ECKART VIEHWEG

Canonical divisors and the additivity of the Kodaira dimension for morphisms or relative dimension one

Compositio Mathematica, tome 35, nº 2 (1977), p. 197-223
<http://www.numdam.org/item?id=CM_1977__35_2_197_0>

COMPOSITIO MATHEMATICA, Vol. 35, Fasc. 2, 1977, pag. 197–223
Noordhoff International Publishing
Printed in the Netherlands

CANONICAL DIVISORS AND THE ADDITIVITY OF THE KODAIRA DIMENSION FOR MORPHISMS OR RELATIVE DIMENSION ONE

Eckart Viehweg

All schemes, varieties and morphisms are defined over the field of complex numbers $\mathbb{C}$.

The following conjecture due to Iitaka is a central problem of the classification theory of algebraic varieties ([21; p. 95], [22]):

CONJECTURE $C_{n,m}$: *Let $\pi : V \to W$ be a surjective morphism of proper, regular varieties, $n = \dim(V)$ and $m = \dim(W)$. Assuming a general fibre $V_w = \pi^{-1}(w)$ is connected, we have the following inequality for the Kodaira dimension*:

$$K(V) \geq K(W) + K(V_w).$$

$C_{2,1}$ is a corollary of Enriques' and Kodaira's classification theory of algebraic surfaces [21; p. 133]. Recently another proof has been given by K. Ueno [23]. I. Nakamura and K. Ueno solved $C_{n,m}$ for analytic fibre bundles $\pi : V \to W$. In this case, V need not be algebraic and equality holds [21]. In [22], Ueno gave a proof of $C_{3,2}$ when $\pi : V \to W$ is a family of elliptic curves with locally meromorphic sections. Some other special cases of $C_{n,1}$ are treated in [21; p. 134] and [23].

In this paper we give an affirmative answer to $C_{n,n-1}$ (π need not be equidimensional).

The case $C_{2,1}$ of "families of curves over a curve" is treated separately (3.7). The proof in this case is rather elementary. In addition we are able to give an explicite description of the canonical divisor of V in terms of the Weierstrass points of the regular fibres and the local behavior of π near the degenerate fibres (3.6). The resulting formulas for the square of the canonical divisor ((3.6) and (4.13)) generalize the formula given by Ueno (see [16; p. 188] and

(4.9)) for families of curves of genus 2. However, as we will see in §4, the "local contributions" are not completely determined by the local invariants of the degenerate fibres ([16], [19]).

In §1 we summarize some known results about the Kodaira dimension and give the reduction of $C_{n,n-1}$ to some statement $C'_{n,n-1}$ (1.6) about the "relative dualizing sheaf". §2 deals with stable curves. We give a description of the relative dualizing sheaf of stable curves using Wronskian determinants (2.10). In §3 and §4 we handle the special case "families of curves over a curve". The proof of $C'_{n,n-1}$ is given in the second half of this paper. §5 contains the proof of some kind of "stable reduction theorem" for higher dimensional base schemes (5.1) and in §6 we use (5.1) and duality theory [6] to reduce the proof of $C'_{n,n-1}$ to stable curves. This special case is handled in §7 and §8.

The proven result is slightly stronger than $C_{n,n-1}$ (see Remark 1.8).

The author wishes to express his gratitude to the Department of Mathematics of the Massachusetts Institute of Technology for the hospitality extended to him.

§1. Kodaira dimension and $\mathscr{L}$-dimension

In this section, X is assumed to be a proper, normal variety and $\mathscr{L}$ to be an invertible sheaf on X.

1.1. DEFINITION:

(i) We set $\mathbb{N}(\mathscr{L}, X) = \{m > 0; \dim_{\mathbb{C}} H^0(X, \mathscr{L}^{\otimes m}) \geq 1\}$.

(ii) For $m \in \mathbb{N}(\mathscr{L}, X)$, we denote by $\Phi_{m,\mathscr{L}} : X \to \mathbb{P}^N$ the rational map given by $\Phi_{m,\mathscr{L}}(x) = (\varphi_0(x), \ldots, \varphi_N(x))$ where $\varphi_0, \ldots, \varphi_N$ is a basis of $H^0(X, \mathscr{L}^{\otimes m})$.

(iii) The *$\mathscr{L}$-dimension* of X is

$$K(\mathscr{L}, X) = \begin{cases} -\infty & \text{if } \mathbb{N}(\mathscr{L}, X) = \emptyset \\ \max\{\dim(\Phi_{m,\mathscr{L}}(X));\ m \in \mathbb{N}(\mathscr{L}, X)\} & \text{if } \mathbb{N}(\mathscr{L}, X) \neq \emptyset. \end{cases}$$

1.2. DEFINITION: Let X be regular and denote the canonical sheaf of X by ω_X. Then $K(X) = K(\omega_X, X)$ is the *Kodaira dimension* of X.

The reader is referred to Ueno [21] for a general discussion of $\mathscr{L}$-dimension and Kodaira dimension. The proofs of the following statements can be found in chapters II and III of [21].

1.3. Properties of the $\mathscr{L}$-dimension:

(i) There exist positive real numbers α, β and m_0, such that for all $m \in \mathbb{N}(\mathscr{L}, X)$, $m \geq m_0$, we have:

$$\alpha \cdot m^{K(\mathscr{L},X)} \leq \dim_{\mathbb{C}} H^0(X, \mathscr{L}^{\otimes m}) \leq \beta \cdot m^{K(\mathscr{L},X)}$$

(ii) Let $f: X' \to X$ be a surjective morphism of normal, proper varieties. Then $K(f^*\mathscr{L}, X') = K(\mathscr{L}, X)$.

(iii) Let $\mathscr{L}' \to \mathscr{L}^{\otimes a}$, $a > 0$, be a non trivial map of invertible sheaves on X. Then $K(\mathscr{L}', X) \leq K(\mathscr{L}, X)$.

1.4. Properties of the Kodaira dimension: X is assumed to be regular.

(i) $K(X)$ depends only on the field of rational functions $\mathbb{C}(X)$ (i.e., $K(X)$ is an invariant of the birational equivalence class of X).

(ii) For every $m \in \mathbb{N}(\omega_X, X)$ we denote the closure of the image of X under Φ_{m,ω_X} by X_m. The induced rational map from X to X_m we also denote by Φ_{m,ω_X}. Assume $K(X) \geq 0$. Then there exists a surjective morphism of regular projective varieties $f: X' \to Y'$ and $m_0 \in \mathbb{N}$, such that for every $m \in \mathbb{N}(\omega_X, X)$, $m \geq m_0$, the following conditions are fulfilled:

(a) f is birationally equivalent to $\Phi_{m,\omega_X}: X \to X_m$ (see remark 1.5).

(b) $\mathbb{C}(Y')$ is algebraically closed in $\mathbb{C}(X)$ and $K(Y') = \dim(Y')$.

(c) There exist closed subvarieties $Z_i \subset Y'$, $Z_i \neq Y'$ for $i \in \mathbb{N}$, such that for every $y \in Y' - \bigcup_{i \in \mathbb{N}} Z_i$ the fibre $X'_y = f^{-1}(y)$ is irreducible, regular and of Kodaira dimension zero.

(iii) Let $g: X \to Y$ be a surjective morphism of proper, regular varieties and X_y a general fibre of g. Then we have:

$$K(X) \leq K(X_y) + \dim(Y).$$

1.5. Remark: Two rational maps (or morphisms) π_i: $V_i \to W_i$, $i = 1, 2$, are called birationally equivalent, if there exist birational maps $\varphi: V_1 \to V_2$ and $\eta: W_1 \to W_2$ such that $\eta \cdot \pi_1 = \pi_2 \cdot \varphi$.

1.4(i) enables us to replace the morphism $\pi: V \to W$ in $C_{n,m}$ by any birational equivalent morphism. Of course, to prove $C_{n,m}$ we may always assume that neither W nor V_w is of Kodaira dimension $-\infty$. Let C be a curve of genus g. Then

$$K(C) = \begin{cases} -\infty & \text{if } g = 0 \\ 0 & \text{if } g = 1 \\ 1 & \text{if } g \geq 2. \end{cases}$$

Hence we may assume, in order to prove $C_{n,n-1}$, that the general fibre of π is a curve of genus $g \geq 1$. The only possible values of $K(V_w)$ are 0 and 1.

1.6. STATEMENT $C'_{n,m}$: *Let $\pi_1 : V_1 \to W_1$ be a surjective morphism of regular, proper varieties with connected general fibre, $n = \dim(V_1)$ and $m = \dim(W_1)$. Then there exists a birationally equivalent morphism $\pi : V \to W$ of regular, proper varieties, such that for a general fibre V_w of π we have the inequality*

$$K(\omega_V \otimes \pi^*\omega_W^{-1}, V) \geq K(V_w).$$

In §8 we are going to prove $C'_{n,n-1}$. The special case $C'_{2,1}$ is proven in §3. The connection with conjecture $C_{n,n-1}$ is given by:

1.7. THEOREM: *Assume that statement $C'_{r,r-1}$ is true for all $r \leq n$. Then $C_{n,n-1}$ is true.*

PROOF: Let $\pi : V \to W$ be a morphism, satisfying the assumptions of $C_{n,n-1}$. Remark 1.5 enables us to assume that $K(\omega_V \otimes \pi^*\omega_W^{-1}, V) \geqslant K(V_w)$. We are allowed, of course, to exclude the trivial cases $K(V_w) = -\infty$ or $K(W) = -\infty$. Choose $m \in \mathbb{N}$ such that $m \in \mathbb{N}(\omega_V, V) \cap \mathbb{N}(\omega_W, W)$ fulfills the condition 1.4(ii), and such that $(\omega_V \otimes \pi^*\omega_W^{-1})^{\otimes m}$ has a non trivial global section. We have an injection

$$\pi_l^* : H^0(W, \omega_W^{\otimes ml}) \to H^0(V, \omega_V^{\otimes ml}) \text{ for every } l \in \mathbb{N}.$$

Using 1.3(i) we get $K(V) \geq K(W)$.

Assume now, that $K(V_w) = 1$ and $K(V) = K(W)$. The injection π_l^* gives a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\Phi_{m,\omega_V}"'] \arrow[d, "\pi"'] & \mathbb{P}^M \arrow[d, "p"] \\ W \arrow[r, "\Phi_{m,\omega_W}"'] & \mathbb{P}^N \end{tikzcd}$$

where p is a projection. The dimension of the images V_m and W_m are equal and $\mathbb{C}(W_m)$ is algebraically closed in $\mathbb{C}(W)$ (1.4(ii)) and therefore in $\mathbb{C}(V)$. Hence p induces a birational map from V_m to W_m.

We can find $u \in W_m$ such that $(p \cdot \Phi_{m,\omega_V})^{-1}(u)$ and $\Phi_{m,\omega_W}^{-1}(u)$ are birationally equivalent to regular varieties V'' and W'' of Kodaira dimension zero (1.4(ii)(c)). Blowing up points of indeterminacy [8], we may assume that π induces a surjective morphism $\pi'': V'' \to W''$. If we choose u in general position, the dimension and the Kodaira dimension of a general fibre of π'' are both one. For $k \in \mathbb{N}(\omega_{W''}, W'') \neq \emptyset$ the sheaf $\omega_{W''}^{\otimes k}$ has a non-trivial section and hence (1.3(iii)) $0 = K(V'') \geq K(\pi''^* \omega_{W''}^{-1} \otimes \omega_{V''}, V'')$. This is a contradiction to $C'_{r,r-1}$, where $r = \dim(V'')$.

1.8. REMARK: Let $K(W) \geq 0$. The argument used at the end of the proof gives the inequality $K(V) \geq \max(K(W) + K(V_w), K(\omega_V \otimes \pi^* \omega_W^{-1}, V))$. In §8 we are going to prove a slightly stronger statement than $C'_{n,n-1}$. Let g be the genus of V_w and M_{go} the coarse moduli scheme of regular curves of genus g. The smooth part of $\pi: V \to W$ induces a rational map $\varphi: W \to M_{go}$. We prove that

$$K(\omega_V \otimes \pi^* \omega_W^{-1}) \geq \max(K(V_w), \dim(\varphi(W))).$$

Hence we get in addition: $K(V) \geq \dim(\varphi(W))$ if $K(W) \geq 0$.

§2. Stable curves and Weierstrass sections

The main references for stable curves are [4], [12] and [5], [11] (genus one).

2.1. DEFINITION: Let S be a scheme and $g \geq 1$.

(i) A *pseudo-stable curve* of genus g over S is a proper, flat morphism $\rho: C \to S$ whose geometric fibres are reduced, connected 1-dimensional schemes C_s of genus g (i.e., $g = \dim_{\mathbb{C}} H^1(O_{C_s})$) with at most ordinary double points as singularities.

(ii) A pseudo-stable curve is called *stable*, if any non singular rational component E of a geometric fibre C_s meets the other components of C_s in more than 2 points.

Example: The only singular stable curve over C of genus 1 is a rational curve with one ordinary double point.

An important advantage in considering stable curves is the existence of moduli schemes. For the definition of fine and coarse moduli schemes see [13; p. 99].

Popp gave a definition of *level μ-structure* for stable curves of genus $g \geq 2$ in [18; p. 235]. We need only two basic properties:

2.2:

(i) Let K be a field of characteristic zero and C a geometrically irreducible, regular curve over K of genus $g \geq 2$. Then there exists a finite algebraic extension K' of K, such that $C \times_K K'$ allows a level μ-structure.

(ii) Let C be a stable curve over $\mathbb{C}$. Then there exists only a finite number of level μ-structures of C over $\mathbb{C}$.

2.3. THEOREM (Popp [18]):

(i) *The coarse moduli space M_g of stable curves of genus $g \geq 2$ exists in the category of algebraic spaces of finite type over $\mathbb{C}$.*

(ii) *The fine moduli space $M_g^{(\mu)}$ of stable curves of genus $g \geq 2$ with level μ-structure ($\mu \geq 3$) and a universal stable curve $\rho_g^{(\mu)}: Z_g^{(\mu)} \to M_g^{(\mu)}$ with level μ-structure exist in the category of algebraic spaces of finite type over $\mathbb{C}$.*

(iii) *M_g and $M_g^{(\mu)}$ are proper over* Spec ($\mathbb{C}$).

Knudson and Mumford obtained a stronger result (see [14]) which is, however, still unpublished:

2.4. THEOREM: *The coarse moduli space M_g is a projective scheme over $\mathbb{C}$.*

For simplicity we are going to use this result. It would be possible, however, to avoid it by working in the category of algebraic spaces in §2, §5 and §7.

2.5. COROLLARY: *$Z_g^{(\mu)}$ and $M_g^{(\mu)}$ are projective schemes.*

PROOF: $M_g^{(\mu)}$ is quasi-finite over M_g (2.2(ii)) and $\rho_g^{(\mu)}$ is a projective morphism [4].

In the case $g = 1$ Deligne and Rapaport gave a definition of *level μ-structure* in [5]. It includes the condition that the stable curve is a generalized elliptic curve [5; p. 178]. For our purpose it is enough to know that the properties 2.2 also hold in this case and that we have the theorem [5]:

2.6. THEOREM: *The coarse moduli scheme M_1 of stable curves of genus 1 and the fine moduli scheme $M_1^{(\mu)}$ of stable elliptic curves with level μ-structure ($\mu \geq 3$) exist as projective curves. $M_1^{(\mu)}$ is a finite Galois cover of M_1. The open part of $M_1^{(\mu)}$ corresponding to the regular curves is affine.*

2.7. LEMMA: *Let S be a normal, proper variety. For $i = 1, 2$ let $\rho_i : C_i \to S$ be stable curves of genus $g \geq 2$. For some open set $U \subset S$ let $\rho_{iU} : C_{iU} \to U$ be the restriction of ρ_i. Then any U-isomorphism $f_U : C_{1U} \to C_{2U}$ can be extended to an S-isomorphism $f : C_1 \to C_2$.*

PROOF: The functor $\mathrm{Isom}_S(C_1, C_2)$ is represented by a scheme $I_S(C_1, C_2)$ which is finite over S [4; p. 84]. f_U induces a morphism $U \to I_S(C_1, C_2)$ over S, which can be extended to S (see [24; II.6.1.13]).

A pseudo-stable curve $\rho : C \to S$ has an invertible *dualizing sheaf* $\omega_{C/S}$ [6].

2.8. LEMMA [4]:

(i) *$\omega_{C/S}$ is compatible with base change.*

(ii) *$\rho_* \omega_{C/S}$ and $R^1\rho_* O_C$ are locally free of rank g and dual to each other.*

(iii) *Assume that S is regular and $C_0 \subseteq C$ the open subscheme on which ρ is smooth. Let $\Omega^1_{C/S}$ be the sheaf of relative differentials. Then $\omega_{C/S}|_{C_0} \cong \Omega^1_{C/S}|_{C_0}$.*

In the second half of this section we are going to describe a divisor D with $\omega_{C/S}^{\otimes g(g+1)/2} \cong O_C(D)$.
Henceforth let S be a normal scheme and $\rho : C \to S$ a pseudo-stable curve with smooth general fibre. Let $\mathscr{L}$ be an invertible sheaf on C, such that $\rho_* \mathscr{L}$ is locally free of rank $r > 0$. Let S_0 be the regular locus of S and $C_0 \subseteq C$ the open part, lying over S_0, on which ρ is smooth. The restriction of ρ to C_0 is denoted by ρ_0.

Since ρ_0 is smooth, every point $x \in C_0$ has a neighbourhood U such that the restriction of ρ_0 to U factors $U \xrightarrow{g} \mathrm{Spec}\,(A[t']) \to \mathrm{Spec}\,(A) \to S_0$ where g is etale. Let t be a parameter on U, lying over t', then $\Omega^1_{C_0/S_0}$ is generated at x by dt. Let $\eta_1, \ldots, \eta_r$ be sections of $\rho_* \mathscr{L}$ in a neighbourhood of $\rho(x)$ and η a generator of $\mathscr{L}$ in a neighbourhood of x. Locally we can write $\eta_i = f_i \cdot \eta$ for $i = 1, \ldots, r$. Define

$$[\eta_1, \ldots, \eta_r] = \det \begin{vmatrix} f_1 \cdot & \cdots & f_r \\ \dfrac{df_1}{dt} & \cdots & \dfrac{df_r}{dt} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \dfrac{d^{(r-1)}f_1}{dt^{(r-1)}} & \cdots & \dfrac{d^{(r-1)}f_r}{dt^{(r-1)}} \end{vmatrix} \cdot dt^{r(r-1)/2} \cdot \eta^r.$$

$[\eta_1, \ldots, \eta_r]$ is independent of the chosen t and η and defines a section of $\mathscr{L}^r \otimes (\Omega^1_{C_0/S_0})^{\otimes r(r-1)/2}$ over some neighbourhood of $\rho^{-1}(\rho(x))$. Now assume, that $\eta_1, \ldots, \eta_r$ and $\eta'_1, \ldots, \eta'_r$ are two bases of $\rho_* \mathscr{L}$ near $\rho(x)$ such that

$$\eta'_i = \sum_{j=1}^{r} a_{ij} \cdot \eta_j \quad \text{for } i = 1, \ldots, r.$$

Then

$$[\eta'_1, \ldots, \eta'_r] = \det |a_{ij}| \cdot [\eta_1, \ldots, \eta_r].$$

Hence $[\eta_1, \ldots, \eta_r] \cdot (\eta_1 \wedge \ldots \wedge \eta_r)^{-1}$ is independent of the chosen basis. Since $C - C_0$ is at least of codimension 2 in C we are able to define:

2.9. Definition: Denote $\mathscr{L}^r \otimes \omega_{C/S}^{\otimes r(r-1)/2} \otimes (\rho^* \bigwedge^r \rho_* \mathscr{L})^{-1}$ by $\mathscr{W}(\mathscr{L})$. The global section $s(\mathscr{L}) = [\eta_1, \ldots, \eta_r] \cdot (\eta_1 \wedge \ldots \wedge \eta_r)^{-1}$ of $\mathscr{W}(\mathscr{L})$ over C is called the *Weierstrass section* of $\mathscr{L}$.

$\rho_* \omega_{C/S}$ is locally free of rank g and hence we have a Weierstrass section $s(\omega_{C/S})$. If, for example, $S = \operatorname{Spec}(\mathbb{C})$, then the divisor of this section is just the usual divisor of Weierstrass points.

$\omega_{C/S}$ and $s(\omega_{C/S})$ are compatible with base change, as long as the assumptions of this section are fulfilled.

For simplicity we use the following notation: Let D be a divisor and $\mathscr{L}$ an invertible sheaf such that $\mathscr{L} \cong O_C(D)$. Then we write $\mathscr{L} \sim D$.

The general fibres of $\rho: C \to S$ are regular. We define $Wp_{C/S}$ to be the closure of the divisor of the Weierstrass points of the general fibres of ρ.

2.10. Theorem (Arakelov [2; p. 1299]): *Let $\rho: C \to S$ be a pseudo-stable curve of genus $g \geq 1$ and S a normal scheme. Let $U = \{s \in S;\ s$ regular point and $\rho^{-1}(s)$ regular curve$\}$. Let d be a divisor on S such that $\wedge^g \rho_* \omega_{C/S} \sim d$. Then there exists a positive divisor $E_{C/S}$ with support in $\rho^{-1}(S - U)$ such that*

$$\omega_{C/S}^{\otimes g(g+1)/2} \sim \rho^* d + Wp_{C/S} + E_{C/S}.$$

Proof: For a (local) base $\omega_1, \ldots, \omega_g$ of $\rho_* \omega_{C/S}$ the section $[\omega_1, \ldots, \omega_g]$ does not vanish identically on a smooth fibre of ρ. Therefore, the divisor of this section is of the form $Wp_{C/S} + E_{C/S}$.

2.11. Remarks:

(i) In the case $g = 1$, there are no Weierstrass points and it is possible to show that $E_{C/S}$ is the zero-divisor if the curve is stable [5; p. 175]. Hence in this case 2.10 reduces to $\rho^* \rho_* \omega_{C/S} \cong \omega_{C/S}$.

(ii) If $\dim(S) = 1$, the support of $Wp_{C/S}$ is finite over S. This is unfortunately no longer true for $\dim(S) \geq 2$ and $g \geq 3$.

§3. The canonical divisor of families of curves over a curve

In §3 and §4 we make the following assumptions:

3.1: *Let W be a regular, proper curve of genus p, let V be a regular, proper surface and $\pi: V \to W$ a surjective morphism whose general fibre is a regular curve of genus $g \geq 1$.*

It follows, that π is flat. A fibre $V_w = \pi^{-1}(w)$ is "degenerate", if it is not reduced or if it has singularities.

$\Delta = \{w \in W;\ V_w$ degenerate fibre$\}$ is a finite set of points. Let $\omega_{V/W} = \omega_V \otimes \pi^* \omega_W^{-1}$ be the dualizing sheaf of π ([6] or §6). We want to describe ω_V. If $g: V^* \to V$ is the proper birational morphism obtained by blowing up a closed point of V and F the exceptional divisor, it is well known, that $\omega_{V^*} = g^*\omega_V \otimes O_{V^*}(F)$. Therefore we may assume that for all $w \in \Delta$ $(\pi^{-1}(w))_{\text{red}}$ has only ordinary double points as singularities.

3.2. Local description ([11] for $g = 1$ and [19] for $g \geq 2$): For $w \in \Delta$ let $\rho: \Gamma \cong V \times_W \operatorname{Spec}(\hat{O}_{w,W}) \to S = \operatorname{Spec}(\hat{O}_{w,W})$ be the induced local family of curves ("$\hat{\ }$" denotes the completion with respect to the maximal ideal).

There exists a cyclic covering S' of S with Galois-group $\langle\sigma\rangle$ such that the normalization Γ' of $\Gamma \times_S S'$ is birationally equivalent to a stable curve $\rho'': \Gamma'' \to S'$. The group $\langle\sigma\rangle$ operates on Γ'', compatible with the operation on S'. Let's call the tuple $(\rho'': \Gamma'' \to S', \sigma)$ a *stable reduction* of $\pi: V \to W$ at w. Denote the closed point of S also by w and the closed point of S' by w'.

Assume that every multiplicity occurring in $\Gamma_w \cong V_w$ divides $n = \operatorname{ord}(\sigma)$. The special fibre $\Gamma'_{w'}$ of Γ' is reduced and has singularities of the (analytic) type $u \cdot v - t^r$, $r \in \mathbb{N}$ (see [12]). Hence the minimal desingularisation Γ_d of Γ' also has a reduced special fibre and is pseudo-stable over S'. The natural maps are denoted by:

$$\begin{tikzcd} \Gamma_d \arrow[r, "h"] \arrow[dr, "\rho_d"'] & \Gamma' \arrow[r, "f"] \arrow[d, "\rho'"] & \Gamma \arrow[r, "i"] \arrow[d, "\rho"] & V \arrow[d, "\pi"] \\ & S' \arrow[r, "f_0"] & S \arrow[r, "i_0"] & W \end{tikzcd}$$

Let $E_d = E_{\Gamma_d/S'}$ be the divisor defined in 2.10. Let l_d be the number of double points of the special fibre $(\Gamma_d)_{w'}$. Define $\delta'_{w'} = \Sigma\,(e_{C'} - n)C'$ where the sum is taken over the set of irreducible components C' of Γ'_w and $e_{C'}$ is the ramification index of C' over Γ.

3.3. DEFINITION: Using the notation of 3.2, we define on Γ (resp. on V):

$$E_w = n^{-1} f_* h_* E_d \quad (E_w = n^{-1} i_* f_* h_* E_d)$$
$$l_w = n^{-1} l_d$$
$$\delta_w = n^{-1} f_* \delta'_{w'} \quad (\delta_w = n^{-1} i_* f_* \delta'_{w'}).$$

E_w may have rational coefficients and l_w may be a rational number. The definition is independent of the chosen S':

3.4. LEMMA:

(i) $\delta_w = \Sigma\,(1 - \mathrm{mult}\,(C)) \cdot C$ *where the sum is taken over all irreducible components* C *of* $\Gamma_w \cong V_w$ *and* $\mathrm{mult}\,(C)$ *is the multiplicity of* C *in* V_w.

(ii) E_w, l_w, *and* δ_w *are independent of the chosen stable resolution.*

PROOF: The description of δ_w in (i) follows from [19, Lemma 7.2]. To prove (ii) we may assume that Γ is already pseudo-stable over S. In this case, however, Γ' is the fibre product $\Gamma \times_S S'$ and has singularities of the type $u \cdot v - t^n$ lying over every double point of Γ_w. The second statement follows from the compatibility of Weierstrass sections with base change and [3; Theorem 2.7].

3.5. REMARK: A stable resolution $(\rho'' : \Gamma'' \to S', \sigma)$ of π in w is called *minimal* if $\mathrm{ord}\,(\sigma) = \mathrm{ord}\,(\sigma_w)$ where σ_w is the restriction of σ to Γ''_w. Such a minimal stable resolution always exists [19]. It is unique up to isomorphism and determines $\rho : \Gamma \to S$. Lemma 3.4 just says that E_w, δ_w and l_w depend only on the minimal stable resolution.

3.6. THEOREM: *We use the notations and assumptions made in* 3.1 *and* 3.3. *Let Wp be the closure of the Weierstrass points of the general fibre of* π. *Then there exists a divisor d on W such that*: (*We denote the intersection numbers by* $(\cdot)$ *or* $(\ )^2$)

(i) $\omega_{V/W}^{\otimes g(g+1)/2} \sim \pi^* d + Wp + \sum_{w \in \Delta} (E_w - \frac{1}{2} g(g+1)\delta_w)$(up to torsion)

(ii) $(\omega_V)^2 = 8(p-1)(g-1) + (\omega_{V/W})^2$

(iii) $(\omega_{V/W})^2 = 12 \deg\,(d) - \sum_{w \in \Delta} (l_w + 2k(\delta_w) + (\delta_w)^2)$

$$= 12 \deg(d) - \sum_{w \in \Delta} [l_w + k(\delta_w) + 2 \cdot g^{-1} \cdot (g+1)^{-1}(Wp \cdot \delta_w) + 2 \cdot g^{-1}(g+1)^{-1}(E_w \cdot \delta_w)]$$

where $k(D) = (\omega_V \cdot O_V(D))$.

(iv) *Define* $\gamma = \chi(O_{Wp}) + (g^3 - g)(p-1)$. *Then we have*

$$[12 \cdot g^2(g+1)^2 + 24 \cdot g(g+1) - 8 \cdot (g^3 - 1)] \cdot \deg(d) =$$

$$-8 \cdot \gamma + \sum_{w \in \Delta} [(g^2(g+1)^2 + 2 \cdot g(g+1)) \cdot l_w + 4(Wp \cdot E_w) + (2 \cdot g^2 + 2 \cdot g + 4) \cdot k(E_w) + 4(Wp \cdot \delta_w) + (2 \cdot g^2 + 2 \cdot g + 4)(E_w \cdot \delta_w)]$$

(v) $\deg(d) \geq 0$, *and* $\deg(d) = 0$ *if and only if there exists a finite cover* W' *of* W *such that* $V \times_W W'$ *is birationally equivalent to a trivial family of curves over* W'. *In this case we may assume* $d = 0$.

3.7. Corollary: $C'_{2,1}$ *is true.*

Proof: If $\deg(d) > 0$, the divisor d is ample on W and hence $C'_{2,1}$ follows from 3.6(i) and 1.3. If $\deg(d) = 0$ it follows from 3.6(v) and 1.3 that

$$K(\omega_{V/W}, V) \geq K(O_V(Wp), V) \geq K(V_w).$$

3.8. Remark: Using the Nakai-Moisezon criterion for ampelness, 1.4(iii) and 3.6 we get more exact results:

$K(V) = 2$ if $g \geq 2$ and $\deg(d) > -(p-1)(g+1)g$,
$K(V) \geq 1$ if $g \geq 2$ and $\deg(d) = -(p-1)(g+1)g$ and
$K(V) = 1$ if $g = 1$ and $\deg(d) > -2 \cdot (p-1)$.

Proof of Theorem 3.6: Using the same kind of construction as in 3.2, we can find a Galois cover W' of W such that the desingularisation V_d of the normalisation V' of $W' \times_W V$ is a pseudo-stable curve of genus g. The natural morphisms are denoted by:

$$\begin{tikzcd} V_d \arrow[r, "h"] \arrow[dr, "\pi_d"'] & V' \arrow[r, "f"] \arrow[d, "\pi'"] & V \arrow[d, "\pi"] \\ & W' \arrow[r, "f_0"] & W \end{tikzcd}$$

Let n be the degree of W' over W and $d' \sim \wedge^g \pi_{d*}\omega_{V_d/W'}$. Define $d = n^{-1} f_{0*} d'$.

Statement 3.6(i) is true for pseudo-stable curves (2.10). We know [3; Theorem 2.7] that $h_* \omega_{V_d/W'} = \omega_{V'/W'}$ is an invertible sheaf. By definition of δ_w and $\omega_{V/W}$, we have:

$$\omega_{V'/W'} = f^*\omega_{V/W} \otimes f^* O_V\left(\sum_{w \in \Delta} \delta_w\right) \tag{3.8}$$

and 3.6(i) follows directly from the definition of the local terms. $\omega_V = \omega_{V/W} \otimes \pi^*\omega_W$ and from 3.6(i) we get

$$(\omega_{V/W} \cdot \pi^*\omega_W) = 2 \cdot g^{-1}(g+1)^{-1}(O_V(Wp) \cdot \pi^*\omega_W) = 4(p-1)(g-1)$$

and hence 3.6(ii).

3.9. LEMMA: *Assuming V is pseudo-stable, we have*:

(i) $\chi(O_V) = (p-1)(g-1) + \deg(d)$

(ii) *Let $e(V)$ be the Euler number of V. Then*

$$e(V) = 4(p-1)(g-1) + \sum_{w \in \Delta} l_w$$

(iii) $(\omega_{V/W})^2 = 12 \deg(d) - \sum_{w \in \Delta} l_w$.

PROOF: The Leray spectral sequence $H^q(W, R^p\pi_* O_V) \Rightarrow H^{q+p}(V, O_V)$ and $\pi_* O_V = O_W$ gives us $\chi(O_V) = (1-p) - \chi(W, R^1\pi_* O_V)$. Hence (i) follows from 2.8(ii) and the Riemann Roch formula for locally free sheaves on curves. Statement (ii) is proven in [9] and (iii) follows from Noether's formula $12\chi(O_V) = e(V) + (\omega_V)^2$ and 3.6(i).

Back to the proof of the Theorem: In general, 3.6(iii) follows from 3.9(iii) and 3.8. To get 3.6(iv), one must simply compare the two equations for $(\omega_{V/W})^2$ you get from 3.6(i) and 3.6(iii). The term $(Wp)^2$ can be eliminated using the genus formula for curves on surfaces.

3.10. LEMMA: *Assume that V is pseudo-stable over W and that the Weierstrass points of the generic fibre of π are $\mathbb{C}(W)$-rational. Then we have*:

(a) *$Wp = \Sigma_{i=1}^r k_i D_i$ where D_i are prime divisors and the support of D_i is isomorphic under π to W.*

$$\text{(b)} \quad -2\gamma = -\sum_{i=1}^r \frac{2(k_i^2 - k_i)}{g(g+1) + 2k_i} \cdot \deg(d) + \sum_{i \neq j} \frac{k_i k_j (g^2 + g + 2)}{g(g+1) + 2k_i} \cdot (D_i \cdot D_j)$$

$$-\sum_{i=1}^{r}\sum_{w\in\Delta}\frac{2(k_i^2-k_i)}{g(g+1)+2k_i}\cdot(E_w\cdot D_i).$$

(c) $\deg(d)\geq 0$ *and* $\deg(d)=0$ *if and only if* $\pi: V\to W$ *is smooth and* $(Wp)_{\text{red}}$ *without singularities.*

PROOF: (a) is proven in [17; p. 1148].

(b) follows from 3.6(i) and the genus formula for curves on surfaces. To prove (c) substitute 3.10(b) in 3.6(iv) and check that the coefficients are positive for $g\geq 1$.

PROOF OF 3.6(v): From the definition of d and 3.10, it follows, that $\deg(d)\geq 0$. If $\deg(d)=0$ we know that $\pi_d: V_d\to W'$ is smooth. From [17; Prop. 5] for $g\geq 2$ or 2.6 for $g=1$, we find that we may assume, that V_d is trivial over W', and therefore $d'=d=0$.

§4. Calculation of the local terms

To calculate the square of the canonical divisor of V using 3.6, we have to know the local contributions. That means: let $\pi: V\to W$ be as in 3.1 and let $w\in\Delta$. The special fibre of π at w can be written $V_w=\Sigma_{i=1}^r \nu_iC_i$, where $\nu_i\in\mathbb{N}-\{0\}$ and C_i is a prime divisor of V. We already know (3.4) that $\delta_w=\Sigma_{i=1}^r(1-\nu_i)C_i$. Hence we have to calculate $E_w=\Sigma_{i=1}^r \mu_iC_i$, $(C_i\cdot Wp)$ and l_w. The remaining term $k(C_i)$ is determined by $\chi(O_{C_i})$ and the intersection theory of the fibre.

4.1: By assumption $(V_w)_{\text{red}}$ has only double points. Let (,) denote the smallest common divisor of two natural numbers. Define $\lambda(x)=(\nu_i,\nu_j)^2\cdot\nu_i^{-1}\cdot\nu_j^{-1}$, if $x\in C_i\cap C_j$ for $i\neq j$, and $\lambda(x)=0$, if x is regular on $(V_w)_{\text{red}}$. Then it easily follows from [12; p. 7] and [19; §6] that $l_w=\Sigma\,\lambda(x)$ where the sum is taken over all closed points of V_w.

4.2. REMARK: If we know for all j, except $j=i$, the multiplicity μ_j of C_j in E_w, we are able to express μ_i in terms of $(C_i\cdot Wp)$ using 3.6(i) and the genus formula for curves on surfaces. Therefore the remaining problem is to calculate either the multiplicity in E_w or the intersection number for "enough" components. We may assume that $\rho:\Gamma\to S$ is pseudostable and $\Gamma_w=V_w$.

4.3. TWISTING $\omega_{\Gamma/S}$: Let $\Delta(o)=\Sigma_{i=1}^r n_iC_i$ be some positive divisor, $\epsilon=\max\{n_i; i=1,\dots,r\}$ and define for $j=1,\dots,\epsilon$ $\Delta(j)=\Sigma_{i=1}^r\max(n_i-j,0)\cdot C_i$, $\mathcal{L}(j)=\omega_{\Gamma/S}\otimes O(\Delta(j))$ and $\mathcal{L}_w(j)=\mathcal{L}(j)\otimes O_{\Gamma_w}$.

Assumption (*):

(a) For $j = 1, \ldots, \epsilon$ we have $\dim_{\mathbb{C}} H^0(\Gamma_w, \mathscr{L}_w(j)) = g$.

(b) For some component of Γ_w (let us say C_1) the canonical map $H^0(\Gamma_w, \mathscr{L}_w(0)) \to H^0(\Gamma_w, \mathscr{L}_w(0) \otimes_{O_{\Gamma_w}} O_{C_1})$ is an isomorphism.

It follows for $j = 1, \ldots, \epsilon$ that $\rho_* \mathscr{L}(j)$ is locally free of rank g.

Now let $\eta_1^{(j)}, \ldots, \eta_g^{(j)}$ be a basis of $\rho_* \mathscr{L}(j)$ and $E(j)$ the part of the divisor of the section $[\eta_1^{(j)}, \ldots, \eta_g^{(j)}]$ of $\mathscr{W}(\mathscr{L}(j))$ with support in the special fibre.

Let $\varphi(j): \mathscr{L}(j+1) \to \mathscr{L}(j)$ be the canonical map and $\varphi_w(j): H^0(\Gamma_w, \mathscr{L}_w(j+1)) \to H^0(\Gamma_w, \mathscr{L}_w(j))$ the induced map. $d(j)$ is defined to be the dimension of the image of $\varphi_w(j)$.

4.4. Lemma: $E(j) = E(j+1) + g \cdot (\Delta(j) - \Delta(j+1)) + (d(j) - g)\Gamma_w$.

Proof: Let E^* be the part of the divisor of the section $[\varphi(j)(\eta_l^{(j+1)}), \ldots, \varphi(j)(\eta_g^{(j+1)})]$ of $\mathscr{W}(\mathscr{L}(j))$ with support in the fibre. Then $E^* = E(j+1) + g \cdot (\Delta(j) - \Delta(j+1))$. We may assume, that the first $d(j)$ sections $\eta_i^{(j+1)}$ generate the image of $\varphi_w(j)$. The rest of the sections vanish on the special fibre with order 1 and the lemma follows.

4.5. Remarks:

(i) If the assumption (*) is fulfilled for some $\Delta(0)$ and some C_1, we are able to calculate μ_1 using 4.4. It is always possible to find such a divisor for every component, if the graph of Γ_w is simply connected and if the double points of Γ_w are in general position. In this case E_w is uniquely determined by the isomorphism-class of Γ_w.

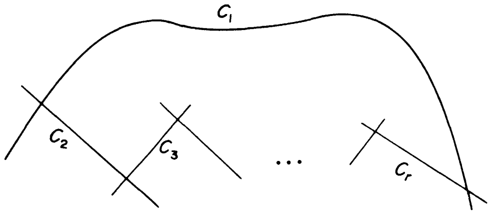
This, however, is no longer true in general. A counter-example is given by a fibre Γ_w with three components and two double points, if one of the double points is a Weierstrass point of one component.

(ii) Under the assumptions and notations of 4.3, the isolated zeros of the Weierstrass section of $\mathscr{L}_w(0)$ outside the singular points of V_w are (with multiplicity) intersection points of V_w and Wp.

4.6. Examples:

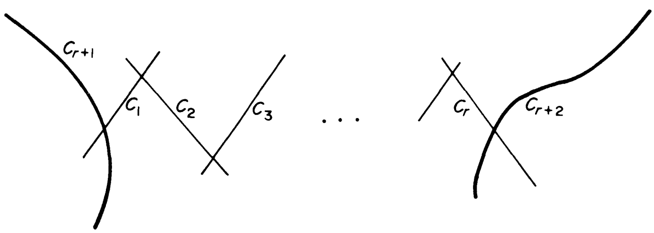
(a) Assume $V_w = C_1 \cup C_2 \cup \cdots \cup C_r$ where C_i is a regular rational curve for $i = 2, \ldots, r$ and C_1 is a curve of genus $g-1$. Assume that the double points are in general position on C_1.

For C_1 we can take $\Delta(0) = 0$. We get $\mu_1 = 0$ and there are $g^3 - g^2$ intersection points of C_1 and Wp outside the singular points of V_w.



Hence $g^2 - g$ intersection points must lie on $C_2 \cup \cdots \cup C_r$. If $g = 2$, it is easy to show (using 4.3 and 4.4) that the remaining 2 intersection points lie on $C_{\frac{1}{2}r+1}$, if 2 divides r.

(b) Assume $V_w = C_1 \cup \cdots \cup C_r \cup C_{r+1} \cup C_{r+2}$ where $C_1, \ldots, C_r$ are regular rational curves and C_{r+i} is a regular curve of genus g_i, $i = 1, 2$ and assume that the double points are in general position.



Then $g = g_1 + g_2$. Take $\Delta(0) = \Sigma_{i=1}^r g_2 \cdot i \cdot C_i + (r+1) \cdot g_2 \cdot C_{r+2}$. From 4.4 we get $\mu_{r+1} = (r+1) \cdot g_2$ and there are $g^2 \cdot g_1 - g_2$ intersection points of C_{r+1} and Wp outside the singular points of V_w. By symmetry and 4.2 we find:

$$E_w = (r+1)g_2 \cdot C_{r+1} + (r+1)g_1 \cdot C_{r+2} + \sum_{i=1}^{r} ((r+1-i)g_2 + ig_1)C_i.$$

4.7. The Case $g = 1$: ($\pi : V \to W$ as in 3.1) In this case we already know (2.11(i)) that $Wp = E_w = 0$ for stable curves, and hence 3.6 reduces.

Let η_W be the maximal multiplicity occuring in $\pi^{-1}(W)$ and define $\delta'_W = (1 - \eta_W) \cdot \eta_W^{-1} \cdot \pi^{-1}(W)$. Then:

$$\omega_{V/W} \sim \pi^* d - \sum_{w \in \Delta} \delta'_W \text{ and } \deg(d) = 12^{-1} \cdot \sum_{w \in \Delta} (l_W + (E_W \cdot \delta_W))$$

4.8. The Case $g = 2$: Using the methods from 4.6, one is able to find E_w for every stable curve of genus 2. It is important, that the components of a singular stable curve of genus 2 are at most of geometric genus 1 and hence "all points are in general position". Let's make the following definition:

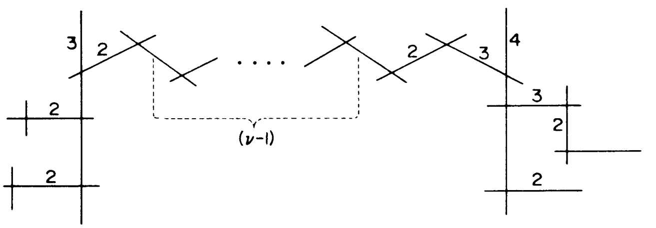
Let $(\rho'' : \Gamma'' \to S', \sigma)$ be a minimal stable resolution (3.5) of π in w. Then let $n_w = (\mathrm{ord}\,(\sigma))^{-1} \cdot \#\{x \in \Gamma''_w;\, x$ singular and $\Gamma''_{w'} - \{x\}$ connected$\}$ and $m_w = (\mathrm{ord}\,(\sigma))^{-1} \cdot \#\{x \in \Gamma''_{w'};\, x$ singular and $\Gamma''_{w'} - \{x\}$ not connected$\}$.

Using that definition one gets $(E_w \cdot W_p) = 6m_w + n_w$ and $k(E_w) = 2m_w$ if V_w is pseudo-stable. The fact that Wp has no singularities outside the degenerate fibres of π (every fibre is hyperelliptic) yields:

4.9. THEOREM (Ueno; see [16]): Let $\pi : V \to W$ be as in 3.1, $g = 2$. Then we have $(\omega_V)^2 = 8(p-1) + \Sigma_{w \in \Delta}\, \eta_w$ where η_w depends only on the local invariants [19] of π in w and

$$\eta_w = \tfrac{12}{5} \cdot m_w + \tfrac{6}{5} \cdot n_w - l_w - 2k(\delta_w) - (\delta_w)^2.$$

4.10. EXAMPLE: One possible fibre is of the form [19; Ex. 8.4]:



where all components are rational. Using 3.4(i) one gets $k(\delta_w) = 0$ and $(\delta_w)^2 = -4$. The description in 4.1 yields $l_w = \nu + 2$. The fibre of the minimal resolution (see [19; 8.4]) is simply connected and we have $n_w = 0$ and $m_w = \nu - \frac{7}{12}$. The local contribution in this example is therefore $\eta_w = \frac{7}{5} \cdot \nu + \frac{3}{5}$.

4.11. REMARK: The fact that for $g > 2$ the local contributions E_w are not completely determined by the local invariants, defined in [19], of the degenerate fibres can be explained in the following way: Let M_g^* be a fine moduli scheme of stable curves of genus g with some suitable additional structure (for example: The Hilbert scheme of three canonical embedded stable curves [4] or $M_g^{(\mu)}$ (see 2.3)) and $\rho : Z_g^* \to M_g^*$ the corresponding universal curve. Let M_{go}^* be the open subscheme of M_g^*, corresponding to regular curves. Then $M_g^* - M_{go}^*$ is the union of irreducible closed subschemes $S_0, \ldots, S_r$ of codimension 1 ($r = \frac{1}{2}g$ or $r = \frac{1}{2}(g-1)$) with the following property [4]: $S_0 \times_{M_g^*} Z_g^*$ is a family of irreducible curves with one double point. $S_i \times_{M_g^*} Z_g^*$ has two components $Z_i^{(1)}$ and $Z_i^{(2)}$ which are families of curves of genus i and $g - i$.

The arguments of 4.4 and 4.6 give the description of $\omega = \omega_{Z_g^*/M_g^*}$:

$$\omega^{\otimes \frac{1}{2}g(g+1)} \otimes \left(\rho^* \overset{g}{\wedge} \rho_* \omega\right)^{-1} \sim Wp_{Z_g^*/M_g^*} + \sum_{i=1}^{r} \left((g-i) \cdot Z_i^{(1)} + i \cdot Z_i^{(2)}\right).$$

Let $\pi : \Gamma \to S$ be a pseudo-stable curve over a local scheme. The corresponding divisor $E_{\Gamma/S} + Wp_{\Gamma/S}$ is then the pullback of the right hand side, but $Wp_{\Gamma/S}$ is not the pullback of $Wp_{Z_g^*/M_g^*}$. One of the reasons is that for $g > 2$ the support of $Wp_{Z_g^*/M_g^*}$ is not finite over M_g^*.

4.12. ADDENDUM: The situation is much better if we restrict ourselves to families of curves with hyperelliptic general fibre. In this case, the number γ occurring in 3.6(iv) is zero and hence $\deg(d)$ depends only on the local behaviour of the family near the degenerate fibres.

Now let $H_{go}^{(\mu)}$ be the subscheme of $M_g^{(\mu)}$ for some $g > 2$ corresponding to hyperelliptic regular curves, $H = H_g^{(\mu)}$ the closure in $M_g^{(\mu)}$ and $\rho : C \to H$ the corresponding family of curves. If $\pi : V \to W$ is a family of stable curves with hyperelliptic general fibre and level μ-structure let $\varphi : W \to H$ be the induced morphism. $\deg(d)$ is nothing but the intersection number of $\varphi(W)$ and $\wedge^g \rho_* \omega_{C/H}$. In this case 3.6(iv) means that $\wedge^g \rho_* \omega_{C/H}$ is numerically equivalent to a divisor with support in $H - H_{go}^{(\mu)}$. Therefore Theorem 4.9 can be generalised:

4.13. THEOREM: *Let $\pi : V \to W$ be as in* 3.1, $g \geq 2$, *and assume that the general fibre is a hyperelliptic curve. Then we have*: $(\omega_V)^2 = 8(p-1)(g-1) + \Sigma_{w \in \Delta} \eta_w$ *where η_w depends only on the local invariants* [19] *of π in w.*

The numbers η_w can be calculated using 3.6, 4.1, 4.6(a) (the remaining 2 intersection points of multiplicity $\frac{1}{2}(g^2 - g)$ lie on $C_{\frac{1}{2}r+1}$ if 2 divides r) and 4.6(b).

§5. "Stable reduction" for higher dimensional base-schemes

In this section we are going to prove the following theorem:

5.1. THEOREM: *Let $\pi_1 : V_1 \to W_1$ be a surjective morphism of proper, regular varieties such that the general fibre of π_1 is a connected curve of genus $g \geq 1$. Then there exists the following commutative diagram of morphisms of proper varieties*:

$$\begin{tikzcd} V \arrow[d, "\pi"] & V' \arrow[l, "h"'] \arrow[r, "f"] \arrow[d, "\pi'"] & V_s \arrow[dl, "\pi_s"] \\ W & W' \arrow[l, "g"] & \end{tikzcd}$$

having the following properties:

(i) $\pi : V \to W$ *is a surjective morphism of regular varieties with connected general fibre and is birationally equivalent* (1.5) *to* $\pi_1 : V_1 \to W_1$.

(ii) $h : V' \to V$ *and* $g : W' \to W$ *are flat covers and* V' *is birational equivalent to* $W' \times_W V$. *The only singularities of* W' *and* V' *are quotient singularities* [20].

(iii) $\pi_s : V_s \to W'$ *is a stable curve of genus g with level* μ*-structure*, $\mu \geq 3$, *and* $f : V' \to V_s$ *is a birational morphism.*

(iv) *Every morphism and every scheme occurring in the diagram is projective.*

PROOF: The smooth fibres of π_1 induce a rational map $\varphi_1 : W_1 \to M_g$ (see §2). M_g is proper and hence after eliminating the points of indeterminacy [8] and replacing the pullback of V_1 by a regular model [8], we may assume that φ_1 is a morphism. Using Chow's lemma we may also assume that π_1, W_1 and V_1 are projective.

For $\mu \geq 3$ we are able (2.2(i)) to find a finite Galois cover $g_1 : W_1' \to W_1$ such that the generic fibre of $V_1 \times_{W_1} W_1'$ over W_1' has a level μ-structure. Let $\Delta(W_1'/W_1)$ be the ramification locus of W_1' in W_1. By "purity of the branch locus" $\Delta(W_1'/W_1)$ is of codimension one. Using "embedded resolution of singularities" [8] we find a sequence of monoidal transformations $\eta : W \to W_1$ such that $\eta^{-1}(\Delta(W_1'/W_1))$ has regular components and at most normal crossings as singularities. Let W' be the normalization of $W \times_{W_1} W_1'$ and $g : W' \to W$ the induced morphism. Since $\Delta(W'/W) \subseteq \eta^{-1}(\Delta(W_1'/W_1))$, it follows from [20; Lemma 2] that g is flat and W' has at most quotient singularities.

Let $M_g^{(\mu)}$ be the fine moduli scheme of stable curves with level μ-structure (2.4). Then the generic fibre of $W' \times_{W_1} V_1$ induces a rational map $\varphi' : W' \to M_g^{(\mu)}$ which is compatible with the morphism $\varphi = \varphi_1 \cdot \eta : W \to M_g$. Since $M_g^{(\mu)}$ is finite over M_g it follows that φ' is a morphism [24; II, 6.1.13] and induces a stable curve $\pi_s : V_s \to W'$.

The morphism π_s is projective [414] and hence V_s is projective. Let G be the Galois group of W' over W. By definition of V_s we have an operation of G on the generic fibre of π_s.

5.2. LEMMA: *The operation of G on the generic fibre of π_s extends to an operation of G on V_s, compatible with the operation of G on W'.*

Since V_s is projective, the quotient exists. The universal property of quotients gives us a morphism $V_s/G \to W'/G \cong W$. If we look only on the generic fibres, we have just made an extension of the base field and then divided by the Galois group of this extension. Therefore V_s/G is birationally equivalent to V_1. Using resolution of singularities, Chow's lemma and embedded resolution of singularities again, we find a regular projective variety V and a projective birational morphism $V \to V_s/G$, such that: Let V' be the normalization of $V \times_W W'$ and $h: V' \to V$ the induced morphism, then $\Delta(V'/V)$ has regular components and only normal crossings as singularities. Hence [20; Lemma 2] h is flat and V' has only quotient singularities.

Finally the birational morphism $f: V' \to V_s$ over W' exists, since V' is also the normalization of $V_s \times_{V_s/G} V$.

PROOF OF LEMMA 5.2: Every $\sigma \in G$ induces an isomorphism σ of W' and there exists a σ invariant open subscheme $U \subseteq W'$ such that

$$\begin{tikzcd} \pi_s^{-1}(U) \arrow[rr, "\sigma'"] \arrow[dr, "\sigma \cdot \pi_s"'] & & \pi_s^{-1}(U) \arrow[dl, "\pi_s"] \\ & U & \end{tikzcd}$$

is commutative. We denote by σ' the isomorphism induced by σ on $\pi_s^{-1}(U)$. For $g \geq 2$ it follows from 2.7 that σ' can be extended to an isomorphism of V_s. For $g = 1$ we know that $M_1^{(\mu)}$ is a curve and it follows directly that there is an isomorphism σ'' of $M_1^{(\mu)}$ such that

$$\begin{tikzcd} W' \arrow[r, "\varphi'"] \arrow[d, "\sigma"] & M_1^{(\mu)} \arrow[d, "\sigma''"] \\ W' \arrow[r, "\varphi'"] & M_1^{(\mu)} \end{tikzcd}$$

is commutative. Therefore in this case the Lemma follows directly from the definition of V_s.

5.3. Corollary: *Under the assumptions and with the notations of* 5.1

(i) V', W' *and* V_s *have rational singularities* [20] *and are Gorenstein schemes* [6; V§9].

(ii) $R^i f_* O_{V'} = 0$ *for* $i \neq 0$ *and* $f_* O_{V'} \cong O_{V_s}$.

Proof: (ii) follows from (i) and [20; Lemma 1]. We know from [20; Prop. 1] that V' and W' have rational singularities. To prove that V_s has rational singularities we may assume (using "flat base change") that W' is regular. Then from [4] or the deformation theory of ordinary double points of curves, it follows that the completion of a singular local ring of V_s has the form

$$\mathbb{C}[[t_1, \ldots, t_r, u, v]]/(u \cdot v - g(t_1, \ldots, t_r)),\ g(t_1, \ldots, t_r) \in \mathbb{C}[[t_1, \ldots, t_r]]$$

and the rationality follows from [20; Prop. 2]. V_s is locally a complete intersection over W' and hence it remains to show that V' and W' are Gorenstein schemes. Both are flat, finite covers of regular schemes. The question is local and hence it is enough to consider the following situation: Let A be a local Gorenstein ring, B a local ring and a free, finite A-module [1; p. 60]. There exists n_0 such that $\operatorname{Ext}^i_A(A/\mathfrak{m}_A, A) = 0$ for $i \geq n_0$ [10; p. 163]. Hence $\operatorname{Ext}^i_A(A/\mathfrak{m}_A, B) = 0$ and from [7; p. 164] we get $\operatorname{Ext}^i_B(B \otimes A/\mathfrak{m}_A, B) = 0$ for $i \geq n_0$. Since B is free over A we get $\operatorname{Ext}^i_B(B/\mathfrak{m}_B, B) = 0$ for $i \geq n_0$ and B is a Gorenstein ring.

§6. Duality theory

In order to compare the Kodaira dimensions of the varieties occurring in 5.1 we need some results of Grothendieck duality theory. Let $f: X \to Y$ be a projective embeddable morphism of noetherian schemes of finite Krull dimension. This is the case if Y is a projective variety and if f is a projective morphism [6; p. 206].

Let $D_{qc}(X)$ be the derived category of quasi-coherent sheaves on X [6; p. 85] and $D^+_{qc}(X)$ (resp. $D^-_{qc}(X)$) the full subcategory of complexes bounded below (resp. above). Then there exists a functor [6; p. 190] $f^!: D^+_{qc}(Y) \to D^+_{qc}(X)$ with the following properties:

6.1: (We denote the derived functors $\boldsymbol{R}f_*$, $\boldsymbol{L}f^*$ and $\otimes$)

(i) For every composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of projective embeddable morphisms there is an isomorphism of functors $(g \cdot f)^! \cong f^! g^!$.

(ii) For every flat base extension $u: Y' \to Y$ there is an isomorphism $v^*f^! = g^!u^*$ where v and g are the two projections of $X \times_Y Y'$.

(iii) Let $D^b_{qc}(Y)_{fTd}$ be the subcategory of $D_{qc}(Y)$ generated by the bounded complexes of finite Tor-dimension [6; p. 97]. There is a functorial isomorphism $f^!(F^{\cdot}) \otimes \boldsymbol{L}f^*(G^{\cdot}) \cong f^!(F^{\cdot} \otimes G^{\cdot})$ for $F^{\cdot} \in D^+_{qc}(Y)$ and $G^{\cdot} \in D^b_{qc}(Y)_{fTd}$ [6; p. 194].

(iv) Let $F^{\cdot} \in D^-_{qc}(X)$ and $G^{\cdot} \in D^+_{qc}(Y)$. There exists a duality isomorphism [6; p. 210]:

$$\Theta^i_f: \mathrm{Ext}^i_X(F^{\cdot}, f^!G^{\cdot}) \to \mathrm{Ext}^i_Y(\boldsymbol{R}f_*F^{\cdot}, G^{\cdot})$$

(v) Assume that f is a flat morphism of Gorenstein schemes of relative dimension m. Then there exists an invertible sheaf $\omega_{X/Y}$ such that $f^!(O_Y)$ is isomorphic in $D^+_{qc}(X)$ to the complex $\omega_{X/Y}[m]$ (see 6.2(i)) [6; p. 298 and 388].

(vi) Under the assumptions of (v), $\omega_{X/Y}$ is compatible with arbitrary base change [6; p. 388].

6.2. Remark:

(i) Let $[m]: D_{qc}(Y) \to D_{qc}(Y)$ be the functor defined by $(G^{\cdot}[m])^r = G^{r+m}$, then $[m]$ is compatible with derived functors. Let G be an invertible sheaf on Y, considered as the trivial complex having G at the 0^{th} place. Then $\boldsymbol{L}f^*(G[m]) \cong (\boldsymbol{L}f^*G)[m] \cong (f^*G)[m]$ and for $F \in D^+_{qc}(Y)$ we have $F^{\cdot} \otimes G[m] \cong (F^{\cdot} \otimes G)[m] \cong (F^{\cdot} \otimes G)[m]$ where $(F^{\cdot} \otimes G)^r = F^r \otimes G$. Under these conditions the isomorphism in 6.1(iii) reduces to $(f^!(F^{\cdot}) \otimes f^*(G))[m] \cong f^!(F^{\cdot} \otimes G)[m]$.

(ii) If $Y = \mathrm{Spec}\,(\mathbb{C})$ and X is a regular and projective variety, then $\omega_{X/Y}$, defined in 6.1(v), is the usual canonical sheaf on X.

We extend the definition of $\omega_{X/Y}$ given in 6.1(v).

6.3. Definition: Let $f: X \to Y$ be a surjective and projective embeddable morphism of irreducible noetherian schemes whose general fibre is of dimension m. If $f^!(O_Y) \cong G[m]$ in $D^+_{qc}(X)$ for an invertible sheaf G, we say that the *dualizing sheaf* of f exists and denote $\omega_{X/Y} = G$.

Statement $C'_{n,m}$ just says, that the Kodaira dimension of the general fibre is smaller than the $\mathscr{L}$-dimension of the dualizing sheaf.

6.4. Lemma: *Let $h: X \to S$ and $g: Y \to S$ be surjective and projective embeddable morphisms of irreducible noetherian schemes of finite Krull dimension and let l (resp. m) be the dimension of a general fibre* *of h (resp. g). Assume that the dualizing sheaves $\omega_{X/S}$ and $\omega_{Y/S}$ exist. Let $f: X \to Y$ be a surjective and projective embeddable morphism over S. Then $\omega_{X/Y}$ exists and is isomorphic to $\omega_{X/S} \otimes f^*\omega_{Y/S}^{-1}$.*

PROOF: We have (6.1(i)) $h^! \cong f^!g^!$ and from 6.2(i) we get: $h^!(O_S) \cong f^!(g^!(O_S) \cong f^!(O_Y \otimes_{O_Y} g^! O_S) \cong f^!(O_Y) \otimes_{O_X} f^* g^!(O_S)$ or $\omega_{X/S}[l] \cong f^!(O_Y) \otimes_{O_X} f^*(\omega_{Y/S})[m]$. Hence $f^!(O_Y) \cong (\omega_{X/S} \otimes f^*\omega_{Y/S}^{-1})[l-m]$.

6.5. COROLLARY: *Using the notations from* 6.4, *we assume that $l = m$, $f_*(O_X) \cong O_Y$ and $R^if_*(O_X) = 0$ for $i \neq 0$. Then there exists an injection $f^*\omega_{Y/S} \to \omega_{X/S}$.*

PROOF: The duality isomorphism (6.1(iv)) for $i = 0$ gives you: $\mathrm{Hom}_X(O_X, f^!O_Y) \cong \mathrm{Hom}_{D_{qc}(Y)}(\boldsymbol{R}f_*O_X, O_Y) \cong \mathrm{Hom}_Y(O_Y, O_Y)$. Therefore there is a non trivial morphism $O_X \to f^!O_Y \cong \omega_{X/S} \otimes f^*\omega_{Y/S}^{-1}$.

6.6. COROLLARY: *Using the notations from* 6.4, *we assume $l = m$ and f finite and birational. Then there exists an injection*

$$\omega_{X/Y} \to O_X.$$

PROOF: We know that $R^jf_*G = 0$ for $j \neq 0$ and any invertible sheaf G on X. The duality isomorphism for $i = 0$ gives $\mathrm{Hom}_X(\omega_{X/Y}, \omega_{X/Y}) \cong \mathrm{Hom}_Y(f_*\omega_{X/Y}, O_Y)$. Therefore $0 \neq \mathrm{Hom}_Y(f_*\omega_{X/Y}, f_*O_X) \cong \mathrm{Hom}_X(f^*f_*\omega_{X/Y}, O_X)$. Since f is affine and birational there is a sheaf ϵ with support in codimension 1 such that $0 \to \epsilon \to f^*f_*\omega_{X/Y} \to \omega_{X/Y} \to 0$ is exact. We have $\mathrm{Hom}_X(\epsilon, O_X) = 0$ and hence

$$0 \neq \mathrm{Hom}_X(f^*f_*\omega_{X/Y}, O_X) \cong \mathrm{Hom}_X(\omega_{X/Y}, O_X).$$

We want to apply these results to the situation described in §5. Remember, in the conclusion of 5.1 we got a diagram of projective morphisms of projective Gorenstein schemes (5.3):

$$\begin{tikzcd}
V \arrow[d, "\pi"] & V' \arrow[l, "h"'] \arrow[r, "f"] \arrow[d, "\pi'"] & V_s \arrow[dl, "\pi_s"] \\
W & W' \arrow[l, "g"] &
\end{tikzcd}$$

From 6.1(v) and 6.4 we know that for every morphism of this diagram the dualizing sheaf exists. The following proposition reduces the proof of $C'_{n,n-1}$ to the case of stable curves.

6.7. Proposition: *Using above notations we have*:

$$K(\omega_{V_s/W'}, V_s) \leq K(\omega_{V'/W'}, V') \leq K(\omega_{V/W}, V).$$

Proof: Using 5.3(ii) it follows that the assumptions of 6.5 are fulfilled for

$$\begin{tikzcd} V' \arrow[rr, "f"] \arrow[dr, "\pi'"'] & & V_s \arrow[dl, "\pi_s"] \\ & W' & \end{tikzcd}$$

and hence the first inequality follows from 1.3(ii) and (iii). The left hand side of the diagram can be written as:

$$\begin{tikzcd} V \arrow[d, "\pi"'] & V_1 = V \times_W W' \arrow[l, "h_1"'] \arrow[d, "p"] & V' \arrow[l, "h_2"'] \arrow[dl, "\pi'"] \\ W & W' \arrow[l, "g"] & \end{tikzcd}$$

Where $h = h_1 \cdot h_2$. Since g is a flat morphism of Gorenstein schemes (5.3) we know from 6.1(vi) that $\omega_{V_1/V} \cong p^*\omega_{W'/W}$. The assumptions of 6.6 are fulfilled by h_2 and we get: $\omega_{V'/V} \otimes \pi'^*\omega_{W'/W}^{-1} \cong \omega_{V'/V_1} \to O_{V'}$ and therefore $\omega_{V'/W'} \cong \omega_{V'/V} \otimes \pi'^*\omega_{W'/W}^{-1} \otimes h^*\omega_{V/W} \to h^*\omega_{V/W}$. Now the second inequality holds by 1.3(ii) and (iii).

§7. The dualizing sheaf for stable curves

For $g \geq 1$ and $\mu \geq 3$ let $M = M_g^{(\mu)}$ be the fine moduli scheme of stable curves with level μ-structure and $\rho: Z \to M$ the corresponding universal curve. Let $M_0 \subseteq M$ be the open subscheme corresponding to the regular curves, $\omega = \omega_{Z/M}$ and $\mathscr{D} = \Lambda^g \rho_* \omega$.

7.1. Proposition: $K(\mathscr{D}, M) = \dim(M)$ *and for some* $m \gg 0$ *we have* (*see* 1.1) $\Phi_{m,\mathscr{D}}|_{M_0}$ *is a finite morphism of* M_0 *on a subscheme of* $\mathbb{P}^N$.

Proof of 7.1 for $g = 1$: We know (2.6) that M is a curve and that

$\rho : Z \to M$ is not smooth. From 3.6 we get $\deg \mathscr{D} > 0$ and hence $\mathscr{D}$ is ample on M.

7.2. REMARK: As far as the author knows, proposition 7.1 for $g \geq 2$ will follow from the announced proof of the projectivity of M_g by Knudson and Mumford (see [14]). In §3 we have already seen, that $\mathscr{D}$ behaves like an ample sheaf: Every curve in M whose general point is contained in M_0 has a positive intersection number with $\mathscr{D}$. It seems reasonable that $\mathscr{D}$ is a good candidate (after some correction along the boundary of M_0), if you are looking for an ample sheaf on M.

Nevertheless, we give the outline of a proof of the weaker statement of 7.1, using analytic methods which can be found in [2] and [15]. Namikawa constructed in [15] a holomorphic morphism from M_g into a projective space, which is an injection on M_{g0}. We just have to show that the induced morphism of M is given by global sections of $\mathscr{D}^{\otimes m}$ for some m. This follows, however, from the methods used by Arakelov to prove [2; Theorem 1.1]:

PROOF OF 7.1 FOR $g \geq 2$: Henceforth we will use the complex topology of M and Z. For sufficiently small $U \subseteq M_0$ we can find cycles $\bar{\alpha}_1, \ldots, \bar{\alpha}_g$ and $\bar{\beta}_1, \ldots, \bar{\beta}_g$ in $R^1\rho_*\mathbb{Z}$ such that for every $t \in U$ the induced cycles (via duality) $\alpha_{1t}, \ldots, \alpha_{gt}, \beta_{1t}, \ldots, \beta_{gt}$ in $H_1(Z_t, \mathbb{Z})$ have the property:

$$(\alpha_{it} \cdot \alpha_{jt}) = (\beta_{it} \cdot \beta_{jt}) = 0 \text{ for } 1 \leq i, j \leq g \text{ and } (\alpha_{it}, \beta_{jt}) = \delta_{ij}.$$

We may assume that $\rho_*\omega$ is free over U. Let $\omega_1, \ldots, \omega_g$ be a basis of $\rho_*\omega$ on U and define:

$$\Omega_1(t, \omega_1, \ldots, \omega_g) = \left| \int_{\beta_{kt}} (\omega_l)_t \right|_{1 \leq k, l \leq g} \quad \text{and}$$

$$\Omega_2(t, \omega_1, \ldots, \omega_g) = \left| \int_{\alpha_{it}} (\omega_j)_t \right|_{1 \leq i, j \leq g}.$$

Both are holomorphic in t, and by replacing $\omega_1, \ldots, \omega_g$ by another basis of $\rho_*\omega$ we may assume that $\Omega_2(t, \omega_1, \ldots, \omega_g)$ is the unit matrix for all $t \in U$. Then $\Omega_1(t) = \Omega_1(t, \omega_1, \ldots, \omega_g)$ is called the period matrix of the fibre Z_t.

For $m \gg 0$ let $s_1, \ldots, s_r$ be a basis of the vector space of Siegel modular forms of weight m and define $\eta = (\omega_1 \wedge \cdots \wedge \omega_g)^{\otimes m}$. Then $s_i(\Omega_1(t)) \cdot \eta$ defines a section of $\mathscr{D}^{\otimes m}$ on the open set U.

If we choose a second system of cycles $\bar{\alpha}'_1, \ldots, \bar{\alpha}'_g, \bar{\beta}'_1, \ldots, \bar{\beta}'_g$ and a basis of $\rho_*\omega$, normalized as above, it follows easily that the section

$s_i(\Omega_1(t)) \cdot \eta$ remains unchanged. Hence we get global sections $\gamma_1, \ldots, \gamma_r$ of $\mathscr{D}$ on M_0 which don't vanish simultaneously. The corresponding holomorphic morphism $M_0 \to \mathbb{P}^{r-1}$ is the same as constructed in [15]. Hence it factors through the coarse moduli scheme M_{g0}, and the induced morphism $M_{g0} \to \mathbb{P}^{r-1}$ is an embedding.

It remains to show that the sections $\gamma_1, \ldots, \gamma_r$ extend to sections of $\mathscr{D}^{\otimes m}$ over M and are not simultaneously zero on the boundary of M_0. Let U' be a small neighbourhood of a point $y \in M - M_0$ in M. Since the sections γ_i are locally obtained from sections of the corresponding sheaf on the Hilbert scheme of three canonically embedded stable curves [15] we may assume that U' is regular. Using Hartog's theorem it is enough to show that the γ_i extend to holomorphic sections of $\mathscr{D}^{\otimes m}$ along a general line through y. This, however, is proven in [2; proof of 1.1]: Arakelov applies the same construction to abelian varieties over a curve, but [2; Lemma 1.4] gives the connection to the case we consider.

§8. The proof of $C'_{n,n-1}$ *and* 1.8

Let $\pi_1 : V_1 \to W_1$ be a surjective morphism of proper, regular varieties, $n = \dim(V_1)$ and $n - 1 = \dim(W_1)$, such that the general fibre of π_1 is a connected curve of genus $g \geq 1$.

Choose a diagram of morphisms of proper schemes as in 5.1. Let $\varphi : W \to M_g$ be the rational map induced by the general fibres of π (or π_1). Let $\varphi_s : W' \to M_g^{(\mu)}$ be the morphism corresponding to $\pi_s : V_s \to W'$. Then $\dim(\varphi(W)) = \dim(\varphi_s(W'))$. We have to prove (1.8):

$$K(\omega_{V/W}, V) \geq \max(K(V_w), \dim(\varphi_s(W'))). \tag{8.1}$$

8.2. Lemma: (8.1) follows from $K(\omega_{V_s/W'}, V_s) \geq \dim(\varphi_s(W'))$.

Proof: The above inequality and 6.7 give us $K(\omega_{V/W}, V) \geq \dim(\varphi_s(W'))$ and hence we have only to consider the case $g \geq 2$ and $\dim(\varphi_s(W')) = 0$. This, however, means that there is a regular curve C of genus g over $\mathbb{C}$ such that $V_s \cong C \times_{\mathrm{Spec}(\mathbb{C})} W'$ and 1.3(ii), 1.5 and 6.7 give us the inequality we need.

Now $\pi_{s*}\omega_{V_s/W'}$ and hence $\wedge^g \pi_{s*}\omega_{V_s/W'}$ are compatible with base change. Therefore (using the notation of 7.1) $\wedge^g \pi_{s*}\omega_{V_s/W'} = \varphi_s^*\mathscr{D}$. Since φ_s maps an open subscheme of W' into M_0, 7.1 yields $K(\wedge^g \pi_{s*}\omega_{V_s/W'}, W') \geq \dim(\varphi_s(W'))$. In 2.10 we proved that there is an injection

$$\pi_s^* \wedge^g \pi_{s*}\omega_{V_s/W'} \hookrightarrow \omega_{V_s/W'}^{\otimes g(g+1)/2},$$

and 1.3(iii) and (ii) prove $K(\omega_{V_s/W'}, V_s) \geq \dim(\varphi_s(W'))$.

REFERENCES

[1] A. ALTMAN and S. KLEIMAN: Introduction to Grothendieck Duality Theory. *Springer Lecture Notes 146.*

[2] S. JU. ARAKELOV: Families of algebraic curves with fixed degeneracies. *Izv. Akad. Nauk. SSSR, Ser. Math. 35* (1971); engl. transl. *Math. USSR Izv. 5* (1971) 1277–1302.

[3] M. ARTIN: Some numerical criteria for contractibility of curves on algebraic surfaces. *Am. J. Math. 84* (1962).

[4] P. DELIGNE and D. MUMFORD: The irreducibility of the space of curves of given genus. *I.H.E.S. Publ. Math. 36* (1969) 75–109.

[5] P. DELIGNE and M. RAPOPORT: Les schémas de modules de courbes elliptique: Modular Functions of one Variable II. *Springer Lecture Notes 349.*

[6] R. HARTSHORNE: Residues and Duality. *Springer Lecture Notes 20.*

[7] P. J. HILTON and U. STAMMBACH: A Course in Homological Algebra. *Springer Graduate Texts in Math. 4.*

[8] H. HIRONAKA: Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math. 79* (1964) 109–326.

[9] B. IVERSEN: Critical points of an algebraic function. *Invent. Math. 12* (1971) 210–224.

[10] I. KAPLANSKY: *Commutative Rings.* The Univ. of Chicago Press 1974.

[11] K. KODAIRA: On compact analytic surfaces II, III. *Ann. of Math. 77* (1963) 563–626 and *78* (1963) 1–40.

[12] A. L. MAYER: *Compactification of the variety of moduli of curves.* Seminar on Degeneration of Algebraic Varieties; conducted by P. A. Griffiths; Princeton 1969–1970.

[13] D. MUMFORD: *Geometric Invariant Theory*; *Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 34.* Springer Verlag 1965.

[14] D. MUMFORD: *Curves and their Jacobians.* Ann Arbor, The Univ. of Michigan Press 1975.

[15] Y. NAMIKAWA: On the canonical holomorphic map from the moduli space of stable curves to the Igusa monoidal transformation. *Nagoya Math. J. 52* (1973) 197–259.

[16] Y. NAMIKAWA: Studies on degeneration; Class. of Alg. Var. and Comp. Compl. Manif. *Springer Lecture Notes 412.*

[17] A. N. PARŠIN: Algebraic curves over function fields I. *Izv. Akad. Nauk SSSR, Ser. Math. 32* (1968), engl. transl. *Math. USSR Izv. 2* (1968) 1145–1170.

[18] H. POPP: Modulraeume algebraischer Mannigfaltigkeiten; Class. of Alg. Var. and Comp. Compl. Manif. *Springer Lecture Notes 412.*

[19] E. VIEHWEG: Invarianten lokaler Familien von Kurven; to appear in *J. reine u. angew. Math.*

[20] E. VIEHWEG: Rational singularities of higher dimensional schemes; to appear in *Proceedings of the AMS.*

[21] K. UENO: Classification Theory of Algebraic Varieties and Compact Complex Spaces. *Springer Lecture Notes 439.*

[22] K. UENO: Classification of algebraic varieties I. *Comp. Math. 27* (1973) 277–342.

[23] K. UENO: Kodaira dimension for certain fibre spaces; to appear.
[24] A. GROTHENDIECK and J. DIEUDONNÉ: Eléments de Géométrie Algébrique. *I.H.E.S. Publ. Math.* 4, . . ., 32.

(Oblatum: 18–VI–1976 & 11–XI–1976)

Institut fuer Mathematik und Informatik
der Universitaet
D-68 Mannheim
A/5 Seminargebaeude