon$*
- (i) *$K$ on $\tilde{C}_t < -M < 0$.*
- (ii) *$\tilde{C}_t$ is homeomorphic to a cylinder.*
- (iii) *$\tilde{C}_t$ is geodesically convex.*
- (iv) *In $\tilde{C}_t$ there is a smooth closed geodesic $\Gamma$ and every other smooth closed geodesic in $C_t$ consists in going around $\Gamma$ more than once.*

The second main result in Part II is a picture of the curves of constant curvature on $B \cap C_t$ in the case that neither branch of $C_0$ has a flex at $P$.

## 1. Proof of Theorem 3

After a unitary transformation we may assume that $P = (0,0)$ and $z = 0$ is tangent to $C$ at $P$. We give the proof for the case that the other tangent at $P$ is $y = 0$. The proof of the more general case is a straightforward generalization of the following argument. Hence we are assuming that

$$f(y,z) = yz + ay^3 + by^2z + cyz^2 + dz^3 + \text{higher order terms} \tag{3}$$

and that in $B, f$ may be factored as $f = f_1 \cdot f_2$ where

$$f_1(y,z) = z + ay^2 + R_1(y,z) \tag{4}$$

$$f_2(y,z) = y + dz^2 + R_2(y,z). \tag{5}$$

Here $R_1$ and $R_2$ have terms of order 2 or more; $R_1$ contains no $y^2$ term and $R_2$ contains no $z^2$ term. Let $C_t : f_t = 0 \cap B$. In this case the cones are defined by

$$D_1 = \{(y, z) \colon |z/y| \leq \alpha\} \; D_2 = \{(y, z) \colon |y/z| \leq \alpha\}$$

for some small positive number $\alpha$.

Part (i) follows immediately from Corollary 2 of Theorem 2.

PROOF OF ii: In the equation defining $C_t$ substitute $z = \beta y$. This is an analytic change of coordinates on $\tilde{C}_t$. Then

$$\tilde{C}_t \colon 0 = t + \beta y^2 + y^3(a + b\beta + c\beta^2 + d\beta^3)$$
$$+ \text{terms of order at least 4 in } y.$$

$$\text{In } \tilde{C}_t, \; \beta = \frac{t}{y^2} \cdot h(y, \beta)$$

where $h$ is analytic and bounded away from 0. Thus $\tilde{C}_t$ is topologically equivalent to the annulus

$$\sqrt{|\alpha|} \leq |\sqrt{\beta}| < \frac{1}{\sqrt{|\beta|}} \text{ via } (y, z) \to \sqrt{\beta} = \frac{z}{y}$$

PROOF OF iii: The following is well-known [4]:

LEMMA 1: *Suppose $M$ is a compact Riemannian surface and $D \subset M$ is open and connected. $D$ is geodesically convex if a geodesic exists through every boundary point of $D$ such that all the points of $D$ in a neighborhood of the boundary point lie on one side of the geodesic.*

Hence it suffices to prove that the geodesic curvature on the boundaries of $\tilde{C}_t$ has the correct constant sign.

LEMMA 2: *Let $E_t$ denote the boundary curve of $\tilde{C}_t$ defined by*

$$E_t \colon f(y, \alpha e^{i\theta} y) + t = 0 \quad 0 \leq \theta \leq 2\pi. \tag{6}$$

*For $t$ sufficiently small the sign of the geodesic curvature $k_g$ on $E_t$ is constant and is given by the sign of $(\alpha^2/1 + \alpha^2) - \frac{1}{2}$.*

PROOF: We may take $y$ to be a local coordinate in a neighborhood of $E_t$. Then we may view $E_t$ as the image of a real analytic function

$$\theta \to y(\theta) \quad 0 \leq \theta \leq 2\pi.$$

Let $h(y)dyd\bar{y}$ give the induced metric. Then with respect to the arc length parameter $s$

$$K_g = \operatorname{Im} \frac{h_y}{h}\frac{dy}{ds} + h\frac{d^2y}{ds^2}\frac{d\bar{y}}{ds}$$

This follows by rewriting the classical formula [5] for $k_g$ in terms of complex coordinates. With respect to the parameter

$$k_g = \frac{1}{\sqrt{h} \cdot |y'|} \operatorname{Im}\left(\frac{h_y}{h} y' + \frac{y''}{y'}\right) \text{ where } y' = \frac{dy}{d\theta}$$

We will show the sign of $\sqrt{h} \cdot |y'| k_g$ is given by the sign of $\alpha^2/1 + \alpha^2 - \frac{1}{2}$ for $t$ sufficiently small by showing

(7) $$\frac{y''}{y} \to \frac{-i}{2} \quad \text{as } t \to 0$$

(8) $$\frac{h_y y'}{h} \to \frac{+i\alpha^2}{1+\alpha^2}$$

We may assume that in a neighborhood of the boundary curve, $C_t$ is the graph of a holomorphic function $z_t(y)$. Then one can calculate using $z = \alpha e^{i\theta} y$ on $E_t$ and writing $z_t(y) = z(y)$

$$y' = \frac{i\alpha e^{i\theta} y}{z_y - \alpha e^{i\theta}}$$

$$\frac{y''}{y'} = i + 2\frac{y'}{y} - \frac{(y')^2 z_{yy}}{i\alpha e^{i\theta} y}$$

Now as $t \to 0$, $z_y \to -\alpha e^{i\theta}$ on $E_t$ so $\dfrac{y'}{y} \to i/2$.

Also as $t \to 0$, $yz_{yy} \to 2\alpha e^{i\theta}$ hence $\dfrac{y''}{y'} \to i/2$.

To evaluate (8) recall that

$$h = \frac{1}{(1 + |y|^2 + |z|^2)^2}(1 + |z_y|^2 + |z - yz_y|^2).$$

Hence $h(y, \alpha e^{i\theta}y) \to 1 + |\alpha|^2$ as $t \to 0$. By calculating $h_y$ one can show $h_y y' \to -i\alpha^2$ as $t \to 0$. Q.E.D.

Finally we must check the meaning of the sign of $k_g$ on $E_t$. Project $E_t$ onto the complex line $z = 0$; the tangent and geodesic curvature vectors of the projected curve $\pi(E_t)$ are just the projections of the tangent and curvature vectors of $E_t$. For $t$ sufficiently small we may assume

$$\pi(E_t)\colon 0 = t + \alpha e^{i\theta}y^2 \text{ and } y' = \frac{-iy}{2}$$

Hence $\pi(E_t)$ is swept out in the clockwise direction. By the lemma $k_g$ is negative on $E_t$. Thus the curvature vector points toward the origin.

The other bounding curve is the image of $E_t$ under a unitary transformation which preserves geodesic curvature. Thus the signs are compatible.

PROOF OF iv: We will prove the following more general

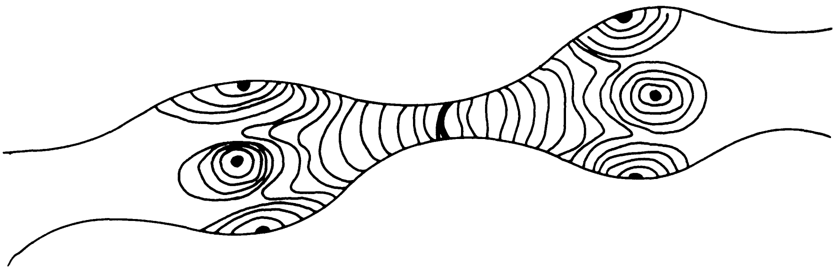
PROPOSITION: *Consider a geodesically convex subdomain $M$ of a Riemannian surface such that $\pi_1(M) \cong \mathbb{Z}$ and the Gaussian curvature is bounded away from zero by a negative constant. Then there exists a unique smooth closed simple geodesic $\Gamma$ and every other smooth closed geodesic in $M$ consists in going around $\Gamma$ more than once.*

PROOF: It is known that if $N$ is a compact Riemannian manifold, then every free homotopy class of loops has a minimal length member which is a smooth closed geodesic. The proof of this result as given in [1] is valid when the hypothesis of compactness is replaced by geodesic convexity. To prove uniqueness, suppose there were two distinct smooth closed geodesics $l_1$ and $l_2$. If $l_1$ and $l_2$ intersect, when lifted to the simply connected covering space they bound a lune $L$ with $\int KdA < 0$ which contradicts Gauss-Bonnet. If $l_1$ and $l_2$ do not intersect, there exists a geodesic $\gamma \subset M$ which realizes the minimum distance between $l_1$ and $l_2$. When $l_1$, $l_2$, and $\gamma$ are lifted to the simply connected covering space, we obtain a geodesic rectangle $R$ where $\int_R KdA < 0$ which again contradicts Gauss-Bonnet.

REMARK: For $t \neq 0$, $H_1(\tilde{C}_t, \mathbb{Z}) = \mathbb{Z}$ and any element of $H_1(\tilde{C}_t, \mathbb{Z})$, $t \neq 0$ is called a vanishing cycle [2]. Theorem 3 shows that there exist unique smooth geodesic representatives of the vanishing cycles.

## 2

I claim that one can draw the following picture of the curves of constant curvature on $B \cap C_t$, for sufficiently small $t$, in the case that neither branch of $C_0$ has a flex at the singular point.



In this case for sufficiently small $t$, $C_t \cap B \cap C_1$ and $C_t \cap B \cap C_2$ each contain exactly three flexes. The dots denote the six flexes. The nipples denote the areas of positive curvature.

The other assumptions made on drawing this picture are $\exists c_0$ depending on $t$ and the coefficients of the third degree terms of the defining equation such that: (1) for $c_0 < c < 2$ the set $K^{-1}(C)$ has six components each of which is a simple closed curve. (2) If $M$ denotes $\min K$ and then $K^{-1}(M)$ is a simple closed curve in $B \cap C_t$ and (3) if $M < c < c_0$, $K^{-1}(c)$ consists of two disjoint simple closed curves, one on each side of $k \equiv M$.

We outline the analytic proof of the picture in the case that the singular point $P = (0, 0)$ and the tangent lines are the axes. Hence we are assuming as in Section 1 that

$$f(y, z) = yz + ay^3 + by^2z + cyz^2 + dz^3 + \text{higher order terms.} \tag{3}$$

We will denote by $Q(\eta)$ the polycylinder

$$Q(\eta) = \{(y, z): |y| < \eta \text{ and } |z| < \eta\}.$$

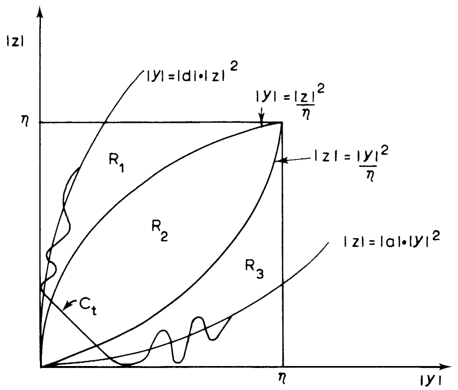
Fix $\eta$ so small that $Q(\eta) \subset B$ and hence $C_0 \cap Q(\eta)$ is the union of two analytic branches defined by (4) and (5). We consider the following subregions of $Q(\eta)$

$$R_1 = Q(\eta) \cap \left\{(y, z): \frac{|y|}{|z|^2} < \frac{1}{\eta}\right\}$$

$$R_2 = Q(\eta) \cap \left\{(y, z): \frac{|y|}{|z|^2} > \frac{1}{\eta} \text{ and } \frac{|z|}{|y|^2} > \frac{1}{\eta}\right\}$$

$$R_3 = Q(\eta) \cap \left\{(y, z): \frac{|z|}{|y|^2} < \frac{1}{\eta}\right\}$$

Hence we have the following diagram of the projection of $Q(\eta)$ into the $|y|$, $|z|$ plane.




We now introduce a change of coordinate on $R_1$ and $R_3$. Let $t = \tau^3$.

$$\text{On } R_1 \text{ let } y' = \frac{y}{\tau^2}\, z' = \frac{z}{\tau}$$

$$\text{On } R_3 \text{ let } y' = \frac{y}{\tau}\, z' = \frac{z}{\tau^2}$$

$R_1$ and $R_3$ are each invariant under these coordinate changes which just expand the regions along the parabolas $y = \gamma z^2$ and $z = \gamma y^2$ respectively, $|\gamma| < 1/\eta$.

LEMMA: *With respect to the* $(y', z')$ *coordinate system in* $R_1$ *as* $t \to 0$

(i) $(C_t \cap R_1)$ *the curve* $0 = 1 + z'y' + d(z')^3$

(ii) $K \to 2 - 4|y'/z'^2|^2$ *on* $C_t \cap R_1$

(iii) *If* $d \neq 0$ *the three flexes on* $C_t \cap R_1$ *approach*

$$(y', z') = (0, \sqrt[3]{-1/d})$$

(iv) *The outer boundary of* $C_t \cap R_1$ *given by*

$$C_t \cap (|y| = \eta) \to \infty.$$

*The convergence in* (i) *and* (ii) *is uniform on compact subsets in the* $(y', z')$ *space. The analogous results hold for* $C_t \cap R_3$.

PROOF: Straightforward computation.

By the previous lemma, then, we can obtain a picture of the curves of constant curvature

$$k = 2 - 4\gamma^2 \quad 0 < \gamma < \frac{1}{\eta}$$

on $C_t \cap R_1$ for $t$ very small by considering the curve

$$C: 0 = 1 + yz + dz^3$$

and the subsets $P_\gamma \subset C$ where

$$P_\gamma = \left\{(y, z) \in C: \left|\frac{y}{z^2}\right| = \gamma \quad 0 < \gamma < \frac{1}{\eta}\right.$$

Note that if $t$ is small enough the image of $C_t$ in the $|y|$, $|z|$ plane lies in an arbitrarily small neighborhood of the image of $C_0$, which if $\eta$ is small enough lies in an arbitrarily small neighborhood of $(|y| = |d|\ |z|^2 \cup |z| = |a|\ \ |y|^2)$.

LEMMA: *On* $R_2 \cap C_t$, *for* $t$ *sufficiently small*

$$K < \left(2 - \frac{2}{\eta^2}\right) + \delta(t)$$

*where* $0 < \delta(t) < 1/\eta^2$ and $\delta(t) \to 0$ as $t \to 0$.

PROOF: Straightforward computation.

LEMMA: *On* $C_t \cap R_1$

(i) *For* $\gamma > |d|$, $P_\gamma$ *is a simple closed curve.*

*If* $d \neq 0$

(ii) *When* $\gamma = |d|$, $P_\gamma$ *is a simple closed curve minus one point at* $\infty$.

(iii) *For* $\gamma < |d|$, $P_\gamma$ *consists of three disjoint simple closed curves*

(iv) $\lim_{\gamma \to 0} P_\gamma = (0, \sqrt[3]{-1/d})$ *which are the flexes.*

PROOF: Obvious.

An analogous lemma holds on $C_t \cap R_3$.

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