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STRATA IN THE DEFORMATION OF REAL ISOLATED SINGULARITIES ARE IN GENERAL NON CONTRACTIBLE

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Introduction

In [4] Looijenga answered positively to a conjecture of Thom on isolated singularities (see [5] page 58) in the case of real simple singularities. The complex form of this conjecture in the case of a simple singularity had been earlier verified (see for example [1] and [2]). There exist some reasons to believe that the conjecture over \mathbb{C} is not true in general and in fact that for elliptic singularities the complement of the (complex) discriminant has a non vanishing second homotopy group (see [3] for an attempt to understand the meaning of this conjectured non-vanishing).

Here we give an example which shows that the conjecture over \mathbb{R} is not true and in fact that a connected component of the complement of the real discriminant¹ for the singularity $x^4 + y^4$ has an infinite fundamental group. This supports, we believe, the idea that the second homotopy group of the complement of the complex discriminant of $x^4 + y^4$ is infinite too.

Consider the semiuniversal deformation of the elliptic singularity $x^4 + y^4$. We will construct a continuous family $\{F_\theta\}_{0 \leq \theta \leq \pi}$ of non singular real curves of the deformation with the following properties:

- i) $F_0 = F_\pi$
- ii) the automorphism of F_0 , induced by the homotopy class of the loop $\theta \rightarrow F_\theta$, is non trivial.

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¹ We follow [4]: the "real discriminant" means the image of the real critical points of the deformation which is contained (generally strictly) in the real part of the complex discriminant.

This will show that there exists a non contractible connected component in the complement of the real discriminant.

Let F_0 be the curve

$$x^4 + y^4 - 2\eta x^2 + \epsilon = 0$$

with $\eta, \epsilon > 0$. One can easily see that for $0 < \epsilon < \eta^2$ this is a compact irreducible curve with two connected components which are separated by the line $x = 0$.

Let F_θ be the curve

$$x^4 + y^4 - 2\eta\varphi^2 + \epsilon = 0$$

where $\varphi = x \cos \theta + y \sin \theta$. Obviously $F_\pi = F_0$. Moreover for each θ , F_θ has the same properties as F_0 with respect to the line $\varphi = 0$, namely it consists of two connected components separated by the line $\varphi = 0$.

This will be shown, through an isotopy principle, by proving that each F_θ is non singular plus the remark that F_θ does not intersect $\varphi = 0$.

Since, as θ varies in $[0, \pi]$, clearly the halfplanes $\varphi > 0$ and $\varphi < 0$ interchange, the components of F_0 will do the same. It will follow that the connected component, in the complement of the (real) discriminant, containing F_0 has a non trivial fundamental group.

Proof that F_θ is non singular

A singular point $P = (x_0, y_0)$ for F_θ is a solution of

$$(+) \quad \begin{cases} x^4 + y^4 - 2\eta\varphi^2 + \epsilon = 0 \\ x^3 - \eta\varphi \cos \theta = 0 \\ y^3 - \eta\varphi \sin \theta = 0 \end{cases}$$

from where

$$x^4 + y^4 = \eta\varphi^2 = \epsilon.$$

One has to show that the system (+) has no solutions for η, ϵ sufficiently small.

Choose $\epsilon < \frac{1}{8}\eta^2$.

On one hand: $(x^2 + y^2)^3 \geq x^6 + y^6 = (\eta\varphi \cos \theta)^2 + (\eta\varphi \sin \theta)^2 = \eta^2\varphi^2 = \eta\epsilon$

On the other: $(x^2 + y^2)^2 \leq 2(x^4 + y^4) \leq 2\epsilon$

Hence $(2\epsilon)^3 \geq (\eta\epsilon)^2$; implying $\epsilon \geq \frac{1}{8}\eta^2$.

Contradiction.

Remark 1

Let Γ be the connected component of $S_R - D_R$ (same notations as in [4]) which contains the curves F_θ .

The loop γ we described before generates an infinite subgroup of $\pi_1(\Gamma, F_0)$. In fact, let G be this subgroup: one has a map $b: G \rightarrow B(2)$ (the braid group in two strings) induced by the map which to each curve F_θ associates the couple of its baricenters, and clearly γ goes to the generator of $B(2)$.

Remark 2

The two components of F_0 induce linearly independent cycles in $H_1(\tilde{F}_0, \mathbb{Z})$, where \tilde{F}_0 denotes the complex fibre, as one easily verifies, for instance, by inspection of the induced ramified covering. It follows that the loop described before also induces a non trivial element in the monodromy of the complex deformations.

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