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TANGENT CONES AT GORENSTEIN SINGULARITIES

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Introduction

If x is a Cohen–Macaulay point of a d-dimensional affine variety V, then the embedding dimension of V at x is bounded above by $e_x + d - 1$, where e_x is the multiplicity of V at x, [1]. In this paper the following result is proved.

THEOREM: *If x is a Gorenstein point of an affine variety V, the tangent cone at x is Gorenstein if the embedding dimension of V at x is d, $d+1$, $e_x + d - 2$ or $e_x + d - 1$. Also, for any $d \geq 0$, there is an affine variety V with Gorenstein point x such that the embedding dimension of V at x is $d + 2 = e_x + d - 3$ and the tangent cone at x is not Gorenstein.*

An immediate application of this result is that the Hilbert function of a Gorenstein singularity x, in the case where the embedding dimension at x is $e_x + d - 2$ with $d \geq 1$, is a polynomial for $n \geq 2$ and is completely determined by the multiplicity at x. In fact,

$$H_x(n) = e_x\binom{n+d-2}{d-1} + \binom{n+d-3}{n}, \qquad n \geq 2.$$

Other applications will be presented in a subsequent paper.

The author proved in [12] an analogous result for Cohen–Macaulay points. Namely that if x is a Cohen–Macaulay point then the tangent cone at x is Cohen–Macaulay if the embedding dimension at x is d, $d+1$ or $e_x + d - 1$. D. Eisenbud and J. Wahl informed the author of the application of [12] to rational surface singularities which have, by a result of M. Artin, [2], embedding dimension $e_x + 1$. In [14], Wahl proves directly that the tangent cone at a rational surface singularity

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is Cohen–Macaulay. He also proves that the minimally elliptic surface singularities of Laufer [7] have Gorenstein tangent cones. This is a special case of the result to be proved in this paper as minimally elliptic singularities are Gorenstein surface singularities x with embedding dimension e_x. The result answers a question posed to the author by D. Eisenbud.

1. Some preliminaries

The techniques which we will use are purely algebraic so we rephrase the problem in terms of local rings and associated graded rings. Henceforth, $(R, \boldsymbol{m})$ is a local ring of dimension $d \geq 0$. (The term "local" includes "Noetherian.") $G(R) = R/\boldsymbol{m} \oplus \boldsymbol{m}/\boldsymbol{m}^2 \oplus \boldsymbol{m}^2/\boldsymbol{m}^3 \oplus \ldots$ is the associated graded ring with maximal homogeneous ideal $\mathcal{M} = \boldsymbol{m}/\boldsymbol{m}^2 \oplus \boldsymbol{m}^2/\boldsymbol{m}^3 \oplus \ldots$. We denote the map $R \to G(R)$ which takes each element x of R to its initial form in $G(R)$ by "–", i.e., $x \to \bar{x}$ in $\boldsymbol{m}^t/\boldsymbol{m}^{t+1}$, where $x \in \boldsymbol{m}^t \setminus \boldsymbol{m}^{t+1}$, some $t \geq 0$. $v_R = \dim_{R/\boldsymbol{m}} \boldsymbol{m}/\boldsymbol{m}^2$ is the embedding dimension of R. If I is an $\boldsymbol{m}$-primary ideal of $(R, \boldsymbol{m})$, $e(I)$ will denote the multiplicity of I. $e(\boldsymbol{m})$ is the multiplicity of R and will be denoted by e_R.

We will need some facts about reductions of ideals, cf. [11]. A minimal reduction for $\boldsymbol{m}$ is a sequence $x_1, \ldots, x_d$ of elements of $\boldsymbol{m}$ such that $\boldsymbol{m}^{r+1} = (x_1, \ldots, x_d)\boldsymbol{m}^r$, for some non-negative integer r. Viewed in $G(R)$, a minimal reduction for $\boldsymbol{m}$ is just a sequence $x_1, \ldots, x_d$ of elements of $\boldsymbol{m}$ with initial forms $\bar{x}_1, \ldots, \bar{x}_d$ forming a system of homogeneous parameters of degree 1 for $G(R)$, i.e., $\bar{x}_1, \ldots, \bar{x}_d$ are elements of $\boldsymbol{m}/\boldsymbol{m}^2$ and $\mathcal{M}^{r+1} \subset (\bar{x}_1, \ldots, \bar{x}_d)G(R)$, for some non-negative integer r. From this point of view, it is well known that, if $R/\boldsymbol{m}$ is infinite – a hypothesis which will never cause us any problem, then there is a system $\bar{x}_1, \ldots, \bar{x}_d$ of homogeneous parameters of degree 1 for $G(R)$; in other words, if $R/\boldsymbol{m}$ is infinite, minimal reductions for $\boldsymbol{m}$ exist, cf. [11]. The integer r that appears above is important, so we make the following definition.

1.1. Definition: Assume that $R/\boldsymbol{m}$ is infinite. The *reduction exponent* $r(R)$ of R is the least integer r such that there is a system of parameters $x_1, \ldots, x_d$ of R with $\boldsymbol{m}^{r+1} = (x_1, \ldots, x_d)\boldsymbol{m}^r$.

It is not hard to see that if $x_1, \ldots, x_d$ is a minimal reduction for $\boldsymbol{m}$, then $e((x_1, \ldots, x_d)R) = e_R$. Thus, if $(R, \boldsymbol{m})$ is a Cohen–Macaulay ring and $x_1, \ldots, x_d$ is a minimal reduction for $\boldsymbol{m}$, then $e_R =$

$\lambda(R/(x_1, \ldots, x_d)R)$, where $\lambda = \lambda_A$ denotes the length of an A-module over an Artinian local ring A.

It is proved in [11] that if $x_1, \ldots, x_d$ is a minimal reduction for $\boldsymbol{m}$, then the elements $x_1, \ldots, x_d$ are analytically independent in $\boldsymbol{m}$. This means that if $f(Y_1, \ldots, Y_d)$ is any form of (arbitrary) degree s with coefficients in R such that $f(x_1, \ldots, x_d) \in \boldsymbol{m}^{s+1}$, then all the coefficients of f are in $\boldsymbol{m}$.

The proof of the result stated in the introduction has two steps. First, we show that for any Cohen–Macaulay local ring $(R, \boldsymbol{m})$, $G(R)$ is Cohen–Macaulay if $r(R) \leq 2$. Then we show that the given hypotheses on $(R, \boldsymbol{m})$ force $r(R) = 2$.

2. The reduction exponent

The following theorem generalizes Theorem 2 in [12].

2.1. THEOREM: *Let $(R, \boldsymbol{m})$ be a d-dimensional Cohen–Macaulay local ring with $R/\boldsymbol{m}$ infinite. If $r(R) \leq 2$, then $G(R)$ is Cohen–Macaulay.*

PROOF: By hypothesis, there is a regular sequence $x_1, \ldots, x_d$ in R with $\boldsymbol{m}^3 = (x_1, \ldots, x_d)\boldsymbol{m}^2$. By [4] or [9], it is sufficient to prove that $\bar{x}_1, \ldots, \bar{x}_d$ is a regular sequence in $G(R)$. We do this by induction on d. There is no problem if $d = 0$, so we let $d = 1$. Suppose $yx_1 \in \boldsymbol{m}^t$, with $t \geq 3$. Then $yx_1 \in x_1\boldsymbol{m}^{t-1}$ and, since x_1 is a nonzero divisor in R, $y \in \boldsymbol{m}^{t-1}$. This shows that $\bar{x}_1$ is a nonzero divisor in $G(R)$.

Assume that $d > 1$. We first check that $\bar{x}_1$ is a nonzero divisor in $G(R)$. Suppose $yx_1 \in \boldsymbol{m}^t$ with $t \geq 3$. Then $yx_1 \in (x_1, \ldots, x_d)^{t-2}\boldsymbol{m}^2$. We have $yx_1 = x_1 g(x_1, \ldots, x_d) + h(x_2, \ldots, x_d)$, where g is a homogeneous polynomial of degree $t-3$ in $x_1, \ldots, x_d$ with coefficients in $\boldsymbol{m}^2$ and h is a homogeneous polynomial of degree $t-2$ in $x_2, \ldots, x_d$ with coefficients in $\boldsymbol{m}^2$. Thus, $(y-g)x_1 = h$. Since the associated graded ring of R with respect to the ideal $\boldsymbol{x}R = (x_1, \ldots, x_d)R$, namely the ring $R/\boldsymbol{x}R \oplus \boldsymbol{x}R/(\boldsymbol{x}R)^2 \oplus (\boldsymbol{x}R)^2/(\boldsymbol{x}R)^3 \oplus \ldots$, is a polynomial ring over $R/\boldsymbol{x}R$, it follows that $y - g \in (x_1, \ldots, x_d)^{t-2}R$. (Actually, by [5], $y - g \in (x_2, \ldots, x_d)^{t-2}R$.) Thus $y = g + \ell$, where ℓ is a homogeneous polynomial of degree $t-2$ in $x_1, \ldots, x_d$. But $yx_1 = gx_1 + \ell x_1$ is in $\boldsymbol{m}^t$ and, since $gx_1 \in \boldsymbol{m}^t$, it follows that $\ell x_1 \in \boldsymbol{m}^t$. By the analytic independence in $\boldsymbol{m}$ of $x_1, \ldots, x_d$, all the coefficients of ℓ are in $\boldsymbol{m}$. Hence $y \in \boldsymbol{m}^{t-1}$. Thus the image of x_1 in $G(R)$ is a nonzero divisor in $G(R)$ and $G(R/x_1R) \cong G(R)/\bar{x}_1G(R)$. R/x_1R is a $(d-1)$-dimensional Cohen–

Macaulay local ring which satisfies the hypotheses of the theorem, so the theorem follows by induction.

2.2. Example: If (R, m) is Cohen–Macaulay and $r(R) = 3$, then $G(R)$ need not be Cohen–Macaulay. Let $R = k[[t^4, t^5, t^{11}]]$, k any (infinite) field and t an indeterminate. Then $m^4 = t^4 m^3$, where $m = (t^4, t^5, t^{11})R$. $R \cong k[[X, Y, Z]]/(XZ - Y^3, YZ - X^4, Z^2 - X^3Y^2)$ and $G(R) \cong k[X, Y, Z]/(XZ, YZ, Z^2, Y^4)$. $G(R)$ is not Cohen–Macaulay.

With regard to this example and similar ones, it is interesting to recall, cf. [10], that if (R, m) is any 1-dimensional complete local domain with $e_R > 1$, then $G(R)$ has nonzero prime nilradical. However, the analysis in [10] does not give information about whether $\mathcal{M}$ belongs to zero in $G(R)$.

3. Gorenstein local rings

If (R, m) is any d-dimensional local Cohen–Macaulay ring, then, by [1], $v_R \leq e_R + d - 1$. If $v_R = d$ or $d + 1$, it is well-known that $G(R)$ is Cohen–Macaulay. In [12], it is proved that for $v_R = e_R + d - 1$, it is also true that $G(R)$ is Cohen–Macaulay. In fact, it is proved that $v_R = e_R + d - 1$ implies that $r(R) = 1$. The following proposition shows that $v_R = e_R + d - 1$ is *not* an interesting case if R is Gorenstein.

3.1. Proposition: *Let (R, m) be a d-dimensional local Cohen–Macaulay ring of embedding dimension $e_R + d - 1$, with $e_R > 1$. Then,*

$$\dim_{R/m} \operatorname{Ext}_R^d(R/m, R) = e_R - 1.$$

Proof: We may assume that R/m is infinite. By [12], $v_R = e_R + d - 1$ implies that there is a regular sequence $x = x_1, \ldots, x_d$ in R with $m^2 = xm$. Let $R^* = R/xR$ and $m^* = m/xR$. Then $m^* = (0 : m^*) = \{a \in R^* \mid am^* = 0\}$. But $\operatorname{Ext}_R^d(R/m, R) \cong \operatorname{Hom}_R(R/m, R/xR) \cong (0 : m^*)$. Now, $\lambda_{R^*}(R^*) = e_R$. Hence, $\dim_{R/m} \operatorname{Ext}_R^d(R/m, R) = \dim_{R/m}(0 : m^*) = \dim_{R^*/m^*} m^* = e_R - 1$.

3.2. Corollary: *If (R, m) is a Gorenstein local ring of embedding dimension $e_R + d - 1$ with $e_R > 1$, then $e_R = 2$.*

Proof: R Gorenstein implies that $\dim_{R/m} \operatorname{Ext}_R^d(R/m, R) = 1$.

We turn to Cohen–Macaulay rings of embedding dimension $e_R + d - 2$. Unlike the case of embedding dimension $e_R + d - 1$, there do exist Gorenstein local rings of embedding dimension $e_R + d - 2$ with e_R any positive integer >2. For example, let e be any positive integer >2 and set $R = k[[t^e, t^{e+1}, \ldots, t^{2e-2}]]$, where k is a field. R is a 1-dimensional complete local domain with $v_R = e - 1$ and $e_R = e$ so $v_R = e_R + d - 2$. The numerical semigroup S generated by $e, e+1, \ldots, 2e-2$ is symmetric because the conductor of S is $2e$ and the number of elements in S which are less than $2e$ is e. It follows from [6] that R is Gorenstein. Let $d > 1$. Then $T = R[[X_1, \ldots, X_{d-1}]]$ is a Gorenstein local ring of dimension d. $e_T = e_R = e$ and $v_T = e_T + d - 2$.

3.3. PROPOSITION: *Let $(R, \mathbf{m})$ be a d-dimensional local Cohen–Macaulay ring. Let $\mathbf{x} = x_1, \ldots, x_d$ be a minimal reduction for $\mathbf{m}$. Then* $\lambda_{R/xR}(\mathbf{m}^2/\mathbf{xm}) = 1$ and $\mathbf{m}^3 \subset \mathbf{xm}$ if and only if $v_R = e_R + d - 2$.

PROOF: We have the exact sequence

$$0 \to \mathrm{Tor}_1^R(R/xR, R/\mathbf{m}) \to \mathbf{m}/\mathbf{xm} \to R/xR \to R/\mathbf{m} \to 0$$

from which it follows that $\lambda_{R/xR}(\mathbf{m}/\mathbf{xm}) = e_R + d - 1$. Since

$$0 \to \mathbf{m}^2/\mathbf{xm} \to \mathbf{m}/\mathbf{xm} \to \mathbf{m}/\mathbf{m}^2 \to 0$$

is exact, $\lambda_{R/xR}(\mathbf{m}^2/\mathbf{xm}) = 1$ if and only if $v_R = e_R + d - 2$. We have $\mathbf{m}^2 \supseteq (\mathbf{m}^3, \mathbf{xm}) \supseteq \mathbf{xm}$. If $v_R = e_R + d - 2$, then $\mathbf{m}^2 \neq \mathbf{xm}$, so $\mathbf{m}^3 \subset \mathbf{xm}$.

In general, $v_R = e_R + d - 2$ does not imply that $\mathbf{m}^3 = \mathbf{xm}^2$, as Example 2.2 shows. However, we will now see that if $(R, \mathbf{m})$ is Gorenstein of embedding dimension $v_R = e_R + d - 2$, then $\mathbf{m}^3 = \mathbf{xm}^2$. In the proof of (3.4) below, we will use the fact that if $(R, \mathbf{m})$ is a 0-dimensional local Gorenstein ring and I is an ideal, then $\lambda_R(R/I) = \lambda_R((0:I))$, cf., for example [3].

3.4. THEOREM: *Let $(R, \mathbf{m})$ be a d-dimensional local Gorenstein ring with embedding dimension $v_R = e_R + d - 2$. Then $G(R)$ is Gorenstein.*

The following lemma will be needed for the proof of (3.4).

3.5. LEMMA: *Let k be any field and V a n-dimensional vector space over k. Let $(\ , \ )$ be a symmetric bilinear form on V. There is a basis $v_1, \ldots, v_n$ for V such that for each i, $1 \le i \le n$, either $(v_i, v_i) = 0$ or $(v_i, v_j) = 0$ for $j \neq i$.*

PROOF: Let $v_1, \ldots, v_n$ be a basis for V. If $(v_i, v_i) = 0$ for all i, the proof is finished, so we assume that $(v_1, v_1) \neq 0$. Let $V_1 = \{x \in V \mid (x, v_1) = 0\}$. $\dim V_1 < \dim V$ so there is, by induction, a basis $w_1, \ldots, w_t$ for V_1 with the required properties. In addition we have $(v_1, w_j) = (w_j, v_1) = 0$ for $1 \leq j \leq t$. To see that $v_1, w_1, \ldots, w_t$ span V, note that if $x \in V$, then $x' = x - [(x, v_1)/(v_1, v_1)]v_1 \in V_1$.

PROOF OF (3.4): We may assume that $R/\boldsymbol{m}$ is infinite and let $\boldsymbol{x} = x_1, \ldots, x_d$ be a minimal reduction for $\boldsymbol{m}$. By (3.3), $\boldsymbol{m}^3 \subset \boldsymbol{xm}$ and $\lambda_{R/\boldsymbol{x}R}(\boldsymbol{m}^2/\boldsymbol{xm}) = 1$. We will first prove that $G(R)$ is Cohen–Macaulay. For this it is sufficient, by (2.1), to show that $\boldsymbol{m}^3 = \boldsymbol{xm}^2$. We prove this by induction on d. Since $d = 0$ is no problem, we begin with $d = 1$. Let $x = x_1$. We may assume that $v_R > 2$. For if $v_R = 2$ then, since $\lambda_{R/\boldsymbol{m}}(\boldsymbol{m}^2/\boldsymbol{m}^3) > 2$, we have $\lambda_{R/\boldsymbol{m}}(\boldsymbol{m}^2/\boldsymbol{m}^3) = 3$ and $\boldsymbol{m}^3 = x\boldsymbol{m}^2$.

The first step is to show that $\boldsymbol{m}^4 \subset x^2\boldsymbol{m}$. Since $\boldsymbol{m}^3 \subset x\boldsymbol{m}$ and $\lambda_{R/xR}(\boldsymbol{m}^2/x\boldsymbol{m}) = 1$, it is sufficient to show that there is some element in $\boldsymbol{m}^2 \backslash x\boldsymbol{m}$ whose square is in $x^2\boldsymbol{m}$. Pass to the 0-dimensional Gorenstein ring $\tilde{R} = R/xR$ with $\tilde{\boldsymbol{m}} = \boldsymbol{m}/xR$ and let $k = \tilde{R}/\tilde{\boldsymbol{m}}$. Let $V = \tilde{\boldsymbol{m}}/\tilde{\boldsymbol{m}}^2$. V is a k-vector space of dimension $e_R - 2 > 1$. V has a non-degenerate symmetric bilinear form given by multiplication in $\tilde{R}$. For $\boldsymbol{m}^2 = (x\boldsymbol{m}, g)$, where g is any element of $\boldsymbol{m}^2 \backslash x\boldsymbol{m}$. $\tilde{\boldsymbol{m}}^2 = \tilde{g}\tilde{R}$ and we define for $\alpha, \beta \in \tilde{\boldsymbol{m}}/\tilde{\boldsymbol{m}}^2$, $(\alpha, \beta) = \bar{u}_{\alpha\beta}$, where for $a \in \tilde{\boldsymbol{m}}$ mapping to α and $b \in \tilde{\boldsymbol{m}}$ mapping to β, $ab = u_{\alpha\beta}\tilde{g}$ and $\bar{u}_{\alpha\beta}$ is the image of $u_{\alpha\beta}$ in k. $\tilde{\boldsymbol{m}}^3 = 0$, so this is well defined. Since R is Gorenstein, this form is also non-degenerate. By (3.5), we can find a minimal basis $x, z_1, \ldots, z_{e_R - 2}$ for $\boldsymbol{m}$ with the property that, for $1 \leq i \leq e_R - 2$, either $z_i^2 \in x\boldsymbol{m}$ or $z_i z_j \in x\boldsymbol{m}$ for $j \neq i$. If there is an i with $z_i^2 \notin x\boldsymbol{m}$, then $z_i^4 \in x^2\boldsymbol{m}$. For if $j \neq i$, there is an $\ell \neq i$, such that $z_j z_\ell \notin x\boldsymbol{m}$. Thus, $z_i^2 = u z_j z_\ell + x\mu$ with $u \in R \backslash \boldsymbol{m}$ and $\mu \in \boldsymbol{m}$. Then, $z_i^4 = u z_j z_\ell z_i^2 + x\mu z_i^2 \in x^2\boldsymbol{m}$. If $z_i^2 \in x\boldsymbol{m}$ for all i, then there is a $j \neq 1$ such that $z_1 z_j \notin x\boldsymbol{m}$. Then $(z_1 z_j)^2 = z_1^2 z_j^2 \in x^2\boldsymbol{m}$.

Thus we have that $\boldsymbol{m}^4 \subset x^2\boldsymbol{m}$. Now we pass to the 0-dimensional Gorenstein ring $R^* = R/x^2R$ with $\boldsymbol{m}^* = \boldsymbol{m}/x^2R$. We have $\lambda_{R^*}(R^*) = 2e_R$, $\lambda_{R^*}(\boldsymbol{m}^*/\boldsymbol{m}^{*2}) = e_R - 1$ and $\lambda_{R^*}(\boldsymbol{m}^{*3}) = 1$. Thus, $\lambda_{R^*}(\boldsymbol{m}^{*2}/\boldsymbol{m}^{*3}) = 2e_R - e_R - 1 = e_R - 1$. Consequently, $\lambda_{R/\boldsymbol{m}}(\boldsymbol{m}^2/\boldsymbol{m}^3) = e_R$ and, since $e_R = \lambda_{R/xR}(R/xR) = \lambda_{R/xR}(\boldsymbol{m}^2/x\boldsymbol{m}^2)$, it follows that $\boldsymbol{m}^3 = x\boldsymbol{m}^2$. This concludes the case $d = 1$.

Assume that $d > 1$. Suppose that $\boldsymbol{m}^3 \not\subset \boldsymbol{xm}^2$. We will show that there is an i, $1 \leq i \leq d$, such that $(\boldsymbol{m}/x_iR)^3 \not\subset (x_1, \ldots, \hat{x}_i, \ldots, x_d)(\boldsymbol{m}/x_iR)^2$. Since R/x_iR is a local Gorenstein ring of dimension $d-1$ and embedding dimension $v_{R/x_iR} = e_{R/x_iR} + (d-1) - 2$, for $e_{R/x_iR} = e_R$, this will contradict the induction hypothesis.[1] Since $\boldsymbol{m}^3 \neq \boldsymbol{xm}^2$, there is an element z in $\boldsymbol{m}^3$

[1] The author is grateful to B. Singh for his simplification of this part of the proof and for several incisive comments on the paper.

with $z \notin xm^2$. Write $z = ax_1 + bx_2 + \cdots + cx_d$ with $a, b, \ldots, c \in m$. By the analytic independence of x in m we have $x^2R \cap m^3 = x^2m \subseteq xm^2$. Therefore, $z \notin xm^2 + x^2R$. Now, if $a, b, \ldots, c \in m^2 + xR$ then $z \in xm^2 + x^2R$, a contradiction. We may therefore assume that $a \notin m^2 + xR$. We then claim that any $i \geq 2$ meets the requirement. To see this, say for $i = d$, suppose $z \in (x_1, \ldots, x_{d-1})m^2 + x_dR$. Then there exists $y \in m^2$ such that $(a - y)x_1 \in (x_2, \ldots, x_d)R$. Therefore $a - y \in (x_2, \ldots, x_d)R$, which shows that $a \in m^2 + xR$, a contradiction. This completes the proof that $m^3 = xm^2$.

It remains to show that $G(R)$ is Gorenstein. $\bar{x}_1, \ldots, \bar{x}_d$ is a regular sequence in $G(R)$ and $G(R/xR) \cong G(R)/(\bar{x}_1, \ldots, \bar{x}_d)G(R)$. Since $G(R)$ is Gorenstein if and only if $G(R)_{\mathcal{M}}$ is Gorenstein, [8], it suffices to show that $G(R/xR)$ is Gorenstein. Let $R_0 = R/xR$ and $m_0 = m/xR$. Then $G(R_0) = R_0/m_0 \oplus m_0/m_0^2 \oplus m_0^2$ and $\mathcal{M}_0 = m_0/m_0^2 \oplus m_0^2$. We must show that $\dim_{R/m} (0 : \mathcal{M}_0) = 1$. But $\bar{y}(m_0/m_0^2) = 0$ means $ym_0 \subseteq m_0^3 = 0$. Hence $(0 : \mathcal{M}_0) = m_0^2$. Since R_0 is Gorenstein, $\dim_{R/m}(m_0^2) = \dim_{R/m}(0 : m_0) = 1$. This concludes the proof of the theorem.

(3.6) EXAMPLE: We show that for any $d \geq 0$ there is a local Gorenstein ring (R, m) of embedding dimension $v_R = d + 2 = e_R + d - 3$ with $G(R)$ not Gorenstein. We begin with $d = 1$. Let k be a field and t an indeterminate. Let $R = k[[t^5, t^6, t^9]]$. The numerical semigroup generated by 5, 6 and 9 is symmetric so, by [6], R is a 1-dimensional local Gorenstein ring. $R \cong k[[X, Y, Z]]/(YZ - X^3, Z^2 - Y^3)$ and $G(R) = k[X, Y, Z]/(YZ, Z^2, Y^4 - ZX^3)$. $3 = v_R = e_R + d - 3 = d + 2$ and $G(R)$ is not Gorenstein. R/t^5R is a similar example for $d = 0$. Examples for $d > 1$ are obtained by adjoining analytic indeterminates $W_1, \ldots, W_{d-1}$ to $k[[t^5, t^6, t^9]]$.

In summary, then, we have the following. If (R, m) is a d-dimensional Gorenstein local ring of multiplicity e_R and embedding dimension v_R, then $G(R)$ is Gorenstein if $v_R = d$, $d + 1$ or $e_R + d - 2$. These are the only embedding dimensions which will always give $G(R)$ Gorenstein.

3.7. COROLLARY: *Let (R, m) be a local Gorenstein ring of multiplicity at most 4. Then $G(R)$ is Gorenstein.*

It follows from (3.4) that the Hilbert function for a d-dimensional local Gorenstein ring of embedding dimension $e_R + d - 2$ is completely determined by e_R. To see this, we recall that the Hilbert sum transforms for any local ring (R, m) are defined inductively as follows. Let n be any non-negative integer.

$$H^0_R(n) = \dim_{R/m}(m^n/m^{n+1})$$

and

$$H^i_R(n) = \sum_{j=0}^{n} H^{i-1}_R(j).$$

3.8. REMARK: Let (R, m) be a d-dimensional local ring. It is known, cf. [13], that if $x_1, \ldots, x_t$ are elements of m/m^2 with images $\bar{x}_1, \ldots, \bar{x}_t$ in $G(R)$ forming a regular sequence, then

$$H^0_R(n) = H^t_{R/(x_1,\ldots,x_t)R}(n),$$

for all $n \geq 0$. Thus, if $G(R)$ is Cohen–Macaulay, there is an Artin local ring R^* of embedding dimension $v_R - d$ and length e_R such that

$$H^0_R(n) = H^d_{R^*}(n),$$

for all $n \geq 0$. (If R/m is infinite, take $R^* = R/(x_1, \ldots, x_d)R$, where $x_1, \ldots, x_d$ is a minimal reduction for m. If R/m is finite, it may be necessary first to pass to the ring $R(u) = R[u]_{mR[u]}$, where u is an indeterminate.)

3.9. COROLLARY: *Let (R, m) be a d-dimensional local Gorenstein ring of embedding dimension $e_R + d - 2$. Then there exists a 0-dimensional local Gorenstein ring (R^*, m^*) with $H^0_{R^*}(1) = e_R - 2, H^0_{R^*}(2) = 1$ and $H^0_{R^*}(n) = 0$, for $n > 2$, such that*

$$H^0_R(n) = H^d_{R^*}(n),$$

for all $n \geq 0$. In fact, for $d \geq 1$,

$$H^0_R(n) = e_R\binom{n+d-2}{d-1} + \binom{n+d-3}{n}, \qquad n \geq 2.$$

PROOF: We may assume that R/m is infinite. In the first part of the proof of (3.4) we saw that (R, m) Gorenstein with $v_R = e_R + d - 2$ implies that there is a regular sequence $x_1, \ldots, x_d$ in R such that $m^3 = (x_1, \ldots, x_d)m^2$ and the images $\bar{x}_1, \ldots, \bar{x}_d$ form a regular sequence in $G(R)$. With $R^* = R/(x_1, \ldots, x_d)R$, the first statement of the corollary follows from (3.8). The second statement follows from the first by double induction on n and d, using the fact that for any local ring (R, m), $H^d_R(n) = H^d_R(n-1) + H^{d-1}_R(n)$.

REFERENCES

[1] S.S. ABHYANKAR: Local rings of high embedding dimension. *Amer. J. Math. 89* (1967) 1073–1077.

[2] M. ARTIN: On isolated rational singularities of surfaces. *Amer. J. Math. 88* (1966) 129–137.

[3] J. HERZOG and E. KUNZ: Der kanonische Modul eines Cohen–Macaulay Rings: Lecture Notes in Mathematics 238. Springer–Verlag, 1971.

[4] M. HOCHSTER and L.J. RATLIFF, JR.: Five theorems on Macaulay rings. *Pac. J. Math. 44* (1973) 147–172.

[5] I. KAPLANSKY: *R*-sequences and homological dimension. *Nagoya Math. J. 20* (1962) 195–199.

[6] E. KUNZ: The value semigroup of a one-dimensional Gorenstein ring. *Proc. Amer. Math. Soc. 25* (1970) 748–751.

[7] H. LAUFER: On minimally elliptic singularities. *Amer. J. Math., 99* (1977) 1257–1259.

[8] J. MATIJEVIC: Three local conditions on a graded ring. *Trans. Amer. Math. Soc. 205* (1975) 275–284.

[9] J. MATIJEVIC and P. ROBERTS: A conjecture of Nagata on graded Cohen–Macaulay rings. *J. Math. Kyoto Univ. 14* (1974) 125–128.

[10] D.G. NORTHCOTT: The neighbourhoods of a local ring. *J. London Math. Soc. 30* (1955) 360–375.

[11] D.G. NORTHCOTT and D. REES: Reductions of ideals in local rings. *Proc. Camb. Phil. Soc. 50* (1954) 145–158.

[12] J.D. SALLY: On the associated graded ring of a local Cohen–Macaulay ring. *J. Math. Kyoto Univ., 17* (1977) 19–21.

[13] B. SINGH: Effect of a permissible blowing-up on the local Hilbert function. *Inventiones math. 26* (1974) 201–212.

[14] J.M. WAHL: Equations defining rational singularities. *Ann. Scient. Ec. Norm. Sup. 10* (1977) 231–264.

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