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SINGULARITY OF PARABOLIC MEASURES*

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Abstract

We show by example that a recent result of Dahlberg on harmonic measure for the Laplace equation can not be extended to parabolic measure for heat equation. The example is based on the non-self-adjointness of the heat operator; the methods are estimations of Green's function and construction of special boundary curves.

Let $f(t)$ be a continuous function on $(-\infty, \infty)$ and $\Omega \subseteq R^2$ be the region $\{(x, t): x > f(t)\}$. Let m be the measure on $\partial\Omega$ defined by $m(E) =$ the Lebesgue measure of $\{t: (f(t), t) \in E\}$. If Ω is Dirichlet regular for the heat equation (or adjoint heat equation), for a fixed point $(y, s) \in \Omega$, the parabolic measure (or adjoint parabolic measure) of a Borel set $E \subseteq \partial\Omega$ at (y, s), denoted by $w^{(y,s)}(E)$ (or $w^{*(y,s)}(E)$), is defined to be the value at (y, s) of the solution of the heat equation (or adjoint heat equation) on Ω with boundary value 1 on E and 0 on $\partial\Omega\backslash E$ in the Brelot-Peron-Wiener sense.

In case $f(t) \equiv 0$, and $\Omega = \{x > 0\}$ it is known that m, $w^{(x_0,t_0)}$, $w^{*(x_0,s_0)}$ are mutually absolutely continuous on $\{(0, t): s_0 \leq t \leq t_0\}$.

Let $\lambda^{(y,s)}$ be the harmonic measure on $\partial\Omega$ at (y, s) corresponding to the Laplace equation $\partial^2/\partial x^2 + \partial^2/\partial t^2 = 0$. It is known by conformal mapping that

THEOREM A: *If f is* Lip 1, *then $\lambda^{(y,s)}$ and m are mutually absolutely continuous on $\partial\Omega$.*

In fact, Dahlberg [1] has proved Theorem A for $x \in R^n$, $n \geq 2$. The proof depends explicitly on the self-adjointness of the Laplace equation. Domains with Lip 1 boundaries are the most general regions on which the boundary behavior of harmonic functions has been extensively studied.

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Because the affine transformations $\{x \to ax + b, t \to a^2 t + c\}$ are the only diffeomorphisms that preserve solutions of heat equation [2], regions with Lip $\frac{1}{2}$ boundaries are very natural for studying solutions of heat equation. In [5], Petrowski proved that if $f(t)$ is Lip $\frac{1}{2}$ then every point on $\partial\Omega$ is a regular point for heat equation.

Richard A. Hunt proposed the problem whether m, w and w^* are mutually absolutely continuous on $\partial\Omega$ if $f(t)$ is Lip $\frac{1}{2}$. In [6], the first author proved the following

THEOREM B: *Suppose that $f(t)$ is* Lip $\frac{1}{2}$ *and E is a set on $\partial\Omega$ with m measure zero, then E is composed of two parts, one with $w^{(y,s)}$ measure zero, the other with $w^{*(y,s)}$ measure zero for each $(y, s) \in \Omega$.*

In this note, we show by example that m, w and w^* can be mutually singular on $\partial\Omega$, for certain Lip $\frac{1}{2}$ function $f(t)$.

1. Lemmas on parabolic functions

We call solutions of heat equation parabolic functions.

For fixed $(y, s) \in R^2$, we denote by $W(x, t; y, s)$ the fundamental solution of heat equation defined by

$$W(x, t; y, s) = \begin{cases} [4\pi(t-s)]^{-1/2} \exp\left[-\dfrac{|x-y|^2}{4(t-s)}\right] & \text{for } t > s \\ 0 & \text{for } t \le s. \end{cases}$$

Suppose Ω is Dirichlet regular for heat equation, we say g is the Green's function for Ω, if for each fixed $(y, s) \in \Omega$, $W(x, t; y, s) - g(x, t; y, s)$ is the bounded parabolic function in Ω with boundary value $W(x, t; y, s)$ for $(x, t) \in \partial\Omega$.

LEMMA 1: *Suppose $D = \{(x, t): x > 2\sqrt{|t|}\}$, $(X, -T)$ is a point in D with $T > 0$ and $g(x, t)$ is the Green's function on D with pole at $(X, -T)$. Then there are constants c and C depending on $(X, -T)$ so that $g(x, 0) \le Cx^2$ for $0 < x < c$.*

PROOF: Let $b = X - 2\sqrt{T}$ and $S = \{(x, t); x = X - b/2$ and $-T \le t \le 0\}$. Let $C_1 = \sup\{g(x, t): (x, t) \in S\}$ and

$$C_2 = \inf\left\{\frac{1}{\sqrt{8\pi T}} - W(x, t; 0, -2T): (x, t) \in S\right\}.$$

The level curve γ defined by $W(x, t; 0, -2T) = 1/\sqrt{8\pi T}$ satisfies the equation:

$$\frac{1}{\sqrt{4\pi(t+2T)}} e^{-(x^2/4(t+2T))} = \frac{1}{\sqrt{8\pi T}}$$

or

$$x^2 = -2(t+2T)\log\left(1+\frac{t}{2T}\right).$$

If $(x, t) \in \gamma$ and $-T \le t < 0$ then $x^2 < -4t$. Therefore if $(x, t) \in D$ and $t \ge -T$, then $(1/\sqrt{8\pi T}) - W(x, t; 0, -2T) \ge 0$. Let V be the region $D \cap \{(x, t): t > -T\} \cap \{x < X - b/2\}$. We observe by the definitions of C_1 and C_2 that

$$C_2 g(x, t) \le C_1\left(\frac{1}{\sqrt{8\pi T}} - W(x, t; 0, -2T)\right)$$

for $(x, t) \in S$. When (x, t) is on the part of ∂V with $t = -T$ or on the part of ∂V with $x = 2\sqrt{|t|}$, the above inequality also holds because the left side is zero and the right side is positive. In view of the maximum principle for the solutions of heat equations [3, Chap. 2], we obtain

$$\begin{aligned} g(x, 0) &\le \frac{C_1}{C_2}\left[\frac{1}{\sqrt{8\pi T}} - W(x, 0; 0, -2T)\right] \\ &= \frac{C_1}{C_2}\frac{1}{\sqrt{8\pi T}}(1 - e^{-x^2/8T}) \\ &\le Cx^2 \end{aligned}$$

for $(x, t) \in S$. When (x, t) is on the part of ∂V with $t = -T$ or on the part of ∂V with $x = 2\sqrt{|t|}$, the above inequality also holds because the

Suppose f is Lip $\frac{1}{2}$ satisfying $|f(t) - f(\tau)| \le M|t-\tau|^{1/2}$. For $a > 0$ we denote by $\Delta(t, a) = \{(f(s), s): |s-t| \le a\}$ and $A(t, a) = (f(t) + 10M\sqrt{a}, t + 2a)$. Under these assumptions, we may reformulate Lemma 1.4 in [4] and Lemma 2.2 in [6] as follows.

LEMMA 2: *There exist positive constants C, c depending on M only, so that*

$$w^{(y,s)}(\Delta(t, r)) \le C w^{(y,s)}(\Delta(t, a)) w^{A(t,a)}(\Delta(t, r))$$

whenever $0 < r < a/2$, $(y, s) \in \Omega$ and $|y - f(t)|^2 + |s - t| > ca$.

LEMMA 3: *Let (x_0, t_0), (y_0, s_0) be two fixed points in Ω with $t_0 > s_0 > a > 0$, and g be the Green's function on Ω. Then there are positive constants C, μ, ρ depending on a, M, (x_0, t_0) and (y_0, s_0) so that*

$$w^{(y_0,s_0)}(\Delta(t, r)) \leq Cr^{1/2} g(x_0, t_0; f(t) + \mu\sqrt{r}, t)$$

whenever $-a < t < a$ and $0 < r < \rho$.

2. A test for singular measures

Suppose that μ is a positive Borel measure on [0, 1], and that μ is not totally singular to Lebesgue measure m; then $d\mu/dx \geq c > 0$ on some set E of measure $m(E) > 0$. Thus

$$\lim \mu([x, x+h])h^{-1} = \mu'(x) \text{ and}$$
$$\lim \mu([x-h, x])^{-1} = \mu'(x)$$

as $h \to 0^+$, when $x \in E$. Letting $h = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ we can apply Egoroff's theorem to find a set $E_0 \subseteq E$, with $m(E_0) > 0$, and a sequence $\epsilon_n > 0$ decreasing to 0, so that

$$|\mu([x, x+h]) - h\mu'(x)| \leq \epsilon_n h, \text{ and}$$
$$|\mu([x-h, x]) - h\mu'(x)| \leq \epsilon_n h,$$

when $(n+1)^{-1} \leq h \leq n^{-1}$, and $x \in E_0$. Let now $r(u)$ be a polygonal function on (0, 1], defined by the conditions $r(n^{-1}) = \epsilon_{n-1}$ for $n \geq 2$ and r is constant on $[\frac{1}{2}, 1]$. Then

$$|\mu([a, b]) - (b-a)\mu'(x)| \leq (b-a)r(b-a)$$

whenever $0 \leq a < x < b \leq 1$ and $x \in E_0$.

Let $0 < \delta < \frac{1}{4}$ and observe that by the Lebesgue density theorem, E_0 must meet one of the sets $[(k+\frac{5}{8})N^{-1}, (k+\frac{3}{4})N^{-1}]$ $(1 \leq k \leq N-2)$ whenever N is sufficiently large. We apply the inequality on μ-measures to intervals $[a, b_1]$, $[a, b_2]$, $[a, b_3]$ with

$$a = (k+\tfrac{1}{2})N^{-1}, \quad b_1 = (k+1-\delta)N^{-1}, \quad b_2 = (k+1+\delta)N^{-1},$$
$$b_3 = (k+1+\tfrac{1}{2})N^{-1}.$$

We write these inequalities as

$$E_i\colon |\mu([a, b_i]) - (b_i - a)\mu'(x)| \leq N^{-1} r(N^{-1}), \quad i = 1, 2, 3.$$

Combination of E_1 and E_2 gives

$$|\mu((b_1, b_2]) - (b_2 - b_1)\mu'(x)| \leq 2N^{-1}r(N^{-1}),$$

and comparison with E_3 yields

$$|\mu((b_1, b_2]) - 2\delta\mu([a, b_3])| \leq 3N^{-1}r(N^{-1}).$$

Let us write $I = [(k+\frac{1}{2})N^{-1}, (k+1+\frac{1}{2})N^{-1}]$, $I_\delta = ((k+1-\delta)N^{-1}, (k+1+\delta)N^{-1}]$, that is $I = [a, b_3]$, $I_\delta = (b_1, b_2]$. We divide the last inequality by $\mu(I)$, which exceeds $c/(2N)$ for $N \geq N_0$. We obtain

$$\mu(I_\delta)/\mu(I) \geq 2\delta + o(1), \quad N \to +\infty.$$

3. Construction of curves

Let $h(t)$ be a function on $[0, 1]$ subject to the following conditions

1) $0 \leq h \leq 1$,
2) $h(t) = 4t^{1/2}$ for $0 \leq t \leq \frac{1}{3}$,
3) $h(t) = h(1-t)$,
4) h is of class C^1 on $[\frac{1}{4}, \frac{3}{4}]$,
5) $|h(t) - h(s)| \leq 4|t-s|^{1/2}$ for $0 \leq s \leq t \leq 1$.

Let $h_n(t)$ be the function on $[0, 1]$ with period $1/n$, such that $h_n(t) = h(nt)/n^{1/2}$ for $0 \leq t \leq 1/n$.

Let $\ell_n(t)$ be a function of class $C^1[0, 1]$, periodic with period $1/n$, such that

6) $0 \leq \ell_n \leq h_n$,
7) $\ell_n(0) = 0$ and $\ell_n(t) = h_n(t)$ for $n^{-3} \leq t \leq n^{-1} - n^{-3}$,
8) $|\ell_n(t) - \ell_n(s)| \leq 5|t-s|^{1/2}$.

We shall choose a sequence (n_j) and set $f_k(t) = \Sigma_1^k \ell_{n_j}(t)$, $f(t) = \Sigma_1^\infty \ell_{n_j}(t)$. We require the following properties of f and f_k:

9) $|f(t) - f(s)| \leq 8|t-s|^{1/2}$.
10) $0 \leq f(t) - f_k(t) \leq n_k^{-3/2}$.
11) The inequalities $3|t-\tau|^{1/2} \leq f_k(t) - f_k(\tau) \leq 6|t-\tau|^{1/2}$ hold whenever $\tau = i/n_k$ $(1 \leq i \leq n_k - 1)$ and $n_k^{-3} \leq |t-\tau| \leq (4n_k)^{-1}$.

To obtain 10) we observe the inequality $0 \leq f(t) - f_k(t) \leq \Sigma_{k+1}^\infty n_j^{-1/2}$, and simply choose $n_{j+1} > 16n_j^3$. This choice is compatible with the rest of the construction and we don't mention it again.

To obtain 11) and 9) we let B_{k-1} be an upper bound for $|f'_{k-1}|$, so that

$$\begin{aligned} |f_k(t) - f_k(\tau) - \ell_{n_k}(t) + \ell_{n_k}(\tau)| &\leq B_{k-1}|t-\tau| \\ &\leq B_{k-1}(4n_k)^{-1/2}|t-\tau|^{1/2}, \end{aligned}$$

for the numbers τ, t mentioned in 11). By 7) and 2) we find

$$\ell_{n_k}(t) - \ell_{n_k}(\tau) = 4|t-\tau|^{1/2}$$

for these numbers, because $n_k\tau$ is an integer, and we obtain 11) by taking $B_{k-1}(4n_k)^{-1/2} < 1$. To obtain 9) we suppose that $|f_p(t) - f_p(s)| \leq (6-p^{-1})|t-s|^{1/2}$, for $p = k-1$. (This is true when $p = 1$). Then

$$|f_k(t) - f_k(s)| \leq |f_{k-1}(t) - f_{k-1}(s)| + |\ell_{n_k}(t) - \ell_{n_k}(s)|.$$

Since $\ell_{n_k} \leq n_k^{-1/2}$, we have the inequality

$$|f_k(t) - f_k(s)| \leq (6-(k-1)^{-1})|t-s|^{1/2} + 2n_k^{-1/2}.$$

Thus the required estimate is valid when $(k^2-k)^{-1}|t-s|^{1/2} \geq 2n_k^{-1/2}$ or $|t-s| \geq 4n_k^{-1}(k^2-k)^2$. But when the last inequality is violated, we can use the estimation

$$|f_k(t) - f_k(s)| \leq B_{k-1}|t-s| + 5|t-s|^{1/2};$$

this yields the inequality in question provided $B_{k-1}|t-s| \leq |t-s|^{1/2} \cdot \frac{1}{2}$, or $|t-s| \leq (2B_{k-1})^{-2}$. One estimate or the other is available for large n_k.

4. The theorem

We retain the notations from §3 and extend the function f, constructed in §3, to $(-\infty, \infty)$ by defining $f(t) = f(0)$ for $t < 0$ and $f(t) = f(1)$ for $t > 1$. We let Ω be $\{(x,t): x > f(t)\}$, w be the parabolic measure on $\partial\Omega$ evaluated at (X, T) and w^* be the adjoint parabolic measure on $\partial\Omega$ evaluated at (Y, S) where (X, T) and (Y, S) are two fixed points in Ω with $T > 1$ and $S < 0$.

We observe, with the aid of maximum principle, that for $E \subseteq \{(f(t), t): 0 < t < 1\}$, if $w(E) = 0$ then $w^{(x,t)}(E) = 0$ for every $(x,t) \in \Omega$; and if $w^*(E) = 0$ then $w^{*(x,t)}(E) = 0$ for every $(x,t) \in \Omega$.

THEOREM: *None of the three measures: m, w and w^* on $\partial\Omega$ is absolutely continuous with respect to another. In fact m, w and w^* are totally singular with respect to each other on* $\{(f(t), t): 0 < t < 1\}$.

We first prove the following lemma and assume as we may that $(X, T) = (10, 100)$.

LEMMA 4: *There are positive absolute constants C and $\rho < \frac{1}{32}$, so that whenever $\tau = i/n_k$ $(1 \le i \le n_k - 1)$, $n_k^{-1} < \delta < \rho$,*

$$I_k = \{(f(t), t): |t - \tau| < (16n_k)^{-1}\} \text{ and}$$
$$E_k = \{(f(t), t): |t - \tau| < \delta n_k^{-1}\}$$

then $w(E_k) \le C\delta^{3/2} w(I_k)$ for sufficiently large k.

PROOF: For a fixed $\tau = i/n_k$, we let $A = (f(\tau) + 5/\sqrt{n_k},\ \tau + 1/8n_k)$ and B be $A + (0, 1/8n_k)$. From 9) and Lemma 2 it follows that for some absolute constant C,

$$w(E_k) \le C w(I_k) w^A(E_k).$$

Let Φ be the map $(x, t) \to (\sqrt{n_k}(x - f(\tau)),\ n_k(t - \tau))$ and G be the Green's function on $\Phi(\Omega)$. We note that $\partial\Phi(\Omega) = \Phi(\partial\Omega)$ is the graph of a Lip $\frac{1}{2}$ function with 8 as an upper bound for the Lip $\frac{1}{2}$ constant and Φ preserves parabolic functions (i.e. v is parabolic on $\Phi(\Omega)$ if and only if $v(\Phi)$ is parabolic on Ω). Let $\bar{w}$ be the parabolic measure on $\partial\Phi(\Omega)$, thus $\bar{w}^{\Phi(A)}(\Phi(E_k)) = w^A(E_k)$. Because $\Phi(A) = (5, \frac{1}{8})$ and $\Phi(B) = (5, \frac{1}{4})$, it follows from Lemma 3 that there exist absolute constants C, μ and $\rho < \frac{1}{32}$, so that

$$\begin{aligned} w^A(E_k) &= \bar{w}^{\Phi(A)}(\Phi(E_k)) \\ &\le C\sqrt{\delta}\, G(\Phi(B); \mu\sqrt{\delta}, 0) \end{aligned}$$

if $0 < \delta < \rho$.

Let $\alpha(t) = 2|t - \tau|^{1/2} + f(\tau) - 10n_k^{-3/2}$. From 9), 10) and 11) it follows that $\alpha(t) \le f(t)$ whenever $|t - \tau| < (4n_k)^{-1}$. Therefore $\Phi(\Omega) \cap \{(x, t): |t| < \frac{1}{4}\} \subseteq \{(x, t): x > 2|t|^{1/2} - 10n_k^{-1}\}$. let $\tilde{G}$ be the Green's function on $\{(x, t): x > 2|t|^{1/2} - 10n_k^{-1}\}$. We recall that $\Phi(B) = (5, \frac{1}{4})$ and $n_k^{-1} < \delta < \rho$, and obtain, by the maximum principle and the adjoint form of Lemma 1, that

$$\begin{aligned} &G(\Phi(B); \mu\sqrt{\delta}, 0) \\ &\le \tilde{G}(\Phi(B); \mu\sqrt{\delta}, 0) = \tilde{G}^*(\mu\sqrt{\delta}, 0; \Phi(B)) \\ &\le C(\mu\sqrt{\delta} + 10n_k^{-1})^2 \le C\delta, \end{aligned}$$

for absolute constants C. Thus $w^A(E_k) \le C\delta^{3/2}$. This proves Lemma 4.

PROOF OF THE THEOREM: From the above lemma and the test for singular measures in §2 we see that w is totally singular to m on

$\{(f(t), t): 0 < t < 1\} \equiv S$, that is, there is a set $E \subseteq S$ of m measure zero but $w(E) = w(S)$. Similarly there is a set $E^* \subseteq S$ of m measure zero but $w^*(E^*) = w^*(S)$. From these properties and Theorem B, we conclude that w and w^* are mutually singular. The theorem follows easily.

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