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# ON DETERMINANTAL IDEALS WHICH ARE SET-THEORETIC COMPLETE INTERSECTIONS*

Giuseppe Valla

Let $A$ be an $r \times s$ $(r \leq s)$ matrix with entries in a commutative noetherian ring $R$ with identity. We shall denote by $(A)$ the ideal generated by its subdeterminants of order $r$. If $(A)$ is a proper ideal of $R$, then the height of $(A)$, abbreviated as $h(A)$, is at most $s - r + 1$ (see [1], Theorem 3). In this paper we prove that there exist elements $f_1, \ldots, f_{s-r+1} \in (A)$ such that $\mathrm{rad}(A) = \mathrm{rad}(f_1, \ldots, f_{s-r+1})$ (where $\mathrm{rad}(I)$ means the radical of the ideal $I$) in each of the following situations:

(1) $A = \|a_{ij}\|$ is an $r \times s$ matrix such that $a_{ij} = a_{kl}$ if $i + j = k + l$.

(2) $A$ is an $r \times (r + 1)$ partly symmetric matrix, where partly symmetric means that the $r \times r$ matrix obtained by omitting the last column is symmetric.

(3) $A = \begin{Vmatrix} a^{p_1} & b^{q_1} & c^{r_1} \\ b^{q_2} & c^{r_2} & a^{p_2} \end{Vmatrix}$ where $(a, b, c)$ is an ideal of height 3 and $p_i, q_i, r_i$ are positive integers not necessarily distinct.

It follows that if $h(A)$ is as large as possible, $s - r + 1$, then the above determinantal ideals are set-theoretic complete intersections.

It is interesting to compare these results with the following theorem due to M. Hochster (never published).

THEOREM: *Let $t < r < s$ be integer, and let $k$ be a field of characteristic 0. Let $A = k[X_{ij}]$ be the ring of polynomials in rs variables, and let $I_t(X)$ be the ideal generated by the $t \times t$ minors of the $r \times s$ matrix $(X_{ij})$. Then $I_t(X)$ is not set theoretically a complete intersection.*

## 1

Let $A = \|a_{ij}\|$ be an $r \times s$ given matrix, where $a_{ij} \in R$ and $r \leq s$. In

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this section we assume that $a_{ij} = a_{kl}$ if $i + j = k + l$, hence we may write

$$A = \begin{Vmatrix} a_1 & a_2 & \cdots & a_s \\ a_2 & a_3 & \cdots & a_{s+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_r & a_{r+1} & \cdots & a_{r+s+1} \end{Vmatrix}$$

We shall denote by $(A)$ the ideal generated by the $r$-rowed minors of $A$ and if $\sigma = (\sigma_1, \ldots, \sigma_r)$ is a set of $r$ integers such that $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_r \leq s$, we put

$$A_\sigma = \begin{Vmatrix} a_{\sigma_1} & a_{\sigma_2} & \cdots & a_{\sigma_r} \\ a_{\sigma_1+1} & a_{\sigma_2+1} & \cdots & a_{\sigma_r+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{\sigma_1+r-1} & a_{\sigma_2+r-1} & \cdots & a_{\sigma_r+r-1} \end{Vmatrix}$$

and $d_\sigma = \det A_\sigma$.

If $i = r, \ldots, s$ let $\mathfrak{A}_i$ be the ideal generated by the $d_\sigma$ with $\sigma_r \leq i$; then $\mathfrak{A}_s = (A)$ and, with a self explanatory notation, $\mathfrak{A}_i = (\mathfrak{A}_{i-1}, d_\sigma)_{\sigma_r = i}$ (where $\mathfrak{A}_{r-1} = (0)$).

Next for all $i = r, \ldots, s$, let $f_i$ be the determinant of the $i \times i$ matrix

$$M_i = \begin{Vmatrix} a_1 & \cdots & a_r & \cdot & \cdots & a_i \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ a_r & \cdots & a_{2r-1} & \cdot & \cdots & a_{i+r-1} \\ \cdot & \cdots & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ a_i & \cdots & a_{i+r-1} & 0 & \cdots & 0 \end{Vmatrix}$$

It is clear that $\mathfrak{A}_r = (f_r)$ and $f_i \in \mathfrak{A}_i$ for all $i = r, \ldots, s$.

THEOREM 1.1: *With the above notations, we have*:

$$\operatorname{rad}(\mathfrak{A}_i) = \operatorname{rad}(\mathfrak{A}_{i-1}, f_i)$$

*for all* $i = r, \ldots, s$.

PROOF: Since $(\mathfrak{A}_{i-1}, f_i) \subseteq \mathfrak{A}_i$ we need only to prove that $\mathfrak{A}_i \subseteq \operatorname{rad}(\mathfrak{A}_{i-1}, f_i)$. This is true if $i = r$, hence we may assume $i > r$. Now $\mathfrak{A}_i = (\mathfrak{A}_{i-1}, d_\sigma)_{\sigma_r = i}$, so it is enough to show that $d_\sigma \in \operatorname{rad}(\mathfrak{A}_{i-1}, f_i)$ for all $\sigma$ such that $\sigma_r = i$. Let $\sigma = (\sigma_1, \ldots, \sigma_r = i)$; then

$$A_\sigma = \left\| \begin{matrix} a_{\sigma_1} & a_{\sigma_2} & \cdots & a_i \\ a_{\sigma_1+1} & a_{\sigma_2+1} & \cdots & a_{i+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{\sigma_1+r-1} & a_{\sigma_2+r-1} & \cdots & a_{i+r-1} \end{matrix} \right\|$$

Hence, by expanding the determinant along the last column, we get $d_\sigma = \sum_{k=0}^{r-1} a_{i+k}c_k$ where $c_k$ is the cofactor of $a_{i+k}$ in $A_\sigma$. Denote by $\lambda_m$ ($m = 1, \dots, i$) the $m$-th row of $M_i$ and let $1 \le \tau_1 < \tau_2 < \cdots < \tau_{i-r} \le i-1$, where $\{\tau_1, \dots, \tau_{i-r}\}$ is the complement of $\{\sigma_1, \dots, \sigma_r = i\}$ in $\{1, 2, \dots, i\}$.

Then if $j = 1, \dots, i-r$ we have $j \le \tau_j \le \tau_{i-r} - (i-r-j) \le i-1-i+r+j = r+j-1$.

Denote by $N_i$ the matrix obtained from $M_i$ by replacing, for all $j = 1, \dots, i-r$, the row $\lambda_{\tau_j}$ by $\sum_{k=0}^{r-1} \lambda_{j+k}c_k$; since, as we have seen, $j \le \tau_j \le r+j-1$, in this linear combination $\lambda_{\tau_j}$ has coefficient $c_{\tau_j-j}$. It follows that

$$\det N_i = \left\{ \prod_{j=1}^{i-r} c_{\tau_j-j} \right) f_i.$$

Denote by $m_{pq}$ the entries of the matrix $M_i$ and by $n_{pq}$ those of $N_i$; then $m_{j+k,l} = a_{j+k+l-1}$ (where $a_t = 0$ if $t > i+r-1$), hence $n_{\tau_j l} = \sum_{k=0}^{r-1} a_{j+k+l-1}c_k$ for all $j = 1, \dots, i-r$ and $l = 1, \dots, i-j+1$. It follows that for all $j = 1, \dots, i-r$ if $1 \le l \le i-j+1$, $n_{\tau_j l}$ is the determinant of the matrix obtained by replacing the last column of $A_\sigma$ by the $(j+l-1)$-th column of $A$. Therefore we get:

(1) $n_{\tau_j l} = 0$ if $j+l-1 \in \{\sigma_1, \dots, \sigma_{r-1}\}$.

(2) $n_{\tau_j l} = d_\sigma$ if $j+l-1 = i$, or, which is the same, $l = i-j+1$.

(3) $n_{\tau_j l} \in \mathfrak{A}_{i-1}$ if $j+l-1 \in \{\tau_1, \dots, \tau_{i-r}\}$ and this because $\tau_{i-r} \le i-1$ and $\sigma_{r-1} \le i-1$.

So we get for all $j = 1, \dots, i-r$: $n_{\tau_j l} \in \mathfrak{A}_{i-1}$ if $l = 1, \dots, i-j$ and $n_{\tau_j, i-j+1} = d_\sigma$. Then we can write

$$\det N_i = \det \left\| \begin{matrix} \cdot & \cdot & \cdots & \cdot & & \cdots & & \cdot & \cdot \\ n_{\tau_1 1} & n_{\tau_1 2} & \cdots & n_{\tau_1 r} & \cdot & \cdots & \cdot & n_{\tau_1 i-1} & n_{\tau_1 i} \\ \cdot & \cdot & \cdots & \cdot & & \cdots & \cdot & \cdot & \\ n_{\tau_2 1} & n_{\tau_2 2} & \cdots & n_{\tau_2 r} & \cdot & \cdots & n_{\tau_2 i-2} & n_{\tau_2 i-1} & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdot & & \\ n_{\tau_{i-r} 1} & n_{\tau_{i-r} 2} & \cdots & n_{\tau_{i-r} r} & n_{\tau_{i-r} r+1} & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & & \cdots & \cdot & \cdot & \cdot \\ a_i & a_{i+1} & \cdots & a_{i+r-1} & \cdot & \cdots & \cdot & \cdot & \cdot \end{matrix} \right\|$$

$$= \det \begin{Vmatrix} 0 & 0 & \cdots & 0 & \cdots & \cdot & 0 & d_\sigma \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & \cdots & d_\sigma & 0 & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & d_\sigma & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdot & \cdot \\ a_i & a_{i+1} & \cdots & a_{i+r-1} & \cdots & \cdot & \cdot & \cdot \end{Vmatrix} \bmod \mathfrak{A}_{i-1}.$$

By expanding the determinant along the first $r$ columns we get

$$\det N_i = \pm\det \begin{Vmatrix} a_{\sigma_1} & a_{\sigma_1+1} & \cdots & a_{\sigma_1+r-1} \\ a_{\sigma_2} & a_{\sigma_2+1} & \cdots & a_{\sigma_2+r-1} \\ & & & \\ \cdot & \cdot & \cdots & \cdot \\ a_i & a_{i+1} & \cdots & a_{i+r-1} \end{Vmatrix} \det \begin{Vmatrix} 0 & 0 & \cdots & \cdot & d_\sigma \\ 0 & 0 & \cdots & d_\sigma & \cdot \\ & & & & \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ d_\sigma & \cdot & \cdots & \cdot & \cdot \end{Vmatrix}$$

$$\bmod \mathfrak{A}_{i-1};$$

but clearly $A_\sigma$ is a symmetric matrix, hence $\det N_i = \pm d_\sigma^{i-r+1} \bmod \mathfrak{A}_{i-1}$. It follows that $d_\sigma \in \operatorname{rad}(\mathfrak{A}_{i-1}, f_i)$, since, as we have seen, $\det N_i \in (f_i)$; this completes the proof.

COROLLARY 1.2: *With $A$ and $f_r, \ldots, f_s$ as before, we have*:

$$\operatorname{rad}(A) = \operatorname{rad}(f_r, \ldots, f_s).$$

PROOF: By Theorem 1.1,

$$\begin{aligned} \operatorname{rad}(A) &= \operatorname{rad}(\mathfrak{A}_s) = \operatorname{rad}(\mathfrak{A}_{s-1}, f_s) = \operatorname{rad}(\operatorname{rad}(\mathfrak{A}_{s-1}) + \operatorname{rad}(f_s)) \\ &= \operatorname{rad}(\operatorname{rad}(\mathfrak{A}_{s-2}, f_{s-1}) + \operatorname{rad}(f_s)) = \operatorname{rad}(\mathfrak{A}_{s-2}, f_{s-1}, f_s) \\ &= \cdots = \operatorname{rad}(\mathfrak{A}_r, f_{r+1}, \ldots, f_s) = \operatorname{rad}(f_r, \ldots, f_s). \end{aligned}$$

REMARK 1.3: If the elements of the matrix $A$ are indeterminates over an algebraically closed field $k$, the ideal $(A)$ is the defining ideal of the locus $V$ of chordal $[r-2]$'s of the normal rational curve of $\mathbb{P}^{s+r-2}$, where if $p \geq 2$ a chordal $[p-1]$ of a manifold is one which meets it in $p$ independent points (see [4] pag. 91 and 229). $V$ is a projective variety in $\mathbb{P}^{s+r-2}$ of dimension $2r-3$ and order $\binom{s}{r-1}$; hence the codimension of $V$ is $s+r-2-2r+3 = s-r+1$ and the above result proves that $V$ is set-theoretic complete intersection. The case $r=2$ is the main result in [5].

## 2

In this section $A$ is a partly symmetric $r \times (r+1)$ matrix whose elements belong to $R$. Therefore we may write

$$A = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1r} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2r} & b_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdots & a_{rr} & b_r \end{matrix} \right\|$$

where the matrix $S = \|a_{ij}\|$ is $r \times r$ symmetric.

Let $B = \left\| \dfrac{A}{b_1 \ldots b_r 0} \right\|$, $f_1 = \det S$ and $f_2 = \det B$; next, for all $i = 1, \ldots, r+1$, denote by $A_i$ the matrix which results when the $i$-th column of $A$ is deleted, and put $d_i = \det A_i$. Then $f_1 = d_{r+1}$, $(A) = (d_1, \ldots, d_{r+1})$ and $f_2 \in (A)$.

THEOREM 2.1: *With the above notations we have*:

$$\operatorname{rad}(A) = \operatorname{rad}(f_1, f_2).$$

PROOF: Since $(f_1, f_2) \subseteq (A)$ and $d_{r+1} = f_1$, it is enough to prove that $(d_1, \ldots, d_r) \subseteq \operatorname{rad}(f_1, f_2)$. Let $i$ be any integer, $1 \leq i \leq r$; by expanding the determinant of $A_i$ along the last column, we get $d_i = \Sigma_{k=1}^r b_k c_{ki}$ where $c_{ki}$ is the cofactor of $b_k$ in $A_i$. Denote by $B'$ the matrix obtained by replacing the $i$-th row of $B$ by the linear combination of the first $r$ rows of $B$ with coefficients $c_{1i}, c_{2i}, \ldots, c_{ri}$. Then it is clear that $\det B' = c_{ii} \det B$ and the $i$-th row of $B'$ is:

$$\left( \sum_{k=1}^r a_{k1} c_{ki}, \ldots, \sum_{k=1}^r a_{kr} c_{ki}, \sum_{k=1}^r b_k c_{ki} \right).$$

But $\Sigma_{k=1}^r a_{kj} c_{ki}$ is the determinant of the matrix obtained by replacing the last column of $A_i$, by the $j$-th column of $A$. Hence $\Sigma_{k=1}^r a_{kj} c_{ki} = 0$ if $j \neq i$, while $\Sigma_{k=1}^r a_{ki} c_{ki} = \pm f_1$. Therefore we get:

$$c_{ii} f_2 = \det B' = \det \left\| \begin{matrix} a_{11} & \cdots & a_{1r} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ a_{i-1,1} & \cdots & a_{i-1,r} & b_{i-1} \\ 0 & \cdots & 0 & d_i \\ a_{i+1,1} & \cdots & a_{i+1,r} & b_{i+1} \\ \cdot & \cdots & \cdot & \cdot \\ a_{r1} & \cdots & a_{rr} & b_r \\ b_1 & \cdots & b_r & 0 \end{matrix} \right\| \mod f_1.$$

By expanding this determinant along the first $r$ columns we get:

$$c_{ii}f_2 = \pm d_i \det \begin{Vmatrix} a_{11} & \cdots & a_{1r} \\ \cdot & \cdots & \cdot \\ a_{i-1,1} & \cdots & a_{i-1,r} \\ a_{i+1,1} & \cdots & a_{i+1,r} \\ \cdot & \cdots & \cdot \\ a_{r1} & \cdots & a_{rr} \\ b_1 & \cdots & b_r \end{Vmatrix} \bmod f_1;$$

But $S$ is symmetric, hence $c_{ii}f_2 = \pm d_i \det A_i^t = \pm d_i \det A_i = \pm d_i^2 \bmod f_1$, and the theorem is proved.

EXAMPLE 2.2: Let $V$ be the rational cubic scroll in $\mathbb{P}^4$; then it is well known that $V$ is the locus where $rk \begin{Vmatrix} X_0 & X_1 & X_3 \\ X_1 & X_2 & X_4 \end{Vmatrix} = 1$. Hence the above theorem shows that $V$ is set-theoretic complete intersection.

## 3

In this last section we will be interested in a particular $2\times 3$ matrix. Suppose $a$, $b$ and $c$ are elements of the ring $R$, such that the ideal they generate is of height 3; next let $p_i$, $q_i$, $r_i$ $(i = 1, 2)$ positive integers not necessarily distinct. Let us consider the $2\times 3$ matrix

$$A = \begin{Vmatrix} a^{p_1} & b^{q_1} & c^{r_1} \\ b^{q_2} & c^{r_2} & a^{p_2} \end{Vmatrix}$$

and put $p = p_1 + p_2$, $q = q_1 + q_2$, $r = r_1 + r_2$ and $f_1 = b^{q_1}a^{p_2} - c^r$, $f_2 = a^p - b^{q_2}c^{r_1}$, $f_3 = a^{p_1}c^{r_2} - b^q$.

We want to show that if $(A) = (f_1, f_2, f_3)$ then $\mathrm{rad}(A)$ is equal to the radical of an ideal generated by 2 elements; but first we shall give some remarks which are useful in the following.

Let $k$ be any integer, $0 \le k \le q$; then we can write

(1) $$kq_1 = tq + s \quad \text{where } 0 \le s \le q-1.$$

Hence we have $kq = kq_1 + kq_2 = tq + s + kq_2$; it follows that

(2) $$q_2(q-k) = (q_2 - k + t)q + s \quad \text{for all } k = 0, \ldots, q.$$

Now, since $q_2(q-k) \geq 0$, we have $(q_2 - k + t)q + s \geq 0$; but $s < q$ by (1), hence

(3) $$q_2 - k + t \geq 0 \quad \text{for all } k = 0, \ldots, q.$$

Then we have also

(4) $0 \leq (q-k)r_1 + r_2(q_2 - k + t) = (q-k)r + tr_2 - q_1 r_2$ for all $k = 0, \ldots, q$.

This allows us to consider the element

$$g = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} a^{kp_2 + tp_1} b^s c^{(q-k)r + tr_2 - q_1 r_2}.$$

THEOREM 3.1: *With the above notations we have*:

$$\operatorname{rad}(A) = \operatorname{rad}(g, f_3).$$

PROOF: We have

$$f_1^q = (b^{q_1} a^{p_2} - c^r)^q = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} a^{kp_2} b^{kq_1} c^{r(q-k)};$$

since by (1) $kq_1 = tq + s$ for all $k = 0, \ldots, q$ we get

$$f_1^q = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} a^{kp_2 + tp_1} b^s c^{r(q-k) + tr_2} \bmod f_3,$$

or $f_1^q = c^{q_1 r_2} g \bmod f_3$. On the other hand

$$f_2^q = (a^p - b^{q_2} c^{r_1})^q = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} a^{kp} b^{q_2(q-k)} c^{r_1(q-k)},$$

hence, using (2) and (3) we get

$$f_2^q = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} a^{kp + p_1(q_2 - k + t)} b^s c^{r_1(q-k) + r_2(q_2 - k + t)} \bmod f_3.$$

But $kp + p_1(q_2 - k + t) = kp_2 + p_1 q_2 + tp_1$, hence, using (4), we get $f_2^q = a^{p_1 q_2} g \bmod f_3$. This proves that $(A) \subseteq \operatorname{rad}(g, f_3)$.

Next we have seen that $f_1^q = c^{q_1 r_2} g \bmod f_3$; hence $c^{q_1 r_2} g \in (A)$. Let $\mathfrak{P}$

be a minimal prime ideal of $(A)$, then $h(\mathfrak{P}) \leq 2$ by [1, Theorem 3], so $c \notin \mathfrak{P}$, because if $c \in \mathfrak{P}$ then $(a, b, c) \subseteq \mathfrak{P}$ which is a contradiction since we have assumed $h(a, b, c) = 3$. It follows that $g \in \operatorname{rad}(A)$; this completes the proof.

EXAMPLE 3.2: Let $k$ be an arbitrary field, $t$ transcendental over $k$. Let $n_1, n_2, n_3$ natural numbers with grestest common divisor 1, and let $C$ be the affine space curve with the parametric equations $X = t^{n_1}$, $Y = t^{n_2}$, $Z = t^{n_3}$. Let $c_i$ be the smallest positive integer such that there exist integers $r_{ij} \geq 0$ with $c_1 n_1 = r_{12} n_2 + r_{13} n_3$, $c_2 n_2 = r_{21} n_1 + r_{23} n_3$, $c_3 n_3 = r_{31} n_1 + r_{32} n_2$. In [2] it is proved that if $C$ is not a complete intersection then $r_{ij} > 0$ for all $i, j$ and $c_1 = r_{21} + r_{31}$, $c_2 = r_{12} + r_{32}$, $c_3 = r_{13} + r_{23}$.

Furthermore if $f_1 = X^{r_{31}} Y^{r_{32}} - Z^{c_3}$, $f_2 = X^{c_1} - Y^{r_{12}} Z^{r_{13}}$ and $f_3 = X^{r_{21}} Z^{r_{23}} - Y^{c_2}$, then the vanishing ideal $I(C) \subseteq k[X, Y, Z]$ of $C$ is $I(C) = (f_1, f_2, f_3)$. Then it is easy to see that $I(C)$ is the ideal generated by the $2 \times 2$ minors of the matrix

$$\begin{Vmatrix} X^{r_{21}} & Y^{r_{32}} & Z^{r_{13}} \\ Y^{r_{12}} & Z^{r_{23}} & X^{r_{31}} \end{Vmatrix}.$$

It follows, by Theorem 3.1, that $C$ is set-theoretic complete intersection. This result has been proved in [3] by completely different methods; see also [6].

Finally we remark that if $C = \{(t^5, t^7, t^8) \in \mathbb{A}^3(k)\}$ then the matrix is $\begin{Vmatrix} X & Y^2 & Z \\ Y & Z^2 & X^2 \end{Vmatrix}$, which is not partly symmetric; so the conclusion that $C$ is set-theoretic complete intersection cannot be drawn from Theorem 2.1.

EXAMPLE 3.3: Let $n, p$ be non-negative integers; we have seen (see Example 3.2) that if

$$C = \{(t^{2n+1}, t^{2n+1+p}, t^{2n+1+2p}) \in \mathbb{A}^3(k)\},$$

the vanishing ideal $I(C)$ in $k[X_1, X_2, X_3]$ is generated by $X_1^{n+p} X_2 - X_3^{n+1}$, $X_1^{n+p-1} - X_2 X_3^n$ and $X_1 X_3 - X_2^2$. Let $\bar{C}$ be the projective closure of $C$ in $\mathbf{P}^3$. Since $C$ has only one point at the infinity, it is well known that the homogeneous ideal of $\bar{C}$ in $k[X_0, X_1, X_2, X_3]$ is generated by the polynomials $X_1^{n+p} X_2 - X_0^p X_3^{n+1}$, $X_1^{n+p+1} - X_0^p X_2 X_3^n$ and $X_1 X_3 - X_2^2$. It is immediately seen that this ideal is generated by the $2 \times 2$ minors of the matrix

$$\begin{Vmatrix} X_1 & X_2 & X_0^p X_3^n \\ X_2 & X_3 & X_1^{n+p} \end{Vmatrix}.$$

Thus, by Theorem 2.1, $\bar{C}$ is set-theoretic complete intersection of the two hypersurfaces $X_1X_3 - X_2^2$ and $X_0^{2p}X_3^{2n+1} + X_1^{2n+2p+1} - 2X_0^pX_1^{n+p}X_2X_3^n$.

EXAMPLE 3.4: If $C = \{(t^3, t^7, t^8) \in \mathbb{A}^3(k)\}$, the vanishing ideal $I(\bar{C}) \subseteq k[X_0, X_1, X_2, X_3]$ of the projective closure $\bar{C}$ of $C$ in $\mathbb{P}^3$, needs five generators and our methods do not apply in order to see if $\bar{C}$ is set-theoretic complete intersection.

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