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A. JOSEPH

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TOWARDS THE JANTZEN CONJECTURE*

A. Joseph**

Abstract

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ its enveloping algebra, Prim $U(\mathfrak{g})$ the set of primitive ideals of $U(\mathfrak{g})$ and $\mathfrak{h}$ a Cartan subalgebra for $\mathfrak{g}$. For $\mathfrak{g}$ simple of type A_{n-1} (Cartan notation), Jantzen [3], 5.9 conjectured that the cardinality of each Prim $U(\mathfrak{g})$ fibre projecting onto a fixed regular integral central character and onto a fixed nilpotent orbit in $\mathfrak{g}^*$ is just the dimension of the appropriate irreducible representation of the symmetric group S_n. Here it is suggested that the appropriate formulation of this conjecture for general $\mathfrak{g}$ involves the dimensions of certain subspaces of polynomials on $\mathfrak{h}^*$ which determine the dimensions of the irreducible finite dimensional representations of parabolic subalgebras of $\mathfrak{g}$. Its reduction to the Jantzen conjecture for type A_{n-1} is essentially a combinatorial result of Garnir [14]. Then through a careful study of ad $\mathfrak{g}$ finite homomorphisms of induced modules (which gives some results of independent interest) the Jantzen conjecture is reduced to two open questions. The first involves the principal series and would give a lower bound (involving the dimensions of the above-mentioned subspaces) on the cardinality of each regular integral fibre. In case A_{n-1} this is just the number of involutions in S_n and coincides with Duflo's upper bound [13], II.2. The second is a problem of Borho [1], 3.3 which whenever the last part of [21], 4.3 holds (for example in type A_{n-1} [25], 4.1) fixes the associated nilpotent orbit.

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** Present address: Department of Mathematics, Tel-Aviv University, Ramat Aviv, Israel.

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1. Introduction

Unless otherwise specified all vector spaces are assumed over the complex field $\mathbb{C}$.

1.1 For each vector space V, let $S(V)$ denote the symmetric algebra over V and V^* the dual of V. For each Lie algebra $\mathfrak{a}$, let $U(\mathfrak{a})$ denote its enveloping algebra and $Z(\mathfrak{a})$ the centre of $U(\mathfrak{a})$. For each associative algebra A let $\mathcal{I}(A)$ (resp. Spec A, Prim A) denote the set of two-sided (resp. prime, primitive) ideals of A and $A^\wedge$ the set of classes of irreducible representations of A, with a similar convention for a group. For $U(\mathfrak{a})^\wedge$ we simply write $\mathfrak{a}^\wedge$. A ring is said to be Noetherian if it is left and right Noetherian.

1.2 Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The principal aim of this paper is the classification of Prim $U(\mathfrak{g})$. Take $I \in \text{Prim } U(\mathfrak{g})$. Then the map $\pi: I \mapsto I \cap Z(\mathfrak{g})$ is a surjection of Prim $U(\mathfrak{g})$ onto Max $Z(\mathfrak{g})$ with fibres of finite cardinality [10], 8.5.7 (b), [13], II, Thm. 1. Give $U(\mathfrak{g})$ the canonical filtration [10], 2.3.1 and identify $\text{gr}(U(\mathfrak{g}))$ with $S(\mathfrak{g})$. Identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form and call $X \in \mathfrak{g}^*$ nilpotent if ad X is nilpotent. As noted in [6], Sect. 7, the zero variety $\mathcal{V}(\text{gr } I)$ of gr I is contained in the cone $\mathcal{N}$ of nilpotent elements of $\mathfrak{g}^*$ which under the adjoint group G is a finite union of orbits. Suppose further that the radical $\sqrt{\text{gr } I}$ of gr I is always a prime ideal. Then since G is algebraic, there is a unique nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$ whose Zariski closure $\bar{\mathcal{O}}$ coincides with $\mathcal{V}(\text{gr } I)$ and hence a map $\mathcal{K}$ of Prim $U(\mathfrak{g})$ into $\mathcal{N}/G$ (c.f. [3], 2.9). This gives rise to the following problem. For each $\hat{\lambda} \in \text{Max } Z(\mathfrak{g}), \mathcal{O} \in \mathcal{N}/G$ determine card$\{\pi^{-1}(\hat{\lambda}) \cap \mathcal{K}^{-1}(\mathcal{O})\}$. For $\mathfrak{g}$ simple of type A_{n-1}: $n = 2, 3, \ldots$, (Cartan notation) and for regular integral $\hat{\lambda}$, J.C. Jantzen conjectured [3], 5.9 that these numbers are just the appropriate dimensions of the irreducible representations of the symmetric group S_n. (Recall that $\hat{S}_n$ is in natural bijection with $\mathcal{N}/G$. The non-regular case is handled by [5], 2.12 and it is generally supposed (c.f. [21], Sect. 4) that the non-integral case mirrors the integral case.)

1.3 In attempting to prove Jantzen's elegant yet mysterious conjecture, it is clearly important to find a reinterpretation which applies to any semisimple Lie algebra. Now although most primitive ideals in type A_{n-1} are not induced ones, they all take the form (c.f. 9.3 and [5], 4.5 d)) of a minimal prime ideal containing an induced one. This leads us to suggest (see 8.2) that for $\mathcal{O}$ polarizable [10], 1. 12. 8, or equivalently for any Richardson orbit $\mathcal{O}$ (see 8.2), the cardinality of $\pi^{-1}(\hat{\lambda}) \cap \mathcal{K}^{-1}(\mathcal{O})$ is the dimension of the space generated by the polynomials on $\mathfrak{h}^*$ which determine all possible dimensions of finite dimensional irreducible representations of an appropriate subset of parabolic subalgebras. Then through a careful study of locally ad $\mathfrak{g}$ finite homomorphisms of induced modules (Sects. 4–7) and [21–25], we are able to reduce the Jantzen conjecture to the following two open questions, 9.1 and 10.2. First to show that for each induced ideal J one has $\sqrt{\operatorname{gr} J} \in \operatorname{Spec} S(\mathfrak{g})$ – a problem suggested by Borho [1], 3.3. Second to show that the simple subquotients of the spherical principal series of different multiplicity (or just of non-commensurable multiplicity, 10.5) in the sense of the Hilbert-Samuel polynomial, necessarily admit different annihilators. Our more general conjecture for an arbitrary semisimple Lie algebra further requires the solution of certain combinatorial questions involving the Weyl group and the root system. In type A_{n-1}, these are resolved through results of Specht [33], Garnir [14], Schensted [31] and Knuth [27].

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2. The Hilbert-Samuel polynomial

To set notation we recall some standard results concerning the Hilbert-Samuel polynomial.

2.1 Let A be an associative algebra which we shall always assume finitely generated and with an identity. Given T, T' subspaces of A we set $TT' = \text{lin span}\{tt' : t \in T, t' \in T'\}$ and for each $k \in \mathbb{N}$, we define T^k inductively through $T^0 = \mathbb{C}$, $T^k = T^{k-1}T$ and set $T^{-1} = 0$. Now suppose that T is a finite dimensional generating subspace of A containing the identity. Then the subspaces $T^{-1} \subset T^0 \subset T^1 \subset \cdots$, define a filtration for A. For each $k \in \mathbb{N}$, set $T_k = T^k/T^{k-1}$ and let

$$\operatorname{gr} A := \bigoplus_{k=0}^{\infty} T_k,$$

denote the associated graded algebra which we shall always assume commutative. If M is a finitely generated left A module, fix a finite dimensional generating subspace M^0 and for each $k \in \mathbb{N}$, set $M^{k-1} = T^{k-1}M^0$, $M_k = M^k/M^{k-1}$. Then

$$\operatorname{gr} M := \bigoplus_{k=0}^{\infty} M_k,$$

is a graded module for $\operatorname{gr} A$ satisfying the hypotheses of [32], Chap.

II, Thm. 3. Through its conclusion there exists a polynomial $q_T(M)$ (the Hilbert-Samuel polynomial) such that $q_T(M)(k) = \Sigma_{\ell=0}^k \dim M_\ell = \dim M^k$, for all k sufficiently large. We set $d(M) = \deg q_T(M)$ and let $e_T(M)/d(M)!$ denote the coefficient of $k^{d(M)}$ in $q_T(M)$. We recall that $d(M)+1$, $e_T(M)$ are positive integers which do not depend on the choice of generating subspace M^0 and $d(M)$ (denoted by $\dim M$ in [25]) does not depend on the choice of generating subspace T (whereas $e_T(M)$ does). We define $d(A)$ (which coincides with Dim A defined in [6]) and $e_T(A)$ through A considered as a left A module. When $A = U(\mathfrak{a})$, for some finite dimensional Lie algebra $\mathfrak{a}$, we shall always take T to be the image of the canonical embedding of $\mathfrak{a} \oplus \mathbf{C}$ in $U(\mathfrak{a})$ (which defines the canonical filtration $\{U(\mathfrak{a})^k : k = 0, 1, \ldots\}$, of $U(\mathfrak{a})$) and we simply write $e(M)$ for $e_T(M)$. We identify $\mathrm{gr}(U(\mathfrak{a}))$ with $S(\mathfrak{a})$.

2.2 Recall the well-known [32], Chap. II, Prop 10

LEMMA: *Let $0 \to M_1 \to M \to M_2 \to 0$, be an exact sequence of finitely generated A modules. Then one of the following hold*

(i) *$d(M_1) < d(M)$ and $d(M_2) = d(M)$, $e_T(M_2) = e_T(M)$.*

(ii) *$d(M_1) = d(M) = d(M_2)$ and $e_T(M) = e_T(M_1) + e_T(M_2)$.*

(iii) *$d(M_2) < d(M)$ and $d(M_1) = d(M)$, $e_T(M_1) = e_T(M)$.*

2.3 Let $\mathfrak{a}$ be a finite dimensional Lie algebra, $A = U(\mathfrak{a})$ and V a left and right $U(\mathfrak{a})$ module (which we can consider as a left $U(\mathfrak{a}) \otimes U(\mathfrak{a})$ module). Set LAnn $V = \{a \in A : aV = 0\}$, RAnn $V = \{a \in A : Va = 0\}$. We shall say that V is ad $\mathfrak{a}$ finite if for each $X \in \mathfrak{a}$, the endomorphism ad $X : v \mapsto Xv - vX$ of V is locally finite. Suppose V is ad $\mathfrak{a}$ finite and finitely generated as a $U(\mathfrak{a}) \otimes U(\mathfrak{a})$ module. Then we can choose a finite dimensional subspace V^0 of V which generates V as a $U(\mathfrak{a}) \otimes U(\mathfrak{a})$ module and satisfies (ad X) $V^0 \subset V^0$, for all $X \in \mathfrak{a}$. Let T denote the image of $\mathfrak{a} \oplus \mathbf{C}$ in $U(\mathfrak{a})$. Then for all $k \in \mathbf{N}$, one has $T^k V^0 = T^{k-1} V^0 T = T^{k-2} V^0 T^2 = \cdots = V^0 T^k$, and in particular that

$$\dim T^k V^0 = \dim\left(\sum_{\ell=0}^{k} T^{k-\ell} V^0 T^\ell\right) = \dim V^0 T^k.$$

It follows that V is finitely generated as a left and as a right $U(\mathfrak{a})$ module and the Hilbert-Samuel polynomials for these three actions coincide. We use $d(V)$, $e(V)$ to denote the common invariants.

An elementary computation gives

LEMMA: *Suppose V is finitely generated as a left and a right $U(\mathfrak{a})$*

module and Ann V = LAnn $V \otimes U(\mathfrak{a}) + U(\mathfrak{a}) \otimes$ RAnn V. *Then*

(i) $d(U(\mathfrak{a}) \otimes U(\mathfrak{a})/\text{Ann } V) = d(U(\mathfrak{a})/\text{LAnn } V) + d(U(\mathfrak{a})/\text{RAnn } V)$,

(ii) $e(U(\mathfrak{a}) \otimes U(\mathfrak{a})/\text{Ann } V) = e(U(\mathfrak{a})/\text{LAnn } V)e(U(\mathfrak{a})/\text{RAnn } V)$.

2.4 The following generalizes [25], 3.1.

PROPOSITION: *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* V *an* ad $\mathfrak{g}$ *finite* $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ *module of finite length. Then*

(i) $d(U(\mathfrak{g})/\text{LAnn } V) = d(U(\mathfrak{g})/\text{RAnn } V) = d(V)$.

(ii) *If* $\sqrt{\text{gr(LAnn } V)}$ *and* $\sqrt{\text{gr(RAnn } V)}$ *are both prime ideals, then they coincide.* (*Recall that* $U(\mathfrak{g})$ *is given the canonical filtration*).

By 2.3, [13], Prop. 7 and [23], 3.2, 3.6, we have

$$\begin{aligned} d(V) &\geq \tfrac{1}{2}(d(U(\mathfrak{g})/\text{LAnn } V) + d(U(\mathfrak{g})/\text{RAnn } V)), \\ &\geq \tfrac{1}{2}(d(V) + d(V)), \text{ by [25], 2.1.} \end{aligned}$$

This gives (i). For (ii) observe that gr(LAnn V) $\subset$ Ann gr V (with gr V prescribed by 2.1 and 2.3) and so by (i), $d(V) = d(U(\mathfrak{g})/\text{LAnn } V) = d(S(\mathfrak{g})/\text{gr(LAnn } V)) \geq d(S(\mathfrak{g})/\text{Ann gr } V) \geq d(\text{gr } V) = d(V)$. Given $\sqrt{\text{gr(LAnn } V)}$ prime, one obtains $\sqrt{\text{gr(LAnn } V)} = \sqrt{\text{Ann gr } V}$ and hence (ii).

3. Induced modules

3.1 Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra for $\mathfrak{g}$, $R \subset \mathfrak{h}^*$ the set of non-zero roots, $R^+ \subset R$ a system of positive roots, $B \subset R^+$ a $\mathbf{Z}$ basis for R, s_α the reflection corresponding to the root α, W the group generated by the s_α: $\alpha \in R$, $P(R)$ the lattice of integral weights. Fix a Chevalley basis for $\mathfrak{g}$ and let X_α denote the element in this basis of weight $\alpha \in R$. Let $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) denote the subalgebra of $\mathfrak{g}$ spanned by the X_α: $\alpha \in R^+$ (resp. $\alpha \in R^-$) and set $\mathfrak{b} := \mathfrak{n} \oplus \mathfrak{h}$. For each subset $B' \subset B$, set $R' = \mathbf{Z}B' \cap R$, $R'^+ = R^+ \cap R'$, $W_{B'}$ the subgroup of W generated by the s_α: $\alpha \in R'$, $w_{B'}$ the unique element of $W_{B'}$ taking B' to $-B'$, $P(R')^{++} = \{\lambda \in \mathfrak{h}^*: 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbf{N}^+, \text{ for all } \alpha \in B'\}$, $B'^{\perp} = \{\lambda \in \mathfrak{h}^*: (\lambda, \alpha) = 0, \text{ for all } \alpha \in B'\}$. Let $\mathfrak{p}_{B'} \supset \mathfrak{b}$ (or simply, $\mathfrak{p}$) denote the subalgebra of $\mathfrak{g}$ with reductive part $\mathfrak{h} \oplus \{\oplus \mathbf{C}X_\alpha: \alpha \in R'\}$, $\mathfrak{m}_{B'}$ (or simply, $\mathfrak{m}$) the nilradical of $\mathfrak{p}_{B'}$ and $\sigma_{B'}$ (resp. ρ if $B' = B$) the half sum of the roots in R'^+. Given $\lambda \in P(R')^{++}$, let $V_{B'}(\lambda)$ denote the simple finite dimensional $\mathfrak{p}_{B'}$ module with highest weight $\lambda - \rho$ and in the notation of [10], 5.1 set $M_{B'}(\lambda) = \text{ind}(V_{B'}(\lambda), \mathfrak{p}_{B'} \uparrow \mathfrak{g})$, $I_{B'}(\lambda) =$

Ann $M_{B'}(\lambda)$. We remark that dim $V_{B'}(\lambda) = \dim V_{B'}(\lambda + \nu)$, for all $\nu \in B'^{\perp}$. When B' is the empty set, $M_{B'}(\lambda)$ coincides with the Verma module $M(\lambda)$ for $\mathfrak{g}, \mathfrak{h}, B, \rho$ as defined in [10], 7.1.14. We let $L(\lambda)$ denote the unique simple quotient of $M(\lambda)$ and set $I(\lambda) = \text{Ann}\, L(\lambda)$. If $M_{B'}(\lambda)$ is defined it is a quotient of $M(\lambda)$ and so $I(\lambda) \supset I_{B'}(\lambda)$.

Let $u \mapsto {}^t u$ (resp. $u \mapsto \check{u}$) denote the involutory antiautomorphism of $U(\mathfrak{g})$ defined through ${}^t X_\alpha = X_{-\alpha}: \alpha \in R$, ${}^t H = H$, for all $H \in \mathfrak{h}$ (resp. $\check{X} = -X$, for all $X \in \mathfrak{g}$). Set $\mathfrak{m}_{B'}^- = {}^t\mathfrak{m}_{B'}$ (or simply, $\mathfrak{m}^-$).

LEMMA: *For each $B' \subset B$, $\lambda \in P(R')^{++}$,*
(i) $d(M_{B'}(\lambda)) = \dim \mathfrak{m}_{B'}$.
(ii) $e(M_{B'}(\lambda)) = \dim V_{B'}(\lambda)$.

Take $M = M_{B'}(\lambda)$, $M^0 = V_{B'}(\lambda)$ in 2.1. Then M identifies with $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^0 = U(\mathfrak{m}^-) \otimes M^0$ and for all $k \in \mathbb{N}$, we have $M^k = U(\mathfrak{m}^-)^k \otimes M^0$, which gives the required assertions.

3.2 Identify $U := U(\mathfrak{g}) \otimes U(\mathfrak{g})$ canonically with $U(\mathfrak{g} \oplus \mathfrak{g})$, set $j(X) = (X, -{}^t X)$, for all $X \in \mathfrak{g}$ and $k := j(\mathfrak{g})$. Given $\lambda, \mu \in \mathfrak{h}^*$, M (resp. N) a subquotient of $M(\lambda)$ (resp. $M(\mu)$), define $\text{Hom}_{\mathbb{C}}(M, N)$ as a U module through $((a \otimes b), x)m = {}^t\check{a}x\check{b}m$, for all $a, b \in U(\mathfrak{g})$, $x \in \text{Hom}_{\mathbb{C}}(M, N)$, $m \in M$. Let $L(M, N)$ denote the subspace of $\text{Hom}_{\mathbb{C}}(M, N)$ of all $\mathfrak{k}$ finite elements (which is a U submodule and ad $\mathfrak{g}$ finite in the sense of 2.3). Given $\lambda, \mu \in P(R')^{++}$, then $L(M_{B'}(\lambda), M_{B'}(\mu))$ is non-trivial iff $\lambda - \mu \in P(R)$, [9], 5.8.

3.3 Call $\lambda \in \mathfrak{h}^*$ *dominant* if $2(\lambda, \alpha)/(\alpha, \alpha) \notin \mathbb{N}^-$, for all $\alpha \in R^+$ and *regular* if $(\lambda, \alpha) \neq 0$, for all $\alpha \in R$. For each $\lambda \in \mathfrak{h}^*$, set $W(\lambda) = \{w \in W: w\lambda = \lambda\}$, $R_\lambda = \{\alpha \in R: 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}\}$, $R_\lambda^+ = R^+ \cap R_\lambda$, W_λ the subgroup of W generated by the $s_\alpha: \alpha \in R_\lambda$ and w_λ the unique element of W_λ taking R_λ^+ to $-R_\lambda^+$. Given $\lambda, \mu \in \mathfrak{h}^*$ such that $\lambda - \mu \in P(R)$, then $W_\lambda = W_\mu$ and we say that λ and μ belong to the same *facette* of $\mathfrak{h}^*$ if $W(\lambda) = W(\mu)$ and there exists $w \in W_\lambda$ such that $w\lambda$ and $w\mu$ are both dominant. Given $\lambda, \mu \in P(R')^{++}$ consider $(M_{B'}(-\lambda) \otimes M_{B'}(-\mu))^*$ as a U module through transposition and let $L_{B'}(\lambda, \mu)$ denote the subspace of all $\mathfrak{k}$-finite elements (which is an ad $\mathfrak{g}$ finite U module and non-trivial iff $\lambda - \mu \in P(R)$). If B' is the empty set we simply write $L(\lambda, \mu)$. If μ is dominant, then [12], Thm. 4.2, $L(\lambda, \mu)$ admits a unique simple quotient $V(\lambda, \mu)$ and we set $V(\lambda, w\mu) = V(w^{-1}\lambda, \mu)$, for all $w \in W_\mu$ (c.f. [12], Thm. 4.1).

LEMMA: *For all $B' \subset B$; $\lambda, \mu \in P(R')^{++}$,*
(i) $e(L_{B'}(\lambda, \mu)) = e(L_{B'}(\mu, \lambda))$.
(ii) *If $M_{B'}(\lambda)$ and $M_{B'}(\mu)$ are simple $U(\mathfrak{g})$ modules, then* $e(L(M_{B'}(\lambda), M_{B'}(\mu)) = e(L(M_{B'}(\mu), M_{B'}(\lambda)))$.

(i) is clear (see proof of [13], Prop. 9). (ii) follows from (i) and [9], 5.5.

REMARK: We shall eventually see (c.f. 6.2 (iii)) that (ii) holds without restriction on simplicity.

3.4 Given $\lambda \in \mathfrak{h}^*$, let $\hat{\lambda}$ denote its orbit under W, which may be identified with the element $I(\lambda) \cap Z(\mathfrak{g})$ of $\operatorname{Max} Z(\mathfrak{g})$. Set $\mathscr{X}_{\hat{\lambda}} = \{I(\mu)\colon \mu \in \hat{\lambda}\}$. Then $\mathscr{X}_{\hat{\lambda}} \subset \pi^{-1}(\hat{\lambda})$ (notation 1.2) and indeed [13], Thm. 1 equality holds. Given $\lambda \in P(R)$ we may further identify $\hat{\lambda}$ with an element of $\mathfrak{k}^\wedge$ (by taking the unique simple $\mathfrak{k}$ module with extreme weight λ) and then $\mathfrak{k}^\wedge = P(R)/W$. Let $P(R)^+$ denote the dominant elements of $P(R)$. We give $P(R)^+$ (which we sometimes identify with $\mathbb{N}^r$: $r = \operatorname{rank} \mathfrak{g}$) the topology induced by the Zariski topology on $\mathfrak{h}^*$.

4. The primeness of the ring $L(M_{B'}(\lambda), M_{B'}(\lambda))$

Retain the notation of Section 3. We start with some standard reasoning.

4.1 A $\mathfrak{k}$-finite U module M is said to admit a formal character (with respect to $\mathfrak{k}$) if each isotypical component $M_{\hat{\nu}}$: $\hat{\nu} \in \mathfrak{k}^\wedge$ has finite multiplicity, which we denote by $[M:\hat{\nu}]$.

LEMMA: *Suppose $M \neq 0$ admits a formal character. Then M admits at least one simple subquotient.*

Choose $\hat{\nu} \in \mathfrak{k}^\wedge$ such that $M_{\hat{\nu}} \neq 0$. Let N' be a submodule of M for which $N'_{\hat{\nu}}$ has minimal non-zero multiplicity and set $N = UN'_{\hat{\nu}}$. By construction every proper submodule of N has no isotypical component of type $\hat{\nu}$. Hence the sum $\bar{N}$ of all proper submodules of N is a proper submodule of N and so $N/\bar{N}$ is the required simple subquotient.

4.2 A U module L is said to admit a central character if there exists $\Lambda \in \operatorname{Max}(Z(\mathfrak{g}) \otimes Z(\mathfrak{g}))$ such that $z - \Lambda(z) \cdot 1$ is nilpotent for every $z \in Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$.

LEMMA: *Let L be a $\mathfrak{k}$ finite U module. If L admits both a formal and a central character, then L has finite length.*

Let $(\hat{\lambda}, \hat{\mu})$: $\lambda, \mu \in \mathfrak{h}^*$ define the central character of L (c.f. 3.4). By [12], Thm. 4.5, the simple subquotients of L form a subset of $\{V(\lambda', \mu')\colon \lambda' \in \hat{\lambda}, \mu' \in \hat{\mu}\}$. Recall [12], 3.4 that $V(\lambda', \mu')$ has a non-zero isotypical component of type $(\lambda' - \mu')\hat{}$ and this can have at most $(\text{card } W)^2$ values. Now if L has infinite length, then by 4.1 it admits infinitely many simple subquotients which therefore cannot all belong to the above set. This contradiction proves the lemma.

REMARK: Obviously L has length $\leq (\text{card } W)^2 \cdot \max\{[M: (\lambda' - \nu')\hat{}]: \lambda' \in \hat{\lambda}, \mu' \in \hat{\mu}\}$.

4.3 PROPOSITION: *For all $\lambda, \mu \in \mathfrak{h}^*$ and every subquotient M (resp. N) of $M(\lambda)$ (resp. $M(\mu)$) one has*

(i) *$L(M, N)$ has finite length as a U module. In particular it is finitely generated as a left or a right $U(\mathfrak{g})$ module* (cf. 2.2).

(ii) *$L(M, M)$ is a Noetherian ring.*

It is clear that $(-\hat{\lambda}, -\hat{\mu})$ defines the central character of $L(M, N)$. Identify $\mathfrak{k}$ with $\mathfrak{g}$. Given F a finite dimensional $\mathfrak{g}$ module consider $M \otimes F$ as a $\mathfrak{g}$ module through $X(m \otimes f) = Xm \otimes f + m \otimes Xf: X \in \mathfrak{g}$, $m \in M$, $f \in F$. We have $\text{Hom}_\mathfrak{g}(F, L(M, N)) = \text{Hom}_\mathfrak{g}(M \otimes F, N)$ up to isomorphism (c.f. [8], 6.2). Now $M \otimes F$ has a formal character with respect to $\mathfrak{h}$ and so taking account of the possible simple subquotients of N (c.f. [10], 7.1.7, 7.4.7, 7.6.1) it follows that $L(M, N)$ admits a formal character. Hence (i) obtains from 4.2 and (ii) from the fact that $U(\mathfrak{g})$ is Noetherian.

REMARK: By [5], 3.6 there is integer $n(\mathfrak{g})$ depending only on $\mathfrak{g}$ which is an upper bound to the length of any Verma module for $\mathfrak{g}$. By [20], 2.2 we then have $\dim \text{Hom}_\mathfrak{g}(M \otimes F, N) \leq n(\mathfrak{g}) \dim F$.

4.4 In the remainder of Sect. 4, we fix $B' \subset B$ and $\lambda \in P(R')^{++}$. For all $\nu \in B'^{\perp}$, the identity map on $U(\mathfrak{m}^-)$ (notation 3.1) induces a $j(\mathfrak{m})$ invariant linear isomorphism $\theta_\lambda^{\lambda-\nu}$ (or simply, θ_ν) of $M_{B'}(\lambda)$ onto $M_{B'}(\lambda - \nu)$. Suppose further that $\nu \in P(R)^+$. Then by [9], 8.4 we have $\theta_\lambda^{\lambda-\nu} \in L(M_{B'}(\lambda), M_{B'}(\lambda - \nu))$ and we let $\Theta_\lambda^{\lambda-\nu}$ (or simply, Θ_ν) denote the unique simple $\mathfrak{k}$ module it generates. It is clear that for all $\lambda, \mu \in P(R')^{++}$, $0 \neq a \in L(M_{B'}(\mu), M_{B'}(\lambda))$ one has $\theta_\nu a \neq 0$.

4.5 The action of $U(\mathfrak{g})$ in $M_{B'}(\lambda)$ defines an embedding of $U(\mathfrak{g})/I_{B'}(\lambda)$ in $L(M_{B'}(\lambda), M_{B'}(\lambda))$ which may be strict [9], 6.5 (see also 10.5) even if $M_{B'}(\lambda)$ is a simple module. Conversely equality can hold [8], 6.10 even if $M_{B'}(\lambda)$ is not simple. In fact since $L(M_{B'}(\lambda), M_{B'}(\lambda))$ is generated by its ad $\mathfrak{m}^-$ invariant elements, it follows that equality holds whenever $\mathfrak{m}^-$ is commutative and $\dim V_{B'}(\lambda) = 1$. Set $P(R')^\vee = \{\lambda \in P(R')^{++}: -w_{B'}\lambda \text{ is dominant}\} = \{\lambda \in P(R')^{++}: 2(\lambda, \alpha)/(\alpha, \alpha) \notin \mathbb{N}^+,$

for all $\alpha \in R^+ \setminus R'^+\}$. For all $\lambda \in P(R')^{++}$, there clearly exists $\nu \in B'^{\perp} \cap P(R)^+$ such that $\lambda - \nu \in P(R')^{\vee}$. The importance of this set derives from the following result of Conze-Berline and Duflo [9], 2.12, 4.7, 6.3.

THEOREM: *For all $B' \subset B$, and all $\lambda \in P(R')^{\vee}$,*
(i) *$M_{B'}(\lambda)$ is a simple $U(\mathfrak{g})$ module.*
(ii) *$U(\mathfrak{g})/I_{B'}(\lambda) = L(M_{B'}(\lambda), M_{B'}(\lambda))$.*

REMARK: (i) is a special case of a result of Jantzen [18].

4.6 A result of Lepowsky [29], Thm. 1.1, states that $M_{B'}(\lambda)$ admits a unique simple submodule if $\dim V_{B'}(\lambda) = 1$. This presumably fails in general. Yet it does have the following important variation:

PROPOSITION: *Consider $M_{B'}(\lambda)$ as a $L(M_{B'}(\lambda), M_{B'}(\lambda))$ module. Then $M_{B'}(\lambda)$ admits a unique simple submodule.*

Choose $\nu \in B'^{\perp} \cap P(R)^+$ such that $\lambda - \nu \in P(R')^{\vee}$ and set $L := L(M_{B'}(\lambda - \nu), M_{B'}(\lambda))\theta_\nu$ which (c.f. 3.2, 4.4) is a non-zero left ideal of $L(M_{B'}(\lambda), M_{B'}(\lambda))$. Now for all $0 \neq m \in M_{B'}(\lambda)$, we have $0 \neq \theta_\nu m \in M_{B'}(\lambda - \nu)$ and so $U(\mathfrak{g})\theta_\nu m = M_{B'}(\lambda - \nu)$, by 4.5 (i). It follows that $N := Lm = L(M_{B'}(\lambda - \nu), M_{B'}(\lambda))M_{B'}(\lambda - \nu)$, is a non-zero $L(M_{B'}(\lambda), M_{B'}(\lambda))$ submodule of $M_{B'}(\lambda)$ which is independent of the choice of m. Consequently for any non-zero simple $L(M_{B'}(\lambda), M_{B'}(\lambda))$ submodule N_0 of $M_{B'}(\lambda)$, we must have $N_0 \supset LN_0 = N$ and so N satisfies the conclusion of the proposition.

4.7 We denote the submodule in the conclusion of 4.6 by $N_{B'}(\lambda)$. Consider $N_{B'}(\lambda)$ as a $U(\mathfrak{g})$ module (the latter given the canonical filtration).

LEMMA:
(i) $d(N_{B'}(\lambda)) = \dim \mathfrak{m}_{B'}$,
(ii) $e(N_{B'}(\lambda)) = \dim V_{B'}(\lambda)$,
(iii) $d(M_{B'}(\lambda)/N_{B'}(\lambda)) < \dim \mathfrak{m}_{B'}$.

Choose $\nu \in B'^{\perp} \cap P(R)^+$ such that $\lambda - \nu \in P(R')^{\vee}$. Set $A = U(\mathfrak{g})/I_{B'}(\lambda - \nu)$, $B = L(M_{B'}(\lambda), M_{B'}(\lambda - \nu))$, $C = L(M_{B'}(\lambda - \nu), M_{B'}(\lambda))$, $V = V_{B'}(\lambda - \nu)$, $M = M_{B'}(\lambda - \nu)$, $D = \operatorname{End} M$. By 4.4, we have $\theta_\nu \in B$. By 4.5 (ii), $A = L(M, M)$ and so $\theta_\nu C$ is a right ideal of A. Furthermore by [26], 4.2, $A^{\mathfrak{m}^-}$ is a prime Goldie ring which by [24], 5.8 is a quotient of $U(\mathfrak{g})^{\mathfrak{m}^-}$ and is hence [42] finitely generated. Recall [9], 8.4 that θ_ν is a $j(\mathfrak{n})$ invariant weight vector of $j(\mathfrak{b})$ weight ν.

(*) $\theta_\nu C^{\mathfrak{m}^-}$ *is an essential right ideal of* $A^{\mathfrak{m}^-}$.

Consider $A^{\mathfrak{m}^-}/\theta_\nu C^{\mathfrak{m}^-}$ as a finitely generated right $A^{\mathfrak{m}^-}$ module. By [39], 2.3 (which trivially generalizes to Goldie rings) it is enough to show that $d(A^{\mathfrak{m}^-}/\theta_\nu C^{\mathfrak{m}^-}) < d(A^{\mathfrak{m}^-})$. This follows easily from the dimensionality estimates (i) and (ii) given below. First for each $\mu \in P(R)^+$, let $A^{\mathfrak{m}^-}_\mu$ (resp. $B^{\mathfrak{m}^-}_\mu$, $C^{\mathfrak{m}^-}_\mu$) denote the subspace of $A^{\mathfrak{m}^-}$ (resp. $B^{\mathfrak{m}^-}, C^{\mathfrak{m}^-}$) of $j(\mathfrak{h})$ weight vectors of weight μ and identify M with $U(\mathfrak{m}^-) \otimes V$. Then $A^{\mathfrak{m}^-}$ is a $j(\mathfrak{h})$ submodule of $D^{\mathfrak{m}^-}$ which is in turn isomorphic to $U(\mathfrak{m}^-) \otimes \operatorname{End} V$ (c.f. 5.8). This gives $\dim A^{\mathfrak{m}^-}_\mu < \infty$. Since $A\theta_\nu \subset B$ and $a\theta_\nu = 0$: $a \in A$ implies $a = 0$, we obtain

(i) $\dim B^{\mathfrak{m}^-}_{\mu+\nu} \geq \dim A^{\mathfrak{m}^-}_\mu$, *for all* $\mu \in P(R)^+$.

Set $\mu^* = -w_B\mu$. Recalling that M is a simple module it follows from [9], 5.5, 5.8 that $\dim C^{\mathfrak{m}^-}_\mu \geq \dim B^{\mathfrak{m}^-}_{\mu^*}$. Yet $\theta_\nu c = 0$: $c \in C$ implies $c = 0$ and so

(ii) $\dim A^{\mathfrak{m}^-}_{\mu+\nu} \geq \dim(\theta_\nu C^{\mathfrak{m}^-}_\mu) \geq \dim B^{\mathfrak{m}^-}_{\mu^*}$, *for all* $\mu \in P(R)^+$.

By [26], 4.2 each regular element $s \in A^{\mathfrak{m}^-}$ is regular in $D^{\mathfrak{m}^-}$. The latter identifies with $U(\mathfrak{m}^-) \otimes \operatorname{End} V$ in which the elements of $U(\mathfrak{m}^-)$ act by right multiplication in $M = U(\mathfrak{m}^-) \otimes V$. Thus for each $m \in M$ we can choose $a \in D^{\mathfrak{m}^-}$ such that $aV = \mathbb{C}m$. If $sm = 0$, then $saV = 0$ and so $saM = 0$. Consequently $sa = 0$, which by the regularity of s implies $a = 0$ and hence $m = 0$.

By (*) we can choose $c \in C^{\mathfrak{m}^-}$ such that $s := \theta_\nu c$ is regular in $A^{\mathfrak{m}^-}$. We have shown that $cm \neq 0$ for all $0 \neq m \in M$ and so $\dim(U(\mathfrak{m}^-)^k c(1 \otimes V)) = \dim(U(\mathfrak{m}^-)^k \otimes V)$, for all $k \in \mathbb{N}$. Since $cM \subset CM \subset N_{B'}(\lambda)$ it follows by 3.1 that $d(N_{B'}(\lambda)) \geq d(M_{B'}(\lambda - \nu)) = \dim \mathfrak{m}_{B'}$ and equality implies $e(N_{B'}(\lambda)) \geq \dim V_{B'}(\lambda - \nu) = \dim V_{B'}(\lambda)$. Yet the opposite inequalities obtain from 3.1 and the fact that $N_{B'}(\lambda)$ is a submodule of $M_{B'}(\lambda)$. This gives (i) and (ii), which combined with 2.2 (i) imply (iii).

4.8 Theorem: *For all* $B' \subset B$, $\lambda \in P(R')^{++}$,

(i) $N_{B'}(\lambda)$ *is a faithful* $L(M_{B'}(\lambda), M_{B'}(\lambda))$ *module.*

(ii) $L(M_{B'}(\lambda), M_{B'}(\lambda))$ *is a prime, Noetherian ring.*

Set $A = L(M_{B'}(\lambda), M_{B'}(\lambda))$, $J = \operatorname{Ann} M_{B'}(\lambda)/N_{B'}(\lambda)$, $I = \operatorname{Ann} N_{B'}(\lambda)$, (computed in A) $J' = J \cap U(\mathfrak{g})/I_{B'}(\lambda)$, $I' = I \cap U(\mathfrak{g})/I_{B'}(\lambda)$. By [25], 2.8, $d(U(\mathfrak{g})/J') = 2d(M_{B'}(\lambda)/N_{B'}(\lambda))$, $d(U(\mathfrak{g})/I') = 2d(N_{B'}(\lambda))$ and so by 4.7 (iii) we have $J \not\subset I$, which by 4.6 gives $JM_{B'}(\lambda) = JN_{B'}(\lambda) = N_{B'}(\lambda)$. Suppose $IM_{B'}(\lambda) \neq 0$. Then by 4.6, $IM_{B'}(\lambda) \supset N_{B'}(\lambda)$ and so $N_{B'}(\lambda) \supset (I \cap J)M_{B'}(\lambda) \supset JIM_{B'}(\lambda) \supset JN_{B'}(\lambda) = N_{B'}(\lambda)$. In particular, $(I \cap J)$ is a non-zero U submodule of A. Set $L = \text{LAnn}(I \cap J)$, $R = \text{RAnn}(I \cap J)$, (computed in $U(\mathfrak{g})/I_{B'}(\lambda)$). Since $IJM_{B'}(\lambda) = IN_{B'}(\lambda) = 0$, we have $L \supset I'$, $R \supset J'$. Yet $0 = L(I \cap J)M_{B'}(\lambda) = LN_{B'}(\lambda)$ and so $L = I'$. By 4.7 (iii), this gives $d(U(\mathfrak{g})/R) \leq d(U(\mathfrak{g})/J') < d(U(\mathfrak{g})/I') = d(U(\mathfrak{g})/L)$. Yet by 4.3 (i), A has finite length as a U module and so by 2.4 (i) we must have $d(U(\mathfrak{g})/R) = d(U(\mathfrak{g})/L)$. This contradiction gives (i). Combined with 4.3 (ii) and 4.6 this gives (ii).

5. Localization

5.1 Let A be a prime, Noetherian ring and S the set of regular elements of A (so then Fract $A = S^{-1}A$). Given M a left A module, set $S^{-1}M := S^{-1}A \otimes_A M$ (or simply, Fract M). We shall say that M is *divisable* (by S) if the map $m \mapsto 1 \otimes m$ of M into $S^{-1}M$ is injective (equivalently, if for each $s \in S$, $0 \neq m \in M$ one has $sm \neq 0$). Obviously any submodule of a divisable module is divisable. In particular any left ideal of A is divisable as a left A module. Suppose in addition that A and M are finitely generated. Then $d(M) \leq d(A)$, by [25], 2.1. Suppose further that M is divisable. Then we have the

LEMMA: $d(M) = d(A)$.

Suppose $d(M) < d(A)$. Choose $0 \neq m \in M$ and set $N = Am$, $L = \text{Ann}\, m$. Then for every left ideal K of A we have $d(K/(K \cap L) \leq d(A/L) = d(N) \leq d(M) < d(A)$. Hence by [39], 2.3 if $K \neq 0$, then $K \cap L \neq 0$ and so ([15], Lemma 7.2.5) $L \cap S \neq \emptyset$. This contradicts the divisability of M.

5.2 Retain the notation and hypotheses of 5.1 and suppose in addition that $d(A) < \infty$. Let rk M denote the maximum number of direct summands of non-zero left A submodules of M. Recall that M is assumed finitely generated and so $S^{-1}M$ is finitely generated as a left $S^{-1}A$ module. By ([15], Lemma 4.3.2, Thms. 2.1.6, 7.2.1) we can write $S^{-1}M$ as a direct sum of $k := \text{rk}\, M$ simple $S^{-1}A$ modules $Q_1, Q_2, \ldots, Q_k$ each isomorphic to a fixed minimal left ideal L of $S^{-1}A$. Let N be any A submodule of M.

LEMMA:

(i) *If* $S^{-1}N \cap M = N$, *then* $d(M/N) < d(M)$.

(ii) $e(M) = e(L \cap A) \cdot \operatorname{rk} M$. *In particular* $\operatorname{rk} M$ *divides* $e(M)$.

(i) Let M^0 be a finite dimensional generating subspace for M. By the hypothesis of (i) there exists $s \in S$ such that $sM^0 \subset N$. Then $d(M/N) \leq d(A/As) < d(A) = d(M)$.

(ii) Set $P_i = A \cap Q_i$: $i = 1, 2, \ldots, k$. We have $S^{-1}P_i = Q_i$ and $d(P_i) = d(M)$ by 5.1. Let N be the direct sum of the P_i (which may be considered as a submodule of M). We have $S^{-1}N = M$ and so by (i) and 2.2 (iii) that $e(M) = e(N)$. Hence it is enough to show that $e(P_i) = e(L \cap A)$, for all i. Set $P = P_i$. Let P^0 be a finite dimensional generating subspace for P and φ the $S^{-1}A$ module isomorphism of $S^{-1}P$ onto L. Choose $s \in S$ such that $s\varphi(P^0) \subset A$. Then $P' := AsP^0$ is an A submodule of P and $S^{-1}P' = S^{-1}P$ by the left Ore condition on A. Hence $d(P/P') < d(P)$ by (i). Again $\varphi(P') = As\,\varphi(P^0) \subset L \cap A$; $S^{-1}(\varphi(P')) = \varphi(S^{-1}P') = L$ and so $d((L \cap A)/\varphi(P')) < d(L \cap A)$ by (i). Hence $e(P) = e(P') = e(L \cap A)$, as required.

5.3 (Notation, Sects. 3, 4). Fix $\lambda_i \in P(R')^{++}$ such that $\lambda_i - \lambda_j \in P(R)$: $i, j = 1, 2, 3$.

LEMMA: *For all* $0 \neq a \in L(M_{B'}(\lambda_1), M_{B'}(\lambda_2))$ *one has* $L(M_{B'}(\lambda_2), M_{B'}(\lambda_3))a \neq 0$.

Choose $\nu \in B'^{\perp} \cap P(R)^+$ such that $M_{B'}(\lambda_2 - \nu)$ is simple and define θ_ν as in 4.4. Then $U(\mathfrak{g})\theta_\nu a M_{B'}(\lambda_1) = M_{B'}(\lambda_2 - \nu)$, by the simplicity of $M_{B'}(\lambda_2 - \nu)$ and then $L(M_{B'}(\lambda_2), M_{B'}(\lambda_3))a \supset L(M_{B'}(\lambda_2 - \nu), M_{B'}(\lambda_3))\theta_\nu a \neq 0$, as required.

5.4 COROLLARY: $d(L(M_{B'}(\lambda_1), M_{B'}(\lambda_2)) = 2 \dim \mathfrak{m}_{B'}$.

Choose $\nu \in B'^{\perp} \cap P(R)^+$ such that $\lambda_2 - \nu \in P(R')^{\vee}$ and a finite dimensional $\mathfrak{k}$ submodule F of $L(M_{B'}(\lambda_2 - \nu), M_{B'}(\lambda_1))$ such that $L(M_{B'}(\lambda_1), M_{B'}(\lambda_2))F \neq 0$. Then $\Theta_\nu L(M_{B'}(\lambda_1), M_{B'}(\lambda_2))\, F$ is a non-zero two-sided ideal of the prime Noetherian ring $U(\mathfrak{g})/I_{B'}(\lambda_2 - \nu)$. By 2.2, [1], 2.4, [25], 2.1 and [6], 3.6, we have $2 \dim \mathfrak{m}_{B'} = d(U(\mathfrak{g})/I_{B'}(\lambda_2 - \nu)) = d(\Theta_\nu L(M_{B'}(\lambda_1), M_{B'}(\lambda_2))\, F) \leq d(L(M_{B'}(\lambda_1), M_{B'}(\lambda_2)))$ where the last step obtains from the fact that Θ_ν, F are finite dimensional and $\mathfrak{k}$ stable. By 4.3 (i), and [25], 2.1, 2.8 we obtain the opposite inequality.

5.5 LEMMA (notation 4.7):

(i) $aN_{B'}(\lambda_1) = 0$: $a \in L(M_{B'}(\lambda_1), M_{B'}(\lambda_2))$ *implies* $a = 0$.

(ii) $N_{B'}(\lambda_2) \subset L(M_{B'}(\lambda_1), M_{B'}(\lambda_2))M_{B'}(\lambda_1)$.

(i) By 5.3, there exists $b \in L(M_{B'}(\lambda_2), M_{B'}(\lambda_1))$ such that $0 \neq ba \in L(M_{B'}(\lambda_1), M_{B'}(\lambda_1))$ and so (i) follows from 4.8 (i). (ii) follows from (i) and 4.6.

5.6 COROLLARY: *For all* $0 \neq a \in L(M_{B'}(\lambda_2), M_{B'}(\lambda_3))$ *one has* $aL(M_{B'}(\lambda_1), M_{B'}(\lambda_2)) \neq 0$.

By 5.5, $aL(M_{B'}(\lambda_1), M_{B'}(\lambda_2))M_{B'}(\lambda_1) \supset aN_{B'}(\lambda_2) \neq 0$.

5.7 In the remainder of Sect. 5 we fix $\lambda, \mu \in P(R')^{++}: \lambda - \mu \in P(R)$.

LEMMA: *Choose* $\nu \in B'^{\perp} \cap P(R)^{+}$ *such that* $M_{B'}(\lambda - \nu)$ *is a simple* $U(\mathfrak{g})$ *module. Then* $L := L(M_{B'}(\lambda - \nu), M_{B'}(\lambda))\theta_\nu$ *contains a regular element of* $L(M_{B'}(\lambda), M_{B'}(\lambda))$.

Taking 4.8 (ii) into account this follows exactly as in the proof of [9], 8.5.

5.8 For all $a \in U(\mathfrak{m}^-)$, $x \in \mathrm{Hom}(V_{B'}(\mu), V_{B'}(\lambda))$, define $r_a \otimes x \in (\mathrm{Hom}(M_{B'}(\mu), M_{B'}(\lambda)))^{j(\mathfrak{m})}$ through $(r_a \otimes x)(b \otimes e) = ba \otimes xe$, for all $b \in U(\mathfrak{m}^-)$, $e \in V_{B'}(\mu)$. It is clear that the map $a \otimes x \mapsto r_a \otimes x$ extends linearly to an isomorphism of $U(\mathfrak{m}^-) \otimes \mathrm{Hom}(V_{B'}(\mu), V_{B'}(\lambda))$ onto $(\mathrm{Hom}(M_{B'}(\mu), M_{B'}(\lambda)))^{j(\mathfrak{m})}$, and we identify $\mathrm{Hom}(V_{B'}(\mu), V_{B'}(\lambda))$ with the image of $1 \otimes \mathrm{Hom}(V_{B'}(\mu), V_{B'}(\lambda))$ under this map.

5.9 THEOREM: *For all* $\lambda, \mu \in P(R')^{++}: \lambda - \mu \in P(R)$, $L(M_{B'}(\mu), M_{B'}(\lambda))$ *and* $L(M_{B'}(\lambda), M_{B'}(\lambda))\,\mathrm{Hom}(V_{B'}(\mu), V_{B'}(\lambda))$ *are finitely generated divisable left* $L(M_{B'}(\lambda), M_{B'}(\lambda))$ *modules and considered as submodules of* $\mathrm{Hom}(M_{B'}(\mu), M_{B'}(\lambda))$ *satisfy*

$$\mathrm{Fract}\, L(M_{B'}(\mu), M_{B'}(\lambda)) = \mathrm{Fract}(L(M_{B'}(\lambda), M_{B'}(\lambda))\,\mathrm{Hom}(V_{B'}(\mu), V_{B'}(\lambda))).$$

$$\text{Set } L = L(M_{B'}(\mu), M_{B'}(\lambda)),$$
$$L' = L(M_{B'}(\lambda), M_{B'}(\mu)), \quad A = L(M_{B'}(\lambda), M_{B'}(\lambda)),$$
$$K = (\mathrm{Hom}(M_{B'}(\mu), M_{B'}(\lambda)))^{j(\mathfrak{m})}.$$

Take $0 \neq a \in L$. By 5.6, there exists $b \in L'$ such that $0 \neq ab \in A$ and so if $s \in A$ is regular, we have $sa \neq 0$. Hence L is divisable. By an argument which exactly parallels [9], 5.10, it follows that for each finite dimensional subspace $T \subset K$, there exists $\nu \in B'^{\perp} \cap P(R)^{+}$ such that $T\theta_\nu \subset L(M_{B'}(\lambda + \nu), M_{B'}(\lambda))$. This and 5.6 establishes the divisability of AK. Choose $a \in K$. By 4.5 and [9], 5.10, there exists $\nu \in B'^{\perp} \cap P(R)^{+}$ such that $\theta_\nu a \in L(M_{B'}(\lambda), M_{B'}(\lambda - \nu))$ and that

$M_{B'}(\lambda - \nu)$ is a simple $U(\mathfrak{g})$ module. By 5.7, we can choose $b \in L(M_{B'}(\lambda - \nu), M_{B'}(\lambda))$ such that $s := b\theta_\nu$ is regular and we have $sa \in L$. Thus Fract $AK \subset$ Fract L. In particular, taking $\mu = \lambda$, we have $(\mathrm{Hom}(M_{B'}(\lambda), M_{B'}(\lambda)))^{j(\mathfrak{m})} \subset \mathrm{Fract}\, A$. Thus by 5.8 we obtain Fract $AK = \mathrm{Fract}(A\, \mathrm{Hom}(V_{B'}(\mu), V_{B'}(\lambda)))$.

It is clear that each $X \in \mathfrak{m}^-$ is locally ad-nilpotent in A and for each $0 \neq a \in U(\mathfrak{m}^-)$, we have $am = 0$: $m \in M_{B'}(\lambda)$, implies $m = 0$. Set $Z := U(\mathfrak{m}^-)^{\mathfrak{n}^-} \setminus \{0\} \subset Z(\mathfrak{m}^-)$. Through the argument of [7], 2.4, each $z \in Z$ is locally ad-nilpotent in A; has trivial right annihilator and so is regular (either by 4.8 (ii) and [15], 7.2.3, or by ad-nilpotence). Thus Z is an Ore subset for both $U(\mathfrak{g})/I_{B'}(\lambda)$ and A. Define $\mathfrak{c}(\mathfrak{m}^-)$ (or simply, $\mathfrak{c}$) as in [24], 2.6. We recall [24], 2.6 that $U(\mathfrak{c})^{\mathfrak{n}} = U(\mathfrak{m}^-)^{\mathfrak{n}}$ which in particular gives that $I_{B'}(\lambda) \cap U(\mathfrak{c}) = 0$. Thus the embedding $U(\mathfrak{g})/I_{B'}(\lambda) \hookrightarrow A$ restricts to an embedding of $U(\mathfrak{c})$ and through the ad-nilpotence of $\mathfrak{m}^-$, by an argument which exactly parallels [24], 3.3, we find that $Z^{-1}L = Z^{-1}(U(\mathfrak{c})L^{\mathfrak{m}^-})$. Combined with our earlier inclusions this proves the theorem.

5.10 Set $m = \dim \mathfrak{m}$ and let $\mathscr{A}_m$ denote the Weyl algebra of index m over $\mathbb{C}$. We note the following result which obtains from 5.9 and the methods of [26].

THEOREM: *For all $B' \subset B$, $\lambda \in P(R')^{++}$,*

(i) $L(M_{B'}(\lambda), M_{B'}(\lambda))^{\mathfrak{m}^-}$ *is a prime, Noetherian ring.*

(ii) Fract $L(M_{B'}(\lambda), M_{B'}(\lambda))^{\mathfrak{m}^-} = \mathrm{Fract}(\mathrm{Hom}(M_{B'}(\lambda), M_{B'}(\lambda)))^{j(\mathfrak{m})}$.

(iii) Fract $L(M_{B'}(\lambda), M_{B'}(\lambda)) = \mathrm{Fract}(\mathscr{A}_m \otimes \mathrm{End}\, V_{B'}(\lambda))$, *up to an isomorphism.*

6. Multiplicities

In 6.1–6.4 we fix $B' \subset B$ and take $\lambda, \mu \in P(R')^{++}$: $\lambda - \mu \in P(R)$ and $\nu \in B'^{\perp} \cap P(R)$.

6.1 (Notation 4.4). In general $\theta_\nu \notin L(M_{B'}(\lambda), M_{B'}(\lambda - \nu))$. Yet $\theta_\nu \in \mathrm{Hom}(V_{B'}(\lambda), V_{B'}(\lambda - \nu))$ and so by 5.9 there exists s regular in $L(M_{B'}(\lambda - \nu), M_{B'}(\lambda - \nu)))$ such that $s\theta_\nu \in L(M_{B'}(\lambda), M_{B'}(\lambda - \nu))$. Let Θ_ν denote the finite dimensional $\mathfrak{k}$ module generated by $s\theta_\nu$. (For $\nu \in P(R)^+$ one can choose $s = 1$, though for our purposes the precise choice of s is irrelevant.)

LEMMA: *For all $0 \neq a \in L(M_{B'}(\mu), M_{B'}(\lambda))$, one has $\Theta_\nu a \neq 0$.*

By the divisibility of $L(M_{B'}(\mu), M_{B'}(\lambda - \nu))$, the relation $s\theta_\nu a = 0$ implies $\theta_\nu a = 0$ and so $a = 0$.

6.2 LEMMA:

(i) $e(L(M_{B'}(\mu), M_{B'}(\lambda)) = e(L(M_{B'}(\mu), M_{B'}(\lambda - \nu))$.

(ii) $e(L(M_{B'}(\mu), M_{B'}(\lambda)) = e(L(M_{B'}(\mu - \nu), M_{B'}(\lambda))$.

(iii) $e(L(M_{B'}(\mu), M_{B'}(\lambda)) = e(L(M_{B'}(\lambda), M_{B'}(\mu))$.

Set $M := L(M_{B'}(\mu), M_{B'}(\lambda - \nu))/\Theta_\nu L(M_{B'}(\mu), M_{B'}(\lambda))$, $J = \Theta_\nu L(M_{B'}(\lambda - \nu), M_{B'}(\lambda))$. By construction $JM = 0$ and by 4.8 and 6.1, J is a non-zero two-sided ideal of the prime, Noetherian ring $A := L(M_{B'}(\lambda - \nu), M_{B'}(\lambda - \nu))$. By 4.3 (i), M is a finitely generated left A module and so by 5.4, [25], 2.1 and [6], 3.6 we obtain $d(M) \le d(A/J) < d(A) = 2 \dim \mathfrak{m}$. By 5.4, 2.2 and 6.1 this gives $e(L(M_{B'}(\mu), M_{B'}(\lambda - \nu)) = e(\Theta_\nu L(M_{B'}(\mu), M_{B'}(\lambda))) \ge e(L(M_{B'}(\mu), M_{B'}(\lambda))$. Replacing ν by $-\nu$ gives (i). An analogous argument (with Θ_ν on the right) gives (ii). Combined with 3.3 (ii) and 4.5 (i) they give (iii).

6.3 PROPOSITION: *Both* $\dim V_{B'}(\mu)$ *and* $\dim V_{B'}(\lambda)$ *divide* $e(L(M_{B'}(\mu), M_{B'}(\lambda)))$.

By 6.2 it suffices to prove the first assertion with $\lambda \in P(R')^\vee$. By 4.5 (ii), 5.2, 5.4 and 5.9, it is enough to show that $\dim V_{B'}(\mu)$ divides $e(U(\mathfrak{g}) \operatorname{Hom}(V_{B'}(\mu), V_{B'}(\lambda)))$. Yet the latter is clearly a direct sum of $\dim V_{B'}(\mu)$ isomorphic $U(\mathfrak{g})$ submodules and so the required assertion is obtained.

6.4 LEMMA: *For all* $\lambda, \mu \in P(R')^{++}$: $\lambda - \mu \in P(R)$, *there exists a positive integer* $c(\mathfrak{g})$ *depending only on* $\mathfrak{g}$ *such that*

$$e(L(M_{B'}(\mu), M_{B'}(\lambda))) \le c(\mathfrak{g}) \dim V_{B'}(\mu) \dim V_{B'}(\lambda).$$

Let T denote the image of $\mathfrak{m}^- \oplus \mathbb{C}$ in $U(\mathfrak{m}^-)$. Set $L = L(M_{B'}(\mu), M_{B'}(\lambda))$, $M = M_{B'}(\mu)$, $N = M_{B'}(\lambda)$, $M^k = T^k \otimes V_{B'}(\mu)$, $N^k = T^k \otimes V_{B'}(\lambda)$, $L^k = \{a \in L: (\operatorname{ad}^k T)a = 0\}$, for all $k \in \mathbb{N}$. If $a \in L^k$, then $aM^k = 0$ implies $aM = 0$ (c.f. [8], 9.9, Eq. (14)). Let E be a finite dimensional generating subspace for L considered as a left $U(\mathfrak{g})$ module and let F denote the image of $\mathfrak{g} \oplus \mathbb{C}$ in $U(\mathfrak{g})/I_{B'}(\lambda)$. By the ad-nilpotence of $\mathfrak{m}^-$, we can assume E to be ad T stable without loss of generality and that there exist $r, s \in \mathbb{N}^+$ such that $F^kE \subset L^{kr+s}$, for all $k \in \mathbb{N}$. Choose $t \in \mathbb{N}$ such that $EM^0 \subset N^t$. Since E is ad $\mathfrak{m}^-$ stable, it follows by induction on $\ell \in \mathbb{N}$ that $EM^\ell \subset N^{\ell+t}$ and so $F^kEM^\ell \subset N^{k+\ell+t}$, for all $k, \ell \in \mathbb{N}$. In particular, $F^kEM^{kr+s} \subset N^{k+kr+s+t}$ and so by our first observation $\dim F^kE \le \dim M^{(k+1)r} \dim N^{k(r+1)+s+t}$. Setting $\dim \mathfrak{m}^- = m$, this gives

$$\dim F^k E \le \dim V_{B'}(\mu) \dim V_{B'}(\lambda)\left[\frac{k^m(r+1)^m}{(m!)}\right]^2 + 0(k^{2m-1}).$$

Taking 5.4 into account, the assertion of the lemma holds with $c(\mathfrak{g}) = (r+1)^{2m}\binom{2m}{m}$, which depends only on $\mathfrak{g}$ (if m, r refer to the case $\mathfrak{m}^- = \mathfrak{n}^-$).

6.5 Fix $r \in \mathbb{N}^+$, P a finite dimensional subspace $\neq 0$ of $\mathbb{Q}[x_1, x_2, \ldots, x_r]$ and Ω a Zariski dense subset of $\mathbb{N}^r$. Assign to the pair P, Ω the integer P^Ω defined as follows. Pick a basis $\{p_i\}_{i=1}^k$ for P with $p_i \in \mathbb{Z}[x_1, x_2, \ldots, x_r]$. Fix $s \in \mathbb{N}$, and let $P^{\lambda,s}$ be the smallest positive integer for which we can write for all $i = 1, 2, \ldots, k$,

$$p_i(\lambda) = \sum_{j=1}^{P^{\lambda,s}} z_{ij} q_j, \quad \text{with } q_j \in \mathbb{Z},\ z_{ij} \in \{-s, -s+1, \ldots, s\}.$$

Set

$$P^\Omega = \sup_{s \in \mathbb{N}} \sup_{\lambda \in \Omega} P^{\lambda,s}.$$

This is independent of the basis chosen. Obviously $P^\Omega \le \dim P$.

LEMMA: *For each $s \in \mathbb{N}^+$, there exists a Zariski dense subset $\Omega' \subset \Omega$ such that $P^{\lambda,s} = \dim P$, for all $\lambda \in \Omega'$. In particular $P^\Omega = \dim P$.*

Fix s. We can write Ω as a union of $(2s+1)(\dim P)^2$ subsets in each of which the z_{ij} are constant. At least one such subset must be Zariski dense and this clearly satisfies the conclusion of the lemma.

6.6 Given $m, n \in \mathbb{N}^+$, let $[m:n]$ denote the greatest common divisor of m, n. Set $\ell = \operatorname{rank} \mathfrak{g}$ and let $\alpha^1, \alpha^2, \ldots, \alpha^{2\ell}$ denote the fundamental weights for $\mathfrak{g} \times \mathfrak{g}$. These form a dual basis to $B \times B$ and in 6.5 we identify $\mathbb{Z}^{2\ell}$ with $P(R) \times P(R)$ and $\mathbb{N}^{2\ell}$ with $P(R)^+ \times P(R)^+$. Given $B' \subset B$, set $D_{B'} = \{w \in W\colon w^{-1}B' \subset R^+\}$. For all $\lambda \in P(R)^{++}$ one has $D_{B'}\lambda \subset P(R')^{++}$. For all $w \in D_{B'}$, define a polynomial $p_{w^{-1}B'}$ on $\mathfrak{h}^*$ through

$$p_{w^{-1}B'}(\lambda) = \prod_{\alpha \in R'^+} (w\lambda, \alpha)/(\sigma_{B'}, \alpha).$$

For all $\lambda \in P(R)^{++}$, one has $p_{w^{-1}B'}(\lambda) = \dim V_{B'}(w\lambda)$, [17], p. 257, Eq. (40). Set $d(\mathfrak{g}) = \Pi_{\alpha \in R^+}(\rho, \alpha)$ (which depends only on $\mathfrak{g}$).

LEMMA: *There exists a dense subset Ω of $P(R)^+ \times P(R)^+$ such that for all $B' \subset B$, $(\mu, \lambda) \in \Omega$ and all $\mu' \in D_{B'}\mu$, $\lambda' \in D_{B'}\lambda$ one has* $[\dim V_{B'}(\mu')\colon \dim V_{B'}(\lambda')] \le d(\mathfrak{g})$.

For each $\mu \in P(R)^{++}$ let p_μ denote the product of the $\dim V_{B'}(\mu')$: $B' \subset B$, $\mu' \in D_{B'}\mu$. Obviously $\Omega := \{(\mu, \lambda): \mu \in P(R)^{++}; \lambda \in \rho + p_\mu P(R)^{++}\}$ is a dense set. For $(\mu, \lambda) \in \Omega$, $w, w' \in D_{B'}$ we have $\dim V_{B'}(w\lambda) = \Pi_{\alpha \in R'^+}(w\rho, \alpha) \bmod \dim V_{B'}(w'\mu)$. Since $\{w^{-1}\alpha: \alpha \in R'^+\} \subset R^+$, the lemma follows.

6.7 Remark (added to revised version): By a recent result of Vogan (40, Thm. 1.1) $e(L(M_{B'}(\mu), M_{B'}(\lambda)))$ depends polynomially on $\mu, \lambda \in \mathfrak{h}^*$. Since $\dim V_{B'}(\mu)$, $\dim V_{B'}(\lambda)$ are also polynomials it follows from 6.3, 6.4 that $e(L(M_{B'}(\mu), M_{B'}(\lambda)))/\dim V_{B'}(\mu) \dim V_{B'}(\lambda)$ is a rational number independent of $\lambda, \mu \in \mathfrak{h}^*$.

7. Translation principles

7.1 Fix $\lambda, \mu \in \mathfrak{h}^*$ and M (resp. N) a subquotient of $M(\lambda)$ (resp. $M(\mu)$). By 4.3 (i), $L(M, N)$ considered as a U module, has finite length which we denote by $\ell(L(M, N))$. Here we combine the translation principles of Jantzen [19] and Zuckerman [34] to show that this is bounded by an integer depending only on $\mathfrak{g}$.

For each $\nu \in P(R)$, let $V(\nu)$ denote the simple, finite dimensional $U(\mathfrak{g})$ module with extreme weight ν. Consider $V(\nu) \otimes V(\nu')$: $\nu, \nu' \in P(R)$ as a U module through $(a \otimes b) \cdot (v \otimes w) = {}^t\check{a}v \otimes bw$, for all $a, b \in U(\mathfrak{g})$, $v \in V(\nu)$, $w \in V(\nu')$. In particular $V(0)$ is the trivial 1-dimensional $U(\mathfrak{g})$ module so we can also consider $V(\nu)$ as a U module through identification with $V(\nu) \otimes V(0)$. Consider $V(\nu) \otimes L(\mu)$ as a $U(\mathfrak{g})$ module through $X(v \otimes w) = Xv \otimes w + v \otimes Xw$, for all $X \in \mathfrak{g}$, $v \in V(\nu)$, $w \in L(\mu)$. Then $V(\nu) \otimes L(\mu)$ is a finite direct sum of $U(\mathfrak{g})$ modules admitting a central character [19], Satz 1 (i), which we call its primary decomposition. Let $\mathscr{P}_{(\mu+\nu)^\wedge}$ denote the projection defined by this decomposition onto the primary component with central character $(\mu + \nu)^\wedge$. The following result is due to Jantzen [19] (as noted explicitly in [5], 2.9).

Theorem: *Suppose μ and $\mu + \nu$ belong to the same facette of $\mathfrak{h}^*$ (see 3.3). Then $\mathscr{P}_{(\mu+\nu)^\wedge}(V(\nu) \otimes L(\mu)) = L(\mu \dot{+} \nu)$, up to a $U(\mathfrak{g})$ module isomorphism.*

7.2 Let $\mathscr{L}$ denote the category of finitely generated U modules admitting a formal character and for each $\Lambda \in \operatorname{Max} \operatorname{Cent} U$, let $\mathscr{L}_\Lambda$ denote the subcategory of $\mathscr{L}$ admitting the central character Λ. Each $L \in \mathscr{L}$ admits a primary decomposition and we let $\mathscr{P}_\Lambda$ denote the projection onto the primary component with central character Λ. Given E a finite dimensional U module and $L \in \mathscr{L}$, then $E \otimes L \in \mathscr{L}$. Moreover $\mathscr{P}_\Lambda$ and $E\otimes$ are exact functors (for details see [34], Sects. 1, 2).

For each $v \in V(\nu)$, $x \in L(M, L(\mu))$ define $f_{v,x} \in \mathrm{Hom}(M, V(\nu) \otimes L(\mu))$ through $f_{v,x}m = v \otimes xm$, for all $m \in M$. For the action of U defined in 3.2 we have $((a \otimes b) \cdot f_{v,x})m = {}^t\check{a} f_{v,x} \check{b} m = {}^t\check{a}(v \otimes x\check{b}m)$, for all $a, b \in U(\mathfrak{g})$. In particular, $j({}^t\check{X}) \cdot f_{v,x} = f_{Xv,x} + f_{v,(\mathrm{ad}\,X)x}$, for all $X \in \mathfrak{g}$ and so $f_{v,x}$ is $\mathfrak{k}$-finite. By 4.3, we have $L(M, L(\mu)) \in \mathscr{L}$ and with $V(\nu)$ considered as a U module, it follows that the map $v \otimes x \mapsto f_{v,x}$ extends linearly to a U module monomorphism of $V(\nu) \otimes L(M, L(\mu))$ into $L(M, V(\nu) \otimes L(\mu))$. Moreover taking account of the action of U we have $\mathscr{P}_{-\hat{\lambda},-(\mu+\nu)\hat{}}L(M, V(\nu) \otimes L(\mu)) = L(M, \mathscr{P}_{(\mu+\nu)\hat{}}(V(\nu) \otimes L(\mu))$. Define the exact functor $\varphi^\mu_{\mu+\nu}$ of $\mathscr{L}_{-\hat{\lambda},-\hat{\mu}}$ into $\mathscr{L}_{-\hat{\lambda},-(\mu+\nu)\hat{}}$ through $\varphi^\mu_{\mu+\nu}L = \mathscr{P}_{-\hat{\lambda},-(\mu+\nu)\hat{}}(V(\nu) \otimes L)$.

Proposition: *Suppose $\mu, \mu + \nu$ belong to the same facette of $\mathfrak{h}^*$. Then*

(i) $\varphi^\mu_{\mu+\nu}L(M, L(\mu)) = L(M, L(\mu + \nu))$, *up to isomorphism*

(ii) $\ell(L(M, L(\mu))) = \ell(L(M, L(\mu + \nu)))$.

(iii) *If E is a strict submodule of $L(M, L(\mu))$, then $\varphi^\mu_{\mu+\nu}E$ identifies through* (i) *with a strict submodule of $L(M, L(\mu + \nu))$.*

(iv) *For each $L \in \mathscr{L}_{-\hat{\lambda},-\hat{\mu}}$ and each* ad $\mathfrak{g}$ *submodule F of $U(\mathfrak{g})$ one has $\varphi^\mu_{\mu+\nu}(LF) = (\varphi^\mu_{\mu+\nu}L)F$.*

Through 7.1 and the above computation we may identify $\varphi^\mu_{\mu+\nu}L(M, L(\mu))$ with a submodule of $L(M, L(\mu + \nu))$. A similar assertion holds for μ replaced by $\mu + \nu$ and ν by $-\nu$. Again since $\mu, \mu + \nu$ belong to the same facette of $\mathfrak{h}^*$, it follows by [34], Thm. 1.2, that $\varphi^\mu_{\mu+\nu}$ and $\varphi^{\mu+\nu}_\mu$ are isomorphisms and mutual inverses. This gives (i)–(iii). (iv) is a trivial consequence of the fact that the second $U(\mathfrak{g})$ factor in U acts trivially in $V(\nu)$.

7.3 For each $w \in V(\nu')$, $y \in L(V(\nu') \otimes L(\lambda), N)$ define $g_{w,y} \in \mathrm{Hom}(L(\lambda), N)$ through $g_{w,y}n = y(w \otimes n)$: $n \in N$ and consider $V(\nu')$ as a U module through identification with $V(0) \otimes V(\nu')$. Then by 7.1 and [34], Thm. 1.2 we obtain as in 7.2 the following

Lemma: *Suppose $\lambda, \lambda + \nu$ belong to the same facette of $\mathfrak{h}^*$. Then $\ell(L(L(\lambda), N)) = \ell(L(L(\lambda + \nu), N))$.*

7.4 Corollary: *There exists a positive integer $e(\mathfrak{g})$ (depending* *only on* $\mathfrak{g}$) *such that for all* $\lambda, \mu \in \mathfrak{h}^*$ *and every subquotient* M (*resp.* N) *of* $M(\lambda)$ (*resp.* $M(\mu)$) *one has* $\ell(L(M, N)) \leq e(\mathfrak{g})$.

Let M' (resp. N') be a maximal proper submodule of M (resp. N) and set $L = L(M, N)$, $L' = \{x \in L: xM \subset N'\}$, $L'' = \{x \in L: xM' = 0\}$. Up to isomorphism, we have the $U(\mathfrak{g})$ module inclusions: $L' \subset L(M, N')$, $L/L' \subset L(M, N/N')$, $L'' \subset L(M/M', N)$, $L/L'' \subset L(M', N)$. Taking [5], 3.16 into account, it follows that we can assume M, N simple without loss of generality. Then by [10], 7.6.1 (ii), and 7.2 and 7.3 it suffices to prove the assertion with $M = L(\lambda)$, $N = L(\mu)$ where (λ, λ), $(\mu, \mu) \leq (\rho, \rho)$. The latter follows from the remarks following 4.2, 4.3.

7.5 (Notation 3.3). The above translation principle gives the following generalization of [23], 5.2.

THEOREM: *Choose* $-\lambda, -\mu \in \mathfrak{h}^*$ *dominant and regular with* $\lambda - \mu \in P(R)$ (*which implies that* $W_\lambda = W_\mu$). *Then for all* $w \in W_\lambda$ *one has*

$$\text{Ann } V(-\mu, -w\lambda) = \check{I}(w_\lambda w^{-1}\mu) \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \check{I}(w_\lambda w\lambda).$$

Since $-\lambda, -\mu$ are dominant we have $M(\lambda) = L(\lambda)$, $M(\mu) = L(\mu)$ and then by [9], 5.3, 5.5 we have $L(-\mu, -w\lambda) = L(M(w\lambda), L(\mu))$, up to isomorphism. Since $V(-\mu, -w\lambda)$ is the unique simple quotient of $L(-\mu, -w\lambda)$ it follows that RAnn $V(-\mu, -w\lambda)$ is just the largest two-sided ideal $U(\mathfrak{g})$ such that $L(M(w\lambda), L(\mu))I \subsetneq L(M(w\lambda), L(\mu))$. Now λ, μ lie in the same facette of $\mathfrak{h}^*$, so taking $\nu = \pm(\lambda - \mu)$ in 7.2, we find that I is just the largest two-sided ideal of $U(\mathfrak{g})$ such that $L(M(w\lambda), L(\lambda))I \subsetneq L(M(w\lambda), L(\lambda))$. That is RAnn $V(-\mu, -w\lambda)$ = RAnn $V(-\lambda, -w\lambda)$ = $I(w_\lambda w\lambda)$, by [23], 5.2. Finally LAnn $V(-\mu, -w\lambda)$ = ^{t}(RAnn $V(-w\lambda, -\mu)$) = ^{t}(RAnn $V(-\lambda, -w^{-1}\mu)$) = ${}^tI(w_\lambda w^{-1}\mu) = I(w_\lambda w^{-1}\mu)$, by say [23], 1.4, 3.1 (i), 3.9 (ii), which proves the theorem.

7.6 The following is a partial answer to a question posed by Borho.

COROLLARY: *Suppose that* $\sqrt{\text{gr } I} \in \text{Spec } S(\mathfrak{g})$ *for all* $I \in \text{Prim } U(\mathfrak{g})$ (c.f. 1.2). *Then for all* $\lambda \in \mathfrak{h}^*$ *regular and all* $\nu \in P(R)$ *for which* λ, $\lambda + \nu$ *lie in the same facette of* $\mathfrak{h}^*$, *one has* $\sqrt{\text{gr } I(\lambda)} = \sqrt{\text{gr } I(\lambda + \nu)}$.

Apply 2.4 (ii) to 7.5.

7.7 Let Σ denote the set of involutions of W.

COROLLARY: ($\mathfrak{g}$ *simple of type* A_{n-1}). *For all* $\lambda, \mu \in P(R)^{++}$, *the* *following two statements are equivalent*

(i) card $\mathscr{X}_{\hat{\lambda}} = \text{card } \Sigma$.

(ii) card$\{\text{Ann } V(w\lambda, w'\mu): w, w' \in W\} = \text{card } W$.

This follows from 7.5 as in the proof of [23], 6.6.

8. The dimension polynomials

8.1 (Notation 6.6). Call $B'' \subset R^+$ a *subbasis* of R if there exists $w \in W$ such that $wB'' \subset B$. Given $B' \subset B$, set $\mathscr{D}_{B'} = \{w^{-1}B': w \in D_{B'}\}$, which is just the set of all subbases of R conjugate to B', and $\mathscr{B}' = \{B'' \in \mathscr{D}_{B'}: B'' \subset B\}$. Let $P_{\mathscr{B}'}$ be the $\mathbf{Q}$ vector space spanned by the $p_{B''}$: $B'' \in \mathscr{D}_{B'}$.

8.2 Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. After Richardson [30] $\mathfrak{m}_{B'}$ admits a unique dense orbit for the action of the subgroup of G with Lie algebra $\mathfrak{p}_{B'}$. Furthermore [2], 3.3 if $X_{B'}$ lies in this orbit, then $GX_{B'}$ is a nilpotent orbit in $\mathfrak{g}^*$ which does not depend on the choice of $B' \in \mathscr{B}'$. It is called the Richardson orbit $\mathcal{O}_{\mathscr{B}'}$ associated with $\mathscr{B}'$.

CONJECTURE: (Notation 1.2). *For all $\lambda \in P(R)^{++}$ and all $B' \subset B$ one has* card$\{\pi^{-1}(\hat{\lambda}) \cap \mathscr{K}^{-1}(\mathcal{O}_{\mathscr{B}'})\} = \dim P_{\mathscr{B}'}$.

Through [5], this holds for types A_1–A_4, B_2, G_2. From unpublished results of Borho and Jantzen it holds for types B_3, C_3, D_4 (excepting possibly if B' is of type $A_1 \times A_1$ or A_3).

8.3 In the next three subsections we shall assume that $\mathfrak{g}$ is simple of type A_{n-1}. Let $P(n)$ denote the set of partitions of n. Given $\xi \in P(n)$, let ξ^* denote its conjugate partition. A subbasis $B'' \subset R^+$ is said to be of type ξ if $ZB'' \cap R$ is a system of type $A_{\xi_1^*-1} \times A_{\xi_2^*-1} \times \cdots \times A_{\xi_s^*-1}$: $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_s^*)$. The following is well-known

LEMMA: *Given $B_1, B_2 \subset R^+$ subbases of type ξ_1, ξ_2 respectively. Then there exists $w \in W$ such that $wB_1 = B_2$ iff $\xi_1 = \xi_2$.*

As noted in the proof of [2], 3.5 c), we have $\mathcal{O}_{\mathscr{B}_1} = \mathcal{O}_{\mathscr{B}_2}$ iff $\mathscr{B}_1 = \mathscr{B}_2$. This and the lemma sets up a bijection between $P(n)$ and the set of Richardson orbits. Given B' of type ξ, we write $\mathcal{O}_\xi$ for $\mathcal{O}_{\mathscr{B}'}$. (Of course it is well-known (c.f. [11], 1.1) that in type A_{n-1} every nilpotent orbit is a Richardson orbit and that the set of all nilpotent orbits is in bijection with $P(n)$ through Jordan canonical form.)

8.4 Let $\xi = (\xi_1, \xi_2, \ldots, \xi_t)$ be a partition of n and $\mathrm{St}(\xi)$ the set of standard tableaux of type ξ. We recall that each $T \in \mathrm{St}(\xi)$ is an array of pairwise distinct entries $t_{ij} \in \{1, 2, \ldots, n\}$: $1 \leq i \leq \xi_j^*$, $1 \leq j \leq \xi_i$ with $t_{ij} < t_{ik}$ if $j < k$ and $t_{ij} < t_{kj}$ if $i < k$. Let $\mathrm{Rt}(\xi)$ denote the set of tableaux satisfying the above requirements with the exception that the t_{ij} need not increase along the rows. For each $T \in \mathrm{Rt}(\xi)$, we define a polynomial $p_T \in \mathbb{Q}[x_1, x_2, \ldots, x_{n-1}]$, through

$$p_T = \prod_{r=1}^{n} \prod_{j=i+1}^{n} \prod_{i=1}^{n} (y_{t_{jr}} - y_{t_{ir}}),$$

where $y_1 = 0$, $y_k = x_1 + x_2 + \cdots + x_{k-1}$, $k = 2, 3, \ldots, n$. Let P_ξ denote the $\mathbb{Q}$ vector space spanned by the p_T: $T \in \mathrm{Rt}(\xi)$.

PROPOSITION: *The set* $\{p_T\colon T \in \mathrm{St}(\xi)\}$ *form a basis for* P_ξ *and the sum* $\Sigma\{P_\xi\colon \xi \in P(n)\}$, *is direct. In particular* $\dim P_\xi = \operatorname{card} \mathrm{St}(\xi)$ *and* $\dim \bigoplus_{\xi \in P(n)} P_\xi = \Sigma_{\xi \in P(n)} \operatorname{card} \mathrm{St}(\xi) = \operatorname{card} \Sigma$.

Specht [33] proved the linear independence of the polynomials which derive from the standard tableaux. Garnir, [14], Thm. III, showed that the remaining polynomials belong to the $\mathbb{Z}$ module generated by the former set. (See also [28], Chap. 0, Sect. 5).

8.5 Fix $\xi \in P(n)$ and a subbasis $B' \subset B$ of type ξ. Set

$$d_\xi = \prod_{\alpha \in R'^+} (\sigma_{B'}, \alpha).$$

For $i \in \{1, 2, \ldots, n-1\}$, let x_i denote the polynomial on $\mathfrak{h}^*$ defined through $x_i(\lambda) = (\alpha_i, \lambda)$. Set $\beta_1 = 0$, $\beta_k = \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}$: $k = 2, 3, \ldots, n$. For each $T \in \mathrm{Rt}(\xi)$, we define $B_T \subset R^+$ through

$$B_T = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (\beta_{t_{j+1,i}} - \beta_{t_{ji}}).$$

LEMMA: *The map* $T \mapsto B_T$ *is a surjection of* $\mathrm{Rt}(\xi)$ *onto the set of subbases of* R *of type* ξ. *Furthermore* $p_{B_T} = d_\xi^{-1} p_T$.

In type A_{n-1}, a subset $B'' \subset R^+$ is a subbasis iff for each pair $\gamma, \delta \in B''$ one has $\gamma - \delta \notin R$. Now given $\gamma, \delta \in R^+$, we can write $\gamma = \alpha_r + \alpha_{r+1} + \cdots \alpha_{s-1} = \beta_s - \beta_r$, $\delta = \alpha_k + \alpha_{k+1} + \cdots + \alpha_{\ell-1} = \beta_\ell - \beta_k$, for some k, ℓ, r, $s \in \{1, 2, \ldots, n\}$ satisfying $s > r$, $\ell > k$. Then $\gamma -$

$\delta \notin R$, iff $r \neq k$ and $s \neq \ell$. Since the entries of T are pairwise distinct, it follows that B_T is a subbasis. Again $\gamma + \delta \in R$, iff either $\ell = r$ or $s = k$. Thus to the i^{th} column of T there corresponds a basis namely $\bigcup_{j=1}^{n} (\beta_{t_{j+1,i}} - \beta_{t_{ji}})$ for a system R_i of type $A_{\xi_i^*-1}$. Given $\gamma \in R_i^+$ (resp. $\delta \in R_j^+$) then r, s (resp. k, ℓ) lie in the i^{th} (resp. j^{th}) column of T. If $i \neq j$, then since the entries of T are pairwise distinct, so are the r, s, k, ℓ. Thus $\gamma - \delta$, $\gamma + \delta$ are not roots and it follows that B_T is a subbasis of type ξ. Conversely every subbasis of type A_t defines a strictly increasing sequence from $\{1, 2, \ldots, n\}$ of length $t + 1$, which we can take to define a column of T. From our previous remarks it is then easy to see that every subbasis of type ξ is of the form B_T: $T \in \text{Rt}(\xi)$. The last part of the lemma is clear.

8.6 Return to the general case. Since $\mathbb{C}p_{B'}$ is a $W_{B'}$ module and $D_{B'}$ identifies naturally with $W/W_{B'}$ it follows that $P_{\mathscr{B}'}$ is a W submodule of $S(\mathfrak{h})_k$ (notation 2.1) with $k = \frac{1}{2}(\text{card}\, R - \dim \mathcal{O}_{\mathscr{B}'})$. After MacDonald [37], $P_{\mathscr{B}'}$ is a simple W module and his argument further shows that if M is a W submodule of $S(\mathfrak{h})_\ell$ isomorphic to $P_{\mathscr{B}'}$, then $\ell \geq k$ and equality implies that $M = P_{\mathscr{B}'}$. Then by a result of Lusztig (c.f. [35], Prop. 1.4) one has $\mathcal{O}_{\mathscr{B}'} = \mathcal{O}_{\mathscr{B}''}$ iff $P_{\mathscr{B}'} = P_{\mathscr{B}''}$. (Strictly speaking the said result is claimed only when the base field has characteristic $p > 0$. Yet it is well-known that whether or not two given nilpotent elements lie in the same G-orbit is independent of p for p sufficiently large and coincides with the result for $p = 0$.)

9. A problem of Borho

9.1 Conjecture: ([1], 3.3). *For all* $B' \subset B$, $\lambda \in P(R')^{++}$ *one has* $\sqrt{\text{gr}\, I_{B'}(\lambda)} \in \text{Spec}\, S(\mathfrak{g})$.

If this holds, then by [1], 2.3 one has $\mathscr{V}(\text{gr}\, I_{B'}(\lambda)) = \bar{\mathcal{O}}_{\mathscr{B}'}$.

9.2 Assume $\mathfrak{g}$ simple of type A_{n-1} and recall that W is then isomorphic to the symmetric group S_n on n elements. After Robinson (c.f. [21], Sect. 5) there exists a bijection Φ of W onto $\cup\{\text{St}(\xi) \times \text{St}(\xi): \xi \in P(n)\}$. For each $w \in W$, we write $\Phi(w) = (A(w), B(w))$ and let $\xi(w)$ denote the partition of n defined by the cardinalities of the rows of $A(w)$ (or $B(w)$).

Lemma: *Suppose* 9.1 *holds. Then for all* $-\lambda, -\mu \in P(R)^{++}$,

(i) $\mathscr{V}(\text{gr}\, I(w\lambda)) = \bar{\mathcal{O}}_{\xi(w)}$.

(ii) $\sqrt{\text{gr LAnn}\, V(-w\lambda, -\mu)} = \sqrt{\text{gr RAnn}\, V(-w\lambda, -\mu)}$.

(i). Set $\xi = \xi(w)$. By [25], 4.1 we have $d(U(\mathfrak{g})/I(w\lambda)) =$

card $R - \Sigma\, \xi_i^*(\xi_i^* - 1)$. Set $m_1 = 0$ and

$$m_i = \sum_{j=1}^{i-1} \xi_j^*: i = 2, 3, \ldots, t.$$

Define $T \in \mathrm{St}(\xi)$ by taking $t_{ij} = i + m_j$ and set $w' = \Phi^{-1}(A(w), T)$. By [22], 9.6 (i), $B' := B \cap -w'R^+$ is a subbasis of type ξ, and so by [1], 2.3 b), we have $d(U(\mathfrak{g})/I_{B'}(w'\lambda)) = 2 \dim \mathfrak{m}_{B'} = \mathrm{card}\, R - \Sigma\, \xi_i^*(\xi_i^* - 1)$. By [21], 4.2, 5.1, we have $I(w\lambda) = I(w'\lambda) \supset I_{B'}(w'\lambda)$ and so $\sqrt{\mathrm{gr}\, I(w\lambda)} \supset \sqrt{\mathrm{gr}\, I_{B'}(w'\lambda)}$. Combined with the above dimensionality estimates and 9.1, this gives (i). By [23], 6.4, $\xi(w_\lambda w) = \xi(w_\lambda w^{-1})$ and combined with (i) and 7.5, this gives (ii).

9.3 The proof of 9.2 (i) uses the fact that the $I(w\lambda)$: $w \in W_\lambda$ are not all distinct and taking $B' := B \cap -wR^+$ we can assume without loss of generality that $d(U(\mathfrak{g})/I_{B'}(w\lambda)) = d(U(\mathfrak{g})/I(w\lambda))$. This clearly further implies that $I(w\lambda)$ is one of the minimal prime ideals containing $I_{B'}(w\lambda)$ and so it ought to be possible to classify Prim $U(\mathfrak{g})$ (for $\mathfrak{g}$ simple of type A_{n-1}) from just the study of induced ideals. This idea is also suggested by the work of Borho-Jantzen (see for example, [5], 4.5 d)) and was the main motivation for the present paper. In principle it further extends to algebras other than type A_{n-1} as is indicated by the second part of [21], Conjecture 4.3 (which holds for algebras up to rank 3 [25], 5.2). This fails in type D_4 (when the (*) condition of [21] is also not satisfied) and is a phenomenon related to the appearance [5], 4.5 e), of non-polarizable orbits in the integral fibres $\mathscr{X}_{\hat{\lambda}}$: $\lambda \in P(R)$ (in the sense of the $\mathscr{X}$ map). Two further consequences of the above condition are noted below.

LEMMA: *Take $B' \subset B$, $\lambda \in P(R')^{++}$ and suppose that $d(U(\mathfrak{g})/I_{B'}(\lambda)) = d(U(\mathfrak{g})/I(\lambda))$. Then*

(i) *$M_{B'}(\lambda)$ is a simple $L(M_{B'}(\lambda), M_{B'}(\lambda))$ module.*

(ii) *Let I be a minimal prime ideal containing $I_{B'}(\lambda)$. Then $d(U(\mathfrak{g})/I) = d(U(\mathfrak{g})/I_{B'}(\lambda))$. Furthermore if 9.1 holds, then $\mathscr{V}(\mathrm{gr}\, I) = \bar{\mathcal{O}}_{\mathscr{B}'}$.*

Set $m = \dim \mathfrak{m}_{B'}$. Suppose (i) is false. Then $L(\lambda)$ is a quotient of $M_{B'}(\lambda)/N_{B'}(\lambda)$ (notation 4.7). Then by 4.7 (iii), [25], 2.8, we have $d(U(\mathfrak{g})/I(\lambda)) = 2d(L(\lambda)) \leq 2d(M_{B'}(\lambda)/N_{B'}(\lambda)) < 2m = d(U(\mathfrak{g})/I_{B'}(\lambda))$, in contradiction to our hypothesis. Hence (i).

(ii) Let $M_{B'}(\lambda) = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{t+1} = 0$, be a composition series for $M_{B'}(\lambda)$ with $L_i := M_i/M_{i+1}$ simple. Set $I_i = \mathrm{Ann}\, L_i$. Then $I_i \in \mathrm{Spec}\, U(\mathfrak{g})$, $2d(L_i) = d(U(\mathfrak{g})/I_i)$ and every minimal prime ideal containing $I_{B'}(\lambda)$ is one of the I_i (though not all are minimal). We suppose further that the composition series is chosen to minimize $\Sigma\{i: d(L_i) < m\}$. Now $d(L_1) = d(L(\lambda)) = m$, by [25], 2.8 and the hypothesis. Thus given $d(L_j) < m$, we let i be the largest positive integer $< j$ such that $d(L_i) = m$ and k the largest positive integer $\geq j$ such that $d(L_r) < m$, for all $r \in \{j, j+1, \ldots, k\}$. By choice of the composition series it follows that M_i/M_{k+1} admits a unique simple quotient L_i and $d(U(\mathfrak{g})/I_i) = 2d(L_i) = 2m > 2d(M_{i+1}/M_{k+1}) = d(U(\mathfrak{g})/\text{Ann } M_{i+1}/M_{k+1})$. It follows from [25], 3.7 that $I_i = \text{Ann } M_i/M_{k+1}$ and in particular that $I_i \subset I_r$: $r \in \{i+1, \ldots, k\}$. This proves (ii).

10. Main theorem and the Jantzen conjecture

10.1 For $\lambda, \mu \in P(R)^{++}$, set $\Lambda = (\hat{\lambda}, \hat{\mu})$ and $\mathscr{H}(\Lambda) = \{V(w\lambda, \mu): w \in W\}$, which by [12], 4.5 contains up to isomorphism every simple $\mathfrak{k}$ finite U module admitting a formal character and with central character Λ. For each $m \in \mathbb{N}$, set $\mathscr{H}_m(\Lambda) = \{V \in \mathscr{H}(\Lambda): d(V) = 2m\}$, $\mathscr{E}_m(\Lambda) = \text{card}\{e(V): V \in \mathscr{H}_m(\Lambda)\}$, $\mathscr{E}_m = \sup\{\mathscr{E}_m(\Lambda): \Lambda \in P(R)^{++} \times P(R)^{++}\}$ and let P_m denote the rational vector space spanned by the polynomials $\{p \otimes q: p, q \in P_{\mathscr{B}'}: \dim \mathfrak{m}_{B'} = m\}$. By 8.6, $\dim P_m = \Sigma\{(\dim P_{\mathscr{B}'})^2: \dim \mathcal{O}_{\mathscr{B}'} = m\}$ (where the sum is over distinct Richardson orbits).

THEOREM: *For each $m \in \mathbb{N}$, one has $\mathscr{E}_m \geq \dim P_m$.*

By 6.3, 6.4, 6.6, there exists a dense subset $\Omega \subset P(R)^{++} \times P(R)^{++}$ such that for each $(\lambda, \mu) \in \Omega$

$$e(L(M_{B'}(w_1\lambda), M_{B'}(w_2\mu)) = (u(B', w_1)/v(B', w_2))p_{w_1^{-1}B'}(\lambda)p_{w_2^{-1}B'}(\mu),$$

for all $B' \subset B$; $w_1, w_2 \in D_{B'}$, where u, v are positive integers $\leq c(\mathfrak{g})d(\mathfrak{g})$. By 2.2 and 4.3 (i) the left hand side is a non-negative integer linear combination of the $\{e(V): V \in \mathscr{H}_m(\Lambda)$ with $m = \dim \mathfrak{m}_{B'}\}$. By 7.4 the coefficients are bounded above by $e(\mathfrak{g})$. Set $f = c(\mathfrak{g})d(\mathfrak{g})e(\mathfrak{g})$ and take a basis $\{p_i\}$ for P_m formed from the $\{f!(p_{w_1^{-1}B'} \otimes p_{w_2^{-1}B'})\}$. Then we can write for all i,

$$p_i(\lambda, \mu) = \sum z_{ij}e(V_j): V_j \in \mathscr{H}_m(\Lambda),$$

where the z_{ij} are non-negative integers $\leq (f+2)!$ By 6.5 this gives the required assertion.

REMARK 1: By the remark following Lemma 6.5, there exists a Zariski dense subset Ω' of $P(R)^{++} \times P(R)^{++}$ such that $\mathscr{E}_m(\Lambda) \geq \dim P_m$, for all $\Lambda \in \Omega'$. By 8.6, $\dim \Sigma_{m \in \mathbb{N}} P_m \leq \Sigma_{\sigma \in \hat{W}} (\dim \sigma)^2 = \text{card } W$, as expected.

REMARK 2: Suppose $\mathfrak{g}$ is simple of type A_{n-1}. Then by 8.4, $\dim \Sigma_{m \in \mathbb{N}} P_m = \Sigma_{\xi \in P(n)}$ (card $\text{St}(\xi))^2 = \text{card } W = \text{card } \mathscr{H}(\Lambda)$, for all $\Lambda \in P(R)^{++} \times P(R)^{++}$. It follows that equality holds in the conclusion of the theorem and that $\mathscr{E}_m(.)$ is locally constant on $P(R)^{++} \times P(R)^{++}$.

10.2 For all $V \in \mathscr{H}(\Lambda)$, we have by 2.4 (i) that $d(V) = \frac{1}{2} d(U/\text{Ann } V)$. Thus if $d(V) \neq d(V')$, then Ann $V \neq$ Ann V'. A rather finer question is contained in the following

CONJECTURE: $e(V) \neq e(V')$ *implies* Ann $V \neq$ Ann V'.

10.3 Take $\mathfrak{g}$ simple of type A_{n-1} and adopt the notation of 9.2.

COROLLARY: *Suppose that* 10.2 *holds. Then for all* $-\lambda \in P(R)^{++}$ *one has* $I(w\lambda) = I(w'\lambda)$ *iff* $A(w) = A(w')$.

Sufficiency follows from [22], 5.1 and 7.9. Then for necessity it suffices to show that card $\mathscr{X}_{\hat{\lambda}} \geq \Sigma_{\xi \in P(n)} \text{card St}(\xi) = \text{card } \Sigma$. Given 10.2, this follows from 7.7, 10.1 and Remark 2 above.

10.4 From the classical theory of the symmetric group, one may identify $P(n)$ with $\hat{S}_n$ so that for all $\xi \in \hat{S}_n$ one has $\dim \xi = \text{card St}(\xi)$. Comparison with [3], 5.9 and applying 9.2 (i) and 10.3 we obtain the

COROLLARY: *To establish the Jantzen conjecture, it suffices to establish* 9.1 *and* 10.2.

10.5 The analogue of 10.2 fails for simple subquotients of Verma modules. For example, take $\mathfrak{g}$ simple of type C_2. Set $B = \{\alpha_1, \alpha_2\}$ with α_1 the short root and $B' = \{\alpha_1\}$. Then $M_{B'}(\alpha_1) = L(\alpha_1)$, $M_{B'}(\alpha_1 + \alpha_2) = L(\alpha_1 + \alpha_2)$ so $e(L(\alpha_1)) = \dim V_{B'}(\alpha_1) = 2$ and $e(L(\alpha_1 + \alpha_2)) = \dim V_{B'}(\alpha_1 + \alpha_2) = 1$, by 3.1 (ii). Yet Ann $L(\alpha_1) =$ Ann $L(\alpha_1 + \alpha_2)$, by [5], 2.20. A similar reasoning using the multiplicity results of Jantzen [20] shows that a corresponding result holds for regular central characters. This bad phenomenon is linked to the failure [9], 6.5, of a question of Kostant: Is $L(M, M) = U(\mathfrak{g})/\text{Ann } M$ for every simple $U(\mathfrak{g})$ module M? In fact by [8], 3.1, $I_{B'}(\alpha_1 + \alpha_1)$ is completely prime. Yet $I_{B'}(\alpha_1) = I_{B'}(\alpha_1 + \alpha_2)$ and so by 5.10 (iii) the embedding $U(\mathfrak{g})/I_{B'}(\alpha_1) \hookrightarrow L(M_{B'}(\alpha_1), M_{B'}(\alpha_1))$ is strict. An indication that the principal series subquotients are better behaved comes from the following. Suppose $V = J/I: J \supsetneq I: I \in \text{Prim}\, U(\mathfrak{g})$. Then $\text{LAnn}\, V = \text{RAnn}\, V = I$ and by 2.2, 2.3 and [6], 3.5 we have $e(V) = e(J/I) = e(U(\mathfrak{g})/I) = \sqrt{e(U/\text{Ann}\, V)}$. Thus simple quotients of the above form satisfy 10.2. Finally we fix a positive integer $h(\mathfrak{g})$ depending only on $\mathfrak{g}$ and then given $V, V' \in \mathcal{H}_m(\Lambda)$ we say that $e(V)$ and $e(V')$ are *commensurable* (relative to $h(\mathfrak{g})$) if one has $e(V) = (u/v)e(V')$ with $u, v \in \{1, 2, \ldots, h(\mathfrak{g})\}$. It is clear that to obtain 10.4 it is enough to show that $\text{Ann}\, V = \text{Ann}\, V'$ implies the commensurability of $e(V)$ and $e(V')$. A refinement of [25], 2.5 and 2.7 (along the lines of 6.4) reduces this question to showing that $(e(V))^2$ divides a $\mathfrak{g}$ fixed multiple of $e(U/\text{Ann}\, V)$. (This has the advantage of being much more weakly dependent on the filtration, changes of which are absorbed by the $\mathfrak{g}$ fixed multiple). A similar remark applies to the simple quotient $L(w\lambda)$ of the Verma module $M(w\lambda)$. For this recall ([13], Lemma 6) that there is a unique minimal ideal $\overline{I(w\lambda)}$ of $U(\mathfrak{g})$ containing $I(w\lambda)$ and $U(\mathfrak{g})/I(w\lambda)$ (and hence $V_w := \overline{I(w\lambda)}/I(w\lambda)$) identifies (c.f. [13], Prop. 10) with a submodule of $L(-w\lambda, -w\lambda)$. Let $V_w^{\perp}$ denote the orthogonal of V_w in $M(w\lambda) \otimes M(w\lambda)$. By say, [23], 5.4 (ii) we have $U(\mathfrak{g})/I(w\lambda) \subset (M(w\lambda) \otimes M(w\lambda))^{\perp}$ and so $(M(w\lambda) \otimes M(w\lambda))/V_w^{\perp} = L(w\lambda) \otimes L(w\lambda)$. Now $e(V_w) = e(U(\mathfrak{g})/I(w\lambda))$, by [6], 3.6, so the question is whether $e(V_w)$ and $e(L(w\lambda) \otimes L(w\lambda))$ are commensurable. This relates to Kostant's problem since the $\mathfrak{k}$ finite part of $(M(w\lambda) \otimes M(w\lambda))^{\perp}$ identifies with the $\mathfrak{k}$ finite part of $(L(w\lambda) \otimes L(w\lambda))^*$ which by the argument of [9], 5.5 identifies with $L(L(w\lambda), L(w\lambda))$.

11. Goldie rank

11.1 Let A be a Noetherian ring (not necessarily prime). We define the Goldie rank $\text{rk}\, A$ through $\text{rk}\, A = \sup\{k \in \mathbb{N}^+ : x^k = 0,\ x^{k-1} \neq 0 : x \in A\}$. (This is one of the many possible definitions of Goldie rank which coincide for prime rings). The origin of the space $P_{\mathcal{B}'}$, 7.6, 8.2 and 8.6 suggests the following

Conjecture: *Take* $-\lambda \in P(R)^+$ *and* $w \in W$ *such that* $I(w\lambda) \in \mathcal{K}^{-1}(\mathcal{O}_{\mathcal{B}'})$. *Then there exists* $p_w \in P_{\mathcal{B}'}$ *such that* $\text{rk}(U(\mathfrak{g})/I(w(\lambda + \nu)) = p_w(w(\lambda + \nu))$, *for all* $-\nu \in P(R)^+$ *and these polynomials form a basis for* $P_{\mathcal{B}'}$.

Remarks: By [9], 8.6 one has $\text{rk}(U(\mathfrak{g})/I(w_{B'}(\lambda))) = p_{B'}(w_{B'}\lambda)$, for all $-\lambda \in P(R)^{++}$. Recalling [5], 2.14, one expects p_w to be divisable by $p_{B'}$ with $B' = B \cap w^{-1}R^-$.

This motivates an additivity principle for Goldie rank analogous to 2.2. A first step in this direction is indicated below.

11.2 Let M be an A module and N a submodule of M. Set $J = \text{Ann } M/N$, $I = \text{Ann } N$. An elementary computation gives

LEMMA:
(i) $I \cap J \supset \text{Ann } M \supset IJ$,
(ii) $\text{rk}(A/\text{Ann } M) \leq \text{rk}(A/I) + \text{rk}(A/J)$.

11.3 PROPOSITION: (*Notation* 11.2). *Suppose that N is the unique proper non-zero submodule of M (which is hence of length* 2) *and that neither $I \supset J$ nor $J \supset I$. Then* $\text{rk}(A/\text{Ann } M) = \text{rk}(A/I) + \text{rk}(A/J)$.

Set $r = \text{rk}(A/I)$, $s = \text{rk}(A/J)$. By the hypothesis, $(J/(I \cap J)$ identifies with a non-zero two-sided ideal of the prime Noetherian ring A/I and so considered as a subring we have Fract A/I = Fract $J/(I \cap J)$. Then by the Faith-Utumi lemma [16], Thm. 4.6, there exists $x_1 \in J$ such that $x_1^r \in I$, $x_1^{r-1} \notin I$. Interchanging I, J gives $x_2 \in I$ such that $x_2^s \in J$, $x_2^{s-1} \notin J$. Set $x = x_1 + x_2$. Then the hypotheses of the lemma give $x^{r-1}(I \cap J)x^{s-1}M = x^{r-1}(I \cap J)M \supset x^{r-1}JIM = x^{r-1}JM = x^{r-1}N \neq 0$. Choose $y \in I \cap J$ such that $x^{r-1}yx^{s-1} \notin \text{Ann } M$. Then $(x+y)^{r+s} \in \text{Ann}$ M, yet $(x+y)^{r+s-1} = x^{r-1}yx^{s-1} \text{ mod Ann } M \notin \text{Ann}|M$ and this establishes the opposite inequality to 11.2 (ii).

REMARK: The assertion also holds if $I = J \supsetneq \text{Ann } M$.

11.4 Take $\lambda \in \mathfrak{h}^*$ and let M be a subquotient of $M(\lambda)$ of length 2. We write $M = M_1 \supsetneq M_2 \supsetneq M_3 = 0$ and set $I_i = \text{Ann } M_i/M_{i+1}$: $i = 1, 2$.

COROLLARY: *There exist* $z_1, z_2 \in \{0, 1\}$ *such that* $\text{rk}(U(\mathfrak{g})/\text{Ann } M) = z_1 \text{rk}(U(\mathfrak{g})/I_1) + z_2 \text{rk}(U(\mathfrak{g})/I_2)$.

This follows from 11.3 and [25], 3.7 by listing all possibilities.

REMARK: We do not know if this holds for any Noetherian ring A. (One of the bad cases is when $I_2 \supsetneq I_1$ and yet $I_1 \neq \text{Ann } M$).

11.5 The next problem is to compute $\text{rk}(U(\mathfrak{g})/I_{B'}(\lambda))$. By [8], 3.3 we have $\text{rk}(U(\mathfrak{g})/I_{B'}(\lambda)) \leq \dim V_{B'}(\lambda)$. Unfortunately as noted in 10.5 equality generally fails. Inspection of the given example shows that this failure is related to the presence of coefficients >1 in the B expansion of a root and so equality might still hold in type A_{n-1}. An indication of this obtains from the following lemma. First we fix $B' \subset B$ and $\lambda \in P(R')^{++}$, setting $M = M_{B'}(\lambda)$, $A = L(M_{B'}(\lambda), M_{B'}(\lambda))$. Let $\mathscr{A}(\mathfrak{m}^-)$ denote the algebra of differential operators on $S(\mathfrak{m}^-)$ with coefficients in $S(\mathfrak{m}^-)$. Since ad $\mathfrak{m}^-$ is locally nilpotent in A, it follows that (c.f. [8] 5.6, 10.3 (i)) we may write $M = S(\mathfrak{m}^-) \otimes V_{B'}(\lambda)$ with A identified as a subalgebra of $\mathscr{A}(\mathfrak{m}^-) \otimes \mathrm{End}\, V_{B'}(\lambda)$. Set $Z = Z(\mathfrak{m}^-)\setminus\{0\}$. Then (c.f. 5.9) Z is an Ore subset for A and for $U(\mathfrak{g})/I_{B'}(\lambda)$, and the map $m \mapsto 1 \otimes m$ of M into $Z^{-1}M := Z^{-1}A \otimes_A M$ is injective. (We remark that $Z(\mathfrak{m}^-)$ identifies with a subalgebra of $S(\mathfrak{m}^-) \otimes 1$). Define $\mathfrak{c}(\mathfrak{m}^-)$ as in [24], 2.6. Then (c.f. 5.9) $U(\mathfrak{c})$ identifies with a subalgebra of A. Set $D = U(\mathfrak{g})/I_{B'}(\lambda)$.

LEMMA: ($\mathfrak{g}$ *simple of type* A_{n+1}). *Suppose* $\mathfrak{m}^-$ *is commutative and set* $K = \mathrm{Fract}\, S(\mathfrak{m}^-)$. *Then up to isomorphism*

(i) $Z^{-1}M = K \otimes V_{B'}(\lambda)$.

(ii) $Z^{-1}A = Z^{-1}U(\mathfrak{c}) \otimes \mathrm{End}\, V_{B'}(\lambda)$, *where* $Z^{-1}U(\mathfrak{c})$ *identifies with* $K\mathscr{A}(\mathfrak{m}^-)$.

(iii) *With respect to the embedding of* D *in* A *and* (ii) *we have* $Z^{-1}D = Z^{-1}U(\mathfrak{c}) \otimes E$, *for some subalgebra* E *of* $\mathrm{End}\, V_{B'}(\lambda)$.

(iv) *If* $\ell(M_{B'}(\lambda)) \leq 2$, *then* $\mathrm{rk}(U(\mathfrak{g})/I_{B'}(\lambda)) = \dim V_{B'}(\lambda)$.

Set $C = Z^{-1}U(\mathfrak{c})$ which (c.f. [24], 2.6) is isomorphic to $K\mathscr{A}(\mathfrak{m}^-)$. We may write $\mathfrak{c} = \ell \oplus \mathfrak{m}_0^-$, with $\ell \subset \mathfrak{h}$, $\mathfrak{m}^- \subset \mathfrak{m}_0^- \subset \mathfrak{n}^-$ (c.f. [24], 2.6). Since $\mathfrak{m}_0^-$ is locally ad-nilpotent on A, it follows as in [24], 3.3 that $Z^{-1}A$ is generated over C by $A^{\mathfrak{m}_0^-}$. By 5.4, [24] 2.6 (ii), 6.7 (iii), and the hypothesis on $\mathfrak{m}^-$, each $a \in A^{\mathfrak{m}_0^-}$ is algebraic over $\mathrm{Fract}\, Z(\mathfrak{m}_0^-)$. Now $A^{\mathfrak{m}_0^-}$ is a direct sum of its ad $\mathfrak{h}$ weight subspaces with weights in $\mathbf{Q}R$. Thus the weights of $A^{\mathfrak{m}_0^-}$ are a linear combination of the weights of $Z(\mathfrak{m}_0^-)$ and our hypothesis on $\mathfrak{g}$ further implies that this is a $\mathbf{Z}$ linear combination (c.f. [24], 4.17). Thus $Z^{-1}A$ is generated over C by $(Z^{-1}A)^{\mathfrak{c}}$ and since C is central simple, this gives $Z^{-1}A = C \otimes (Z^{-1}A)^{\mathfrak{c}}$, up to isomorphism. In particular, $d((Z^{-1}A)^{\mathfrak{c}}) = d(A) - \dim \mathfrak{c} = 0$ (c.f. [6], 6.1) and so $(Z^{-1}A)^{\mathfrak{c}}$ is finite dimensional over $\mathbf{C}$. By 4.8 and [4], 4.5 it is a prime ring. By 5.10 (iii), it is isomorphic to $\mathrm{End}\, V_{B'}(\lambda)$. Hence (ii). Let F be the one-dimensional lowest weight subspace of $V_{B'}(\lambda)$. M is generated by $U(\mathfrak{g})$ over F and $U(\mathfrak{c})(1 \otimes F) = U(\mathfrak{m}^-) \otimes F$ and so (i) obtains from (ii) in an obvious fashion. The argument given in (ii) shows that $Z^{-1}D$ is generated over C by $(Z^{-1}D)^{\mathfrak{c}}$ which hence identifies with a subalgebra of $\mathrm{End}\, V_{B'}(\lambda)$. Hence (iii). Under the hypothesis of (iv), M is an indecomposable $U(\mathfrak{g})$ module of length ≤ 2. Hence $Z^{-1}M$ is an indecomposable B module of length ≤ 2. By 11.5, it follows that $V_{B'}(\lambda)$ is an indecomposable E module of length

≤ 2. Since $\mathbf{C}$ is algebraically closed, this gives the required assertion (by say 11.3, [15], Thm. 1.1.1, Lemma 2.1.5).

EXAMPLE: Take $n = 4$, $\lambda \in P(R)^{++}$. Set $B = \{\alpha_1, \alpha_2, \alpha_3\}$ taking the usual numbering in the Dynkin diagram and set $s_{\alpha_i} = s_i$: $i = 1, 2, 3$. By [5], 4.17, card $\mathscr{X}_{\hat{\lambda}} = 10$. For just two of these ideals, namely $I(s_1s_3\lambda)$ and $I(s_2\lambda)$, the Goldie ranks of the corresponding quotient algebras fail to be given by [9], 8.6. By [25], 3.7 the first of these coincides with the induced ideal $I_{\{\alpha_1,\alpha_3\}}(s_2s_1\lambda)$ and so by (iv) we obtain $\text{rk}(U(\mathfrak{g})/I(s_1s_3\lambda)) = (\alpha_2, \lambda)(\alpha_1 + \alpha_2 + \alpha_3, \lambda)$. By [22], Thm. 5.1 (or see [25], Fig. 1), one has $I(s_2\lambda) = I(s_1s_2\lambda)$. Set $M = M_{\{\alpha_2,\alpha_3\}}(s_1s_2\lambda)$. The results of Jantzen [20] show that M admits a unique proper simple submodule N and up to isomorphism, $M/N = L(s_1s_2\lambda)$, $N = L(s_1s_2s_1\lambda)$. Since neither $I(s_1s_2\lambda) \subset I(s_1s_2s_1\lambda)$ nor $I(s_1s_2\lambda) \supset I(s_1s_2s_1\lambda)$, we may apply 11.3 and (iv) which combined with [9], 8.6 gives $\text{rk}(U(\mathfrak{g})/I(s_2\lambda)) = \frac{1}{2}(\alpha_1, \lambda)(\alpha_3, \lambda)(\alpha_1 + 2\alpha_2 + \alpha_3, \lambda)$. From this one can easily check that 11.1 holds for $\mathfrak{g}$ simple of type A_{n-1}: $n = 2, 3, 4$. Excepting 5 cases (out of 26) a similar calculation verifies 11.1 in type A_4.

11.6 In general 11.5 (ii) fails because the weights of $A^{\mathfrak{m}_{\bar{0}}}$ can be half-integer linear combinations of the weights of $Z(\mathfrak{m}_{\bar{0}})$ (c.f. [24], 6.8, 6.15) and this permits the strict inequality $\text{rk}(U(\mathfrak{g})/I_{B'}(\lambda)) < \dim V_{B'}(\lambda)$. Insight into this phenomenon obtains from the following example. Set $\mathscr{A}_1 = \mathbf{C}[x, d/dx]$, take V to be a two-dimensional vector space and set $A = \mathscr{A}_1 \otimes \text{End } V$, $M = \mathbf{C}[x] \otimes V$, considered as an A module in the obvious fashion. Let $D(\lambda)$: $\lambda \in \mathbf{C}$ denote the subalgebra of A generated by

$$y_\lambda := \begin{pmatrix} 0 & 2x\,d/dx + \lambda \\ 2d/dx & 0 \end{pmatrix} \quad z := \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}.$$

It is easily verified that M is a simple $D(\lambda)$ module for all $\lambda \in \mathbf{C}$. Yet $y_1z - zy_1 = 1$ and so $D(1)$ is integral. That is $1 = \text{rk } D(1) = \frac{1}{2}\text{rk } A$. Here we consider $\mathfrak{c}$ as the Lie algebra generated by y_1z and $z^2 = x$. Set $Z = \mathbf{C}[x] - \{0\}$, which is an Ore subset for both A and $D(\lambda)$. Note that $Z^{-1}A$ is not generated over $Z^{-1}U(\mathfrak{c})$ by $(Z^{-1}A)^{\mathfrak{c}}$. Finally when $\lambda \neq 1$, we have $Z^{-1}A = Z^{-1}D(\lambda)$, which shows that the good value of the Goldie rank is recovered by an infinitesimal change in λ.

11.7 As remarked in 1.2 and as indicated by the results of [5, 13, 21–23, 25] the structure of $\mathscr{X}_{\hat{\lambda}}$ should depend only on W_λ and so in particular the non-integral fibres should satisfy the obvious analogues of 8.2 and 11.1. For example, take $\mathfrak{g}$ simple of type C_2 and write $B = \{\alpha_1, \alpha_2\}$ with α_2 long. Take $-\lambda \in \mathfrak{h}^*$ dominant and regular such that $B_\lambda = \{\alpha_1, \alpha_1 + \alpha_2\}$. Set $\sigma_{B_\lambda} = \frac{1}{2}(2\alpha_1 + \alpha_2)$. Then by [36], 4.4 we have $2\,\mathrm{rk}(U(\mathfrak{g})/I(w_\lambda\lambda)) = (\alpha_1, w_\lambda\lambda)(\alpha_1 + \alpha_2, w_\lambda\lambda)/(\alpha_1, \sigma_{B_\lambda})(\alpha_1 + \alpha_2, \sigma_{B_\lambda}) = p_{B_\lambda}(w_\lambda\lambda)$. Apart from the mysterious fixed "correction factor" of $\frac{1}{2}$ this coincides with the Goldie ranks for the finite codimensional primitive ideals in type $A_1 \times A_1$ (i.e. of type B_λ). It is the only known value of the Goldie rank for $U(\mathfrak{g})/I(w_\lambda\lambda)$ when B_λ is not a subbasis.

11.8 Let Ω_R denote the subset of $\hat{W}$ defined by the Richardson orbits through 8.6. To each $\xi \in \hat{W}$ define $\xi^* \in \hat{W}$ through $\xi^*(w) = \xi(w)\det w$: $w \in W$. The idea that emerges from 8.2, 8.6 and 11.1 is that the regular integral fibres $\mathscr{X}_{\hat{\lambda}}$: $\lambda \in P(R)^{++}$ are parametrized by a subset Ω of $\hat{W}$ containing Ω_R. In particular one should have $\mathrm{card}\,\mathscr{X}_{\hat{\lambda}} = \Sigma\{\dim \xi : \xi \in \Omega\}$. A conjecture of Borho and Jantzen [5], 2.19 suggests that $\Omega = \Omega^*$. Since $\Omega_R \neq \Omega_R^*$ in general (for example in type D_4) the simplest hypothesis is that $\Omega = \Omega_R \cup \Omega_R^*$. In type D_4 this predicts that $\mathrm{card}\,\mathscr{X}_{\hat{\lambda}} = 36$, for $\lambda \in P(R)^{++}$ in agreement with the results of Borho and Jantzen (private communication). It also "explains" the mysterious appearance of a non-polarizable nilpotent orbit $\mathcal{O}$ in the integral fibre. (More precisely $\bar{\mathcal{O}}$ is the zero variety of $\mathrm{gr}\, I$ for some $I \in \mathscr{X}_{\hat{\lambda}}$: $\lambda \in P(R)^+$ in type D_4 (c.f. [5], 4.5)). Moreover it suggests a possible generalization of the situation described in 8.6. Namely for each $\sigma \in \hat{W}$, let $n(\sigma)$ be the smallest integer such that σ occurs as a subrepresentation of the W module $S(\mathfrak{h})_{n(\sigma)}$. Then does σ occur with multiplicity one in $S(\mathfrak{h})_{n(\sigma)}$? (Unfortunately not. In type E_7 multiplicity 2 can occur even for $\sigma \in \Omega_R^*$ [41]). In addition let $\mathscr{S}: \hat{W} \to \mathcal{N}/G$ be the Springer surjection (defined through [38], 6.10 taking $\mathscr{S} = p\xi^{-1}$ where p is the projection onto the first factor in Σ). Then is $n(\sigma) = \frac{1}{2}(\mathrm{card}\, R - \dim \mathscr{S}(\sigma^*))$? (This is true for the representations defined by a $P_{\mathscr{B}'}$: $B' \subset B$, as noted in 8.6). If so then one can complete conjecture 11.1 in an obvious fashion to include the possible appearance of non-polarizable orbits in the integral fibres.

11.8 Reconsider the example of 10.5. Set $V = U(\mathfrak{g})/I_{B'}(\alpha_1)$, $L = L(M_{B'}(\alpha_1), M_{B'}(\alpha_1))$, and recall that the embedding $V \hookrightarrow L$ is strict. Set $V' = L/V$. An easy calculation shows that $V = V(-(\alpha_1 + \alpha_2), -(\alpha_1 + \alpha_2))$, $V' = V(-(\alpha_1 + \alpha_2), -\alpha_1)$ up to isomorphism and that $\mathrm{Ann}\, V = \mathrm{Ann}\, V'$ (computed in U). Hence the truth of 10.2 implies $e(V) = e(V')$ which might seem extraordinary in view of the relation $\mathrm{rk}\, L = 2\,\mathrm{rk}\, V$. Nevertheless the example in 11.6 shows how this phenomenon can occur. Adopt the notation of 11.6 and recall that $D(1)$ is isomorphic to the Weyl algebra $\mathscr{A}_1$. Take $A = \mathscr{A}_1 \otimes \mathrm{End}\, V$: $\dim V = 2$ and let e be the matrix with one in the upper left hand corner and zeros elsewhere. Let B be the left $D(1)$ submodule of A generated over e and $1 - e$. Since $ey = y(1 - e)$ and

$ez = z(1-e)$, B is a subalgebra of A containing $D(1)$. It hence admits M as a faithful simple module and so is prime, Noetherian. Since $ey \neq 0$ and yet $(ey)^2 = 0$, one has $2 = \operatorname{rk} B = 2\operatorname{rk}(D(1))$. On the other hand $e(B) = 2e(D(1)) = 2$ for the filtration on $D(1)$ induced by the canonical filtration of $\mathscr{A}_1$.

Index of notation

Symbols frequently used in the text are given below in order of appearance.

1.1 $\mathbb{C}$, $S(V)$, V^*, $U(\mathfrak{a})$, $Z(\mathfrak{a})$, $\mathscr{I}(A)$, Spec A, Prim A, $A\hat{}$, $\mathfrak{a}\hat{}$.

1.2 $\mathfrak{g}$, π, $\mathscr{V}$, $\mathscr{N}$, G, $\mathscr{O}$, $\mathscr{K}$.

2.1 $d(M)$, $e(M)$, $U(\mathfrak{a})^k$.

2.3 LAnn V, RAnn V.

3.1 $\mathfrak{h}$, R, R^+, B, s_α, W, $P(R)$, X_α, $\mathfrak{n}$, $\mathfrak{n}^-$, $\mathfrak{b}$, B', R', $W_{B'}$, $w_{B'}$, $P(R')^{++}$, $B'^{\perp}$, $\mathfrak{p}_{B'}$, $\mathfrak{m}_{B'}$, $\sigma_{B'}$, ρ, $V_{B'}(\lambda)$, $M_{B'}(\lambda)$, $I_{B'}(\lambda)$, $M(\lambda)$, $L(\lambda)$, $I(\lambda)$, ${}^t u$, $\check{u}$, $\mathfrak{m}^-_{B'}$.

3.2 j, $\mathfrak{k}$, $L(M, N)$.

3.3 $W(\lambda)$, R_λ, W_λ, w_λ, $L_{B'}(\lambda, \mu)$, $L(\lambda, \mu)$, $V(\lambda, \mu)$.

3.4 $\mathscr{K}_{\hat{\lambda}}$, $P(R)^+$.

4.4 θ_ν, Θ_ν.

4.5 $P(R')$.

6.6 $D_{B'}$, $p_{w^{-1}B'}$.

7.1 $V(\nu)$.

7.7 Σ.

8.1 $\mathscr{D}_{B'}$, $\mathscr{B}'$, $P_{\mathscr{B}'}$.

8.2 $\mathscr{O}_{\mathscr{B}'}$.

8.3 ξ, ξ^*.

8.4 St(ξ), Rt(ξ), P_ξ.

9.2 Φ, $A(w)$, $B(w)$, $\xi(w)$.

10.1 $\mathscr{H}(\Lambda)$, $\mathscr{H}_m(\Lambda)$.

11.1 rk A.

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Centre de mathématiques
Bâtiment 425
91405 Orsay
France