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ON THE LOCAL STRUCTURE OF A GENERIC CENTRAL SET

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1. Introduction

Let Q be a closed subset of R^n, $n \geq 2$. A closed ball B, $\mathring{B} \subset R^n \setminus Q$, which is not a proper subset of another ball B_1, $\mathring{B}_1 \subset R^n \setminus Q$, is called a maximal ball. The set consisting of the centers of all maximal balls is called the central set of Q and is denoted by $C(Q)$. The notion of the central set (for $Q = \partial G$ – the boundary of an open bounded domain G in R^n) was introduced and studied by D. Milman in [8], [9], [10].

For $z \in R^n$ and $y \in Q$ let $\rho(z, y) = |z - y|^2 = (z - y, z - y)$, where (,) denotes the scalar produce in R^n. Suppose, that Q is a smooth submanifold of R^n. Then for each $z \in R^n$, $\rho(z, \cdot)$: $Q \to R$ is a smooth function on Q. The cut locus of Q is defined by R. Thom in [11] as the set, consisting of all z for which $\rho(z, \cdot)$ has either at least two equal absolute minima or one degenerate absolute minimum. As shown in [9], in this case $C(Q)$ coincides with the cut locus of Q.

Under the name skeleton the notion of the central set is also studied in the Theory of Pattern Recognition (see e.g. [6]).

The local topological structure of the central set can be rather complicated even for Q – *a* C^∞ – smooth submanifold of R^n. But if this manifold Q is embedded into R^n generically, the pathologies can be avoided and the local structure of $C(Q)$ can be studied by methods of Singularities Theory (see [11], [2], [3]). In some sense the study of the generic central sets is equivalent to the study of the Maxwell set in the space of smooth functions ([11]).

In this paper a further topological description of the central set of a generic smooth submanifold in R^n is given. The method used is based on the transversality theorem of Looijenga ([7], see also [12]) and on

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the notion of a generalized differential of a Lipschitzian function, developed in Mathematical Programming ([4], [5]). We apply the inverse function theorem of F. Clarke ([5]) to the mappings defined by (non-smooth) functions of type $\delta(z) = \min_{y \in Q} \rho(z, y)$.

(The similar notions of generalized derivatives are used by D. Milman [10] to study the properties of the central function of the boundary ∂G, which associates to $y \in G$ the center $z \in C(\partial G)$ of the maximal ball, containing y).

In section 2 the required definitions and results concerning the generalized differentials are given.

In section 3 we consider the central set $C(Q)$ of an arbitrary closed subset $Q \subset R^n$. Let $\delta(z) = \min_{y \in Q} \rho(z, y)$ denote by $B(z)$ the closed ball of radius $[\delta(z)]^{1/2}$ centered at z and let $T(z) = B(z) \cap Q$. Let $\mu(z)$ be the number of points in $T(z)$ if $T(z)$ is a finite set, and $\mu(z) = \infty$ otherwise.

The structure of $C(Q)$ at $z \in C(Q)$ is determined on the one hand by the global geometry of the set $T(z)$ and, on the other hand, by the character of tangency of $B(z)$ and Q at each point of $T(z)$. It turns out that the typical model for the "global" part of $C(Q)$ at z is the central set of the boundary of a regular simplex: Let $\Delta^{\mu-1}$ be a regular μ-1-dimensional simplex, c – its center. The central set $C(\partial\Delta^{\mu-1})$ consists of all μ-2-simplices formed by $(\mu-3)$-faces of $\partial\Delta^{\mu-1}$ and c. It is the cone over $(\mu-3)$-skeleton of $\partial\Delta^{\mu-1}$. Denote by C^μ the interior part $C(\partial\Delta^{\mu-1}) \cap \mathring{\Delta}^{\mu-1}$. In theorem 1 the conditions of $T(z)$ are given under which in a neighborhood of z the subset $C_g(z) \subset C(Q)$ can be selected, such that $C_g(z)$ is homeomorphic to $C^\mu \times I^{n-\mu+1}$, where $I = (0, 1)$, $\mu \leq n+1$.

In sections 4 and 5 we consider the case where Q is a smooth submanifold of R^n embedded generically (in the sense which will be made precise below). We prove that here the conditions of theorem 1 are naturally satisfied for any $z \in C(Q)$ with $\mu(z) \geq 2$ and then (theorem 2) in a neighborhood of any $z \in C(Q)$ with $\mu(z) \geq 2$ there is the subset $C_g(z) \subset C(Q)$, homeomorphic to $C^{\mu(z)} \times I^{n-\mu(z)+1}$.

Theorem 3 of section 5 gives a complete topological description of $C(Q)$ in a neighborhood of $z \in C(Q)$, for which the tangency of $B(z)$ and Q at any point of $T(z)$ is either of the first order, or of the third order in one direction (the function $\rho(z, \cdot)$ has global minima of types A_1 or A_3 only). This covers, in particular, all the possibilities for $n = 2, 3$, described also in [11], [3], and, together with the case where $\rho(z, \cdot)$ has the only global minimum of type A_5, all the possibilities for $n = 4$.

REMARK 1: All the results of section 3 and 4 remain true also for the cut locus of a point in a compact Riemannian manifold with a generic metric. The only difference is that instead of the distance function we consider the energy function on the paths space of this manifold and instead of Looijenga's theorem use Buchner's transversality theorem ([2]).

REMARK 2: The results of this paper can be extended in the following directions:

a) The central set $C(Q)$ of an arbitrary closed set $Q \subset R^n$ can be described in terms of differential properties of the function $\delta(z)$. In this description the second order generalized differentials appear.
b) For Q – the closed generic polyhedron in R^n $C(Q)$ in a neighborhood of any $z \in C(Q)$ is homoeomorphic to $C^{\mu(z)} \times I^{n-\mu(z)+1}$.
c) The structure of the central set of a generic smooth manifold in R^n can be described, using above methods, also for points where more complicated singularities of $\rho(z, \cdot)$ then A_1 and A_3 arise.

These results will appear separately.

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2. Differential properties of Lipschitzian functions

We need some results of F. Clarke ([4], [5]): Let $f: R^n \to R^m$ satisfy a Lipschitz condition in a neighborhood of a point $z_0 \in R^n$. Thus for some constant K for all z_1 and z_2 near z_0, we have:

$$|f(z_1) - f(z_2)| \leq K|z_1 - z_2|.$$

Let $L_{n,m}$ be the space of linear mappings $\ell: R^n \to R^m$. The differential $df(z) \in L_{n,m}$ exists almost everywhere near z_0.

DEFINITION 1: The generalized differential $\partial f(z_0)$ is the convex hull of all $\ell \in L_{n,m}$ of the form $\ell = \lim_{z_i \to z_0} df(z_i)$, where z_i converges to z_0 and f is differentiable at z_i for each i.

DEFINITION 2: $\partial f(z_0)$ is said to be of maximal rank (or non-degenerate) if every ℓ in $\partial f(z_0)$ is of maximal rank (equal to $\min(n, m)$).

Since we are only interested in a local behavior of mappings, we will consider the germs $f:(R^n, z_0)\to(R^m, w_0)$.

The following property of generalized differentials can be easily proved:

PROPOSITION 1: *Let* $f:(R^n, z_0)\to(R^m, w_0)$ *and* $g:(R^m, w_0)\to(R^p, y_0)$ *be the germs of Lipschitzian mappings. Then* $\partial(g\circ f(z_0)$ *is contained in the convex hull of the set* $\partial g(w_0)\circ\partial f(z_0)\subset L_{n,p}$, *consisting of* $\ell_2\circ\ell_1$, $\ell_1\in\partial f(z_0)$, $\ell_2\in\partial g(w_0)$. ■

The Inverse Function Theorem ([5], theorem 1).

Let $f:(R^n, z_0)\to(R^n, w_0)$ be a Lipschitzian germ and let $\partial f(z_0)$ be nondegenerate. Then there exists a Lipschitzian germ $g:(R^n, w_0)\to(R^n, z_0)$, with $f\circ g = Id$, $g\circ f = Id$. In particular, f is a homeomorphism of some neighborhood of z_0 onto the neighborhood of w_0. ■

The homeomorphism $f:G_1\to G_2$ of open sets $G_1, G_2\subset R^n$, which is Lipschitzian near each point $z\in G_1$, and whose inverse f^{-1} is Lipschitzian near each $w\in G_2$, will be called an L-homeomorphism.

We need the following form of the implicit function theorem (which is obtained by usual construction from the inverse function theorem):

COROLLARY 1: *Let* $f:(R^n, z_0)\to(R^m, w_0)$, $n\geq m$, *be the Lipschitzian germ with* $\partial f(z_0)$ *of maximal rank. Then there exists a germ of an* L*-homeomorphism*

$$\varphi:(R^n, z_0)\to(R^m\times R^{n-m}, w_0\times 0),$$

such that the diagram

$$\begin{array}{ccc} (R^n, z_0) & \xrightarrow{\varphi} & (R^m\times R^{n-m}, w_0\times 0) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & (R^m, w_0) & \end{array}$$

commutes, where π *is the projection on the first factor.* ■

Now we turn to distance functions. It is convenient to consider instead of the differential of the real function its gradient (using the scalar product (,) in R^n), which will be denoted also by df. The generalized gradient $\partial f(z_0)$ of a real Lipschitzian function f is now a convex set in the vector space R^n.

Denote by $\hat{D}$ the convex hull of a set $D \subset R^n$, and let for $A, B \subset R^n$ $v(A, B)$ denotes the set of vectors of the form $v = 2(b-a)$, $a \in A$, $b \in B$.

Let Q be a closed subset of R^n, $\delta(z) = \min_{y \in Q} \rho(z, y)$ and $T(z)$ as above.

PROPOSITION 2: ([4]), *theorem* 2.1).
Let $z \in R^n \smallsetminus Q$. Then $\partial\delta(z) = v(\hat{T}(z), z)$. The first differential $d\delta(z)$ exists if and only if $T(z)$ consists of one point. ■

3. The global part of $C(Q)$ at z_0

Let $z_0 \in C(Q)$ and let $T(z_0)$ be the union of disjoint close nonempty subsets $T_1, \ldots, T_\mu$, $\mu \geq 2$ (then, $z \in R^n > Q$). We can find open sets W_ℓ and V_ℓ, $\ell = 1, \ldots, \mu$, such that $T_\ell \subset \bar{W}_\ell \subset V_\ell$ and V_ℓ are disjoint. Now define $\delta_\ell(z)$ as the distance function of $Q \cap \bar{W}_\ell : \delta_\ell(z) = \min_{y \in Q \cap \bar{W}_\ell} \rho(z, y)$. The germ of δ_ℓ at z_0 depends only on T_ℓ and Q, but not on W_ℓ.

LEMMA 1: *There exists a neighborhood U_0 of z_0 such that for $z \in U_0$*

$$\delta(z) = \min_{\ell = 1, \ldots, \mu} \delta_\ell(z).$$

PROOF: $\delta(z) = \min_{y \in Q} \rho(x, y)$ and for $z = z_0$ this minimum is attained on $T(z_0)$. Then for z sufficiently close to z_0 $\min_{y \in Q} \rho(z, y)$ is attained in the neighborhood $W = \bigcup_{\ell=1}^{\mu} W_\ell$ of $T(z_0)$. Hence

$$\delta(z) = \min_{y \in Q \cap \bar{W}} \rho(z, y) = \min_{\ell = 1, \ldots, \mu} \left(\min_{y \in Q \cap \bar{W}_\ell} \rho(z, y) \right) = \min_{\ell = 1, \ldots, \mu} \delta_\ell(z). \quad ■$$

Let $C_g(z_0) \subset U_0$ consist of points $z \in U_0$ satisfying one of the relations

$$\delta_i(z) = \delta_j(z) \leq \min_{\ell \neq i,j} \delta_\ell(z), \quad i < j \leq \mu. \tag{1}$$

We call $C_g(z_0)$ the global part of $C(Q)$ at z_0 (with respect to the partition $T(z_0) = \bigcup_{\ell=1}^{\mu} T_\ell$).

LEMMA 2: *$C_g(z_0) \subset C(Q)$. The complement $U_0 \cap C(Q) \setminus C_g(z_0)$ is the disjoint union of sets $C_\ell(z_0)$, $\ell = 1, \ldots, \mu$, and for each ℓ $C_\ell(z_0) \subset C(Q \cap \bar{W}_\ell)$.*

PROOF: For $z \in C_g(z_0)$ $\min_{y \in Q} \rho(z, y) = \min_{\ell=1,\ldots,\mu} \delta_\ell(z)$ is attained as two or more different points, then the ball $B(z)$ is maximal and $z \in C(Q)$. Now $U_0 \setminus C_g(z_0) = \bigcup_{\ell=1}^{\mu} U_\ell$ where $U_\ell = \{z \in U_0 / \delta_\ell(z) < \min_{j \neq \ell} \delta_j(z)\}$ are disjoint open sets and clearly

$$C_\ell(z) = C(Q) \cap U_\ell = C(Q \cap \bar{W}_\ell) \cap U_\ell. \qquad \blacksquare$$

Now let $2 \leq \mu \leq n + 1$. Suppose that the sets T_ℓ satisfy the following condition: each μ points $y_1, \ldots, y_\mu$, $y_\ell \in \hat{T}_\ell$, are the vertices of a nondegenerate simplex in R^n. As above, $C^\mu = C(\partial \Delta^{\mu-1}) \cap \mathring{\Delta}^{\mu-1}$, c – the center of $\Delta^{\mu-1}$.

THEOREM 1: *There is a neighborhood U of z_0 such that the triple $(U, C_g(z_0) \cap U, z_0)$ is L-homeomorphic to the triple $(\mathring{\Delta}^{\mu-1} \times I^{n-\mu+1}, C^\mu \times I^{n-\mu+1}, c \times 0)$. (Where I, as above, is the interval $(0, 1)$.)*

PROOF: We shall find a Lipschitzian germ $H : (R^n, z_0) \to (\Delta^{\mu-1}, c)$ such that

(i) $H^{-1}(C^\mu) = C_g(z_0)$

(ii) $\partial H(z_0)$ is of maximal rank.

By (ii) and corollary 1, there exists a germ of L-homeomorphism

$$\varphi : (R^n, z_0) \to (\Delta^{\mu-1} \times R^{n-\mu+1}, c \times 0)$$

such that the diagram

$$\begin{tikzcd} (R^n, z_0) \arrow[r, "\varphi"] \arrow[dr, "H"'] & (\Delta^{\mu-1} \times R^{n-\mu+1}, c \times 0) \arrow[d, "\pi"] \\ & (\Delta^{\mu-1}, c) \end{tikzcd}$$

commutes. By (i) φ maps $C_g(z_0)$ on $\pi^{-1}(C^\mu) = C^\mu \times R^{n-\mu+1}$. This proves the theorem for germs. Passing to representatives we obtain the required form of the theorem.

It remains to construct the mapping H. Let us consider $R^{\mu} = \{(u_1, \ldots, u_{\mu})\}$ and the simplex $\Delta^{\mu-1} \subset R^{\mu}$ defined by $\Sigma_{\ell=1}^{\mu} u_{\ell} = 1$, $u_{\ell} \geq 0$. The central set C^{μ} consists of points $(u_1, \ldots, u_{\mu}) \in \mathring{\Delta}^{\mu-1}$, satisfying one of the relations

$$u_i = u_j \leq \min_{\ell \neq i,j} u_{\ell}, \quad i < j \leq \mu. \tag{2}$$

Now define $H : (R^n, z_0) \to (\Delta^{\mu-1}, c)$ as

$$H(z) = \left(1/\sum_{\ell=1}^{\mu} \delta_{\ell}(z)\right)(\delta_1(z), \ldots, \delta_{\mu}(z)) \in \Delta^{\mu-1} \subset R^{\mu}.$$

The property (i) of H follows from (1) and (2). In order to prove (ii) note that $H = h_1 \circ h$, where $h : (R^n, z_0) \to (R^{\mu}, d)$,

$$h(z) = (\delta_1(z), \ldots, \delta_{\mu}(z)), \quad d = (\delta(z_0), \ldots, \delta(z_0)),$$

$$h_1 : (R^{\mu}, d) \to (\Delta^{\mu-1}, c),$$

$$h_1(u_1, \ldots, u_{\mu}) = (1/\sum_{\ell=1}^{\mu} u_{\ell})(u_1, \ldots, u_{\mu}).$$

LEMMA 3: *Let* $y_1, \ldots, y_{\mu} \in T(z_0)$ *and let* $h_{(y_1, \ldots, y_{\mu})}: (R^n, z_0) \to (R^{\mu}, d)$ *be defined by* $h_{(y_1, \ldots, y_{\mu})}(z) = (\rho(z, y_1), \ldots, \rho(z, y_{\mu}))$. *Then the following conditions are equivalent*:

(*a*) $h_1 \circ h_{(y_1, \ldots, y_{\mu})}: (R^n, z_0) \to (\Delta^{\mu-1}, c)$ *is a submersion*

(*b*) $h_{(y_1, \ldots, y_{\mu})}: (R^n, z_0) \to (R^{\mu}, d)$ *is transversal to the diagonal* D *of* R^{μ}

(*c*) *the points* $y_1, \ldots, y_{\mu}$ *from a nondegenerate simplex in* R^n.

PROOF: Since the kernel of $dh_1(d)$ coincides with the diagonal D, (*a*) and (*b*) are equivalent. D is defined in R^{μ} by $u_2 - u_1 = 0, \ldots, u_{\mu} - u_1 = 0$, hence $h_{(y_1, \ldots, y_{\mu})}$ is transversal to D if and only if the vectors $\operatorname{grad}_z(\rho(z, y_{\ell}) - \rho(z, y_1))(z_0) = 2(z_0 - y_{\ell}) - 2(z_0 - y_1) = -2(y_{\ell} - y_1)$, $\ell = 1, \ldots, \mu$, are linearly independent, and this is equivalent to (*c*). ■

Now by proposition 1, $\partial H(z_0)$ is contained in the convex hull of $\partial h_1(d) \circ \partial h(z_0)$, which coincides with $dh_1(d) \circ \partial h(z_0)$, since $dh_1(d)$ is the usual differential. By proposition 2 any $\ell \in \partial h(z_0)$ is of the form $\ell = dh_{(y_1, \ldots, y_{\mu})}(z_0)$, where $y_{\ell} \in \hat{T}_{\ell}$, $\ell = 1, \ldots, \mu$. By condition of the theorem $y_1, \ldots, y_{\mu}$ form a nondegenerated simplex and hence by lemma 3 all the elements of $\partial H(z_0)$ are of maximal rank. ■

COROLLARY 2: *Suppose that for $z_0 \in C(Q)$ the set $T(z_0)$ can be decomposed into the union of two nonempty closed subsets T_1 and T_2 with $\hat{T}_1 \cap \hat{T}_2 = \emptyset$. Then for some neighborhood U of z_0 and $C_g(z_0)$ – the global part of $C(Q)$ at z_0 with respect to the partition $T(z_0) = T_1 \cup T_2$ – the triple $(U, C_g(z_0) \cap U, z_0)$ is L-homeomorphic to the triple $(D^n, D^{n-1}, 0)$ where D^q is an open unit q-dimensional disk centered at* 0.

PROOF: Since any two different points y_1, y_2 in R^n form a non-degenerated 1-simplex, conditions of theorem 1 are satisfied with $\mu = 2$, $T(z_0) = T_1 \cup T_2$. Then

$$(U, C_g(z_0) \cap U, z_0) \cong (\mathring{\Delta}^1 \times I^{n-1}, c \times I^{n-1}, c \times 0) \cong (D^n, D^{n-1}, 0). \quad \blacksquare$$

4. The global part of the central set of a generic submanifold

The precise definitions of the notions used in this and the following section, can be found e.g. in [1], [12].

Let M be a compact smooth (C^∞) k-dimensional manifold, $k \leq n-1$. The space of embeddings of M into R^n, $\mathrm{Emb}(M, R^n)$, will be considered with the Whitney topology. A set in $\mathrm{Emb}(M, R^n)$ containing a countable intersection of dense open sets is called a residual set. The residual set in $\mathrm{Emb}(M, R^n)$ is dense.

Let $J^r(M, R)$ be the space of r-jets of smooth real functions on M and ${}_sJ^r(M, R)$ – the space of s-multijets. For smooth $\psi: M \to R$ let $J^r(\psi): M \to J^r(M, R)$ and ${}_sJ^r(\psi): M^{(s)} \to {}_sJ^r(M, R)$ be the jet and s-multijet extensions of ψ.

Let $i: M \to R^n$ be the embedding. The distance function ρ can now be defined on $M: \rho^i(z, x) = |(i(x) - z|^2, z \in R^n, \quad x \in M$. As shown in [9] the central set $C(i) = C(i(M))$ coincides with the cut locus of i, defined in [11] as the set consisting of $z \in R^n$ for which $\rho^i(z, \cdot): M \to R$ has either at least two equal global minima or one degenerated global minimum.

Let $\hat{\rho}^i_s$ be the multijet extension with respect to x of $\rho^i(z, x)$:

$$\hat{\rho}^i_s = {}_sJ^r(\rho^i): M^{(s)} \times R^n \to {}_sJ^r(M, R).$$

The Transversality Theorem. ([7], see also [12], theorem B).

For any submanifold W of ${}_sJ^r(M, R)$, invariant under addition of constants, the set of embeddings $i: M \to R^n$ with $\hat{\rho}^i_s$ transversal to W, is residual.

Invariance here means that for any $(\eta_1, \ldots, \eta_s) \in W$ and $c \in R$, $(\eta_1 + c, \ldots, \eta_s + c) \in W$. ∎

Now fix some $r \geq 0$ and let $W^s \subset {}_sJ^r(M, R)$ be the submanifold consisting of multijets $(\eta_1, \ldots, \eta_s)$ with the equal values $\bar{\eta}_1 = \bar{\eta}_2 = \cdots = \bar{\eta}_s$ ($\bar{\eta} \in R$ denotes the target of the jet $\eta \in J^r(M, R)$). By the transversality theorem the set of embeddings $i: M \to R^n$ with $\hat{\rho}^i_s$ transversal to W^s, $s = 1, \ldots, n+2$, is residual.

These embeddings will be called generic in this section.

Proposition 3: *Let $i: M \to R^n$ be a generic embedding. Then for each $z \in C(i)$ the points of $T(z)$ are the vertices of a non-degenerated simplex in R^n. In particular, $\mu(z) \leq n+1$.*

Proof: Let $z_0 \in C(i)$ and let $y^0_1, \ldots, y^0_s \in T(z_0)$, $Y^0_\ell = i(x^0_\ell)$, $s \leq n+2$. The multijet $\hat{\rho}^i_s((x^0_1, \ldots, x^0_s) \times z_0) \in {}_sJ^r(M, R)$ belongs to W^s. This manifold is defined in ${}_sJ^r(M, R)$ by $\bar{\eta}_1 = \cdots = \bar{\eta}_s$ and since the target $\bar{\eta}_\ell$ of $\hat{\rho}^i_s((x_1, \ldots, x_s)xz)$ is equal to $\rho^i(z, x_\ell)$, the transversality of $\hat{\rho}^i_s$ to W^s is equivalent to the transversality of the mapping $\tilde{h}: M^{(s)} \times R^n \to R^s$, $\tilde{h}((x_1, \ldots, x_s) \times z) = (\rho^i(z, x_1), \ldots, \rho^i(z, x_s))$ to the diagonal D of R^s. Now $d\tilde{h} = (d_{x_1}\tilde{h}, \ldots, d_{x_s}\tilde{h}, d_z\tilde{h})$ and since $x^0_1, \ldots, x^0_s$ are minima of $\rho^i(z_0, \cdot)$ on M, $d_{x_\ell}\tilde{h} = 0$ at $(x^0_1, \ldots, x^0_s) \times z_0$, $\ell = 1, \ldots, s$. Hence $d_z\tilde{h}: R^n \to R^s$ is transversal to D, and since $d_zh((x^0_1, \ldots, x^0_s) \times z_0) = dh_{(y^0_1, \ldots, y^0_s)}(z_0)$, by lemma 3 the points $y^0_1, \ldots, y^0_s$ form a nondegenerated simplex in R^n. This is true for any $s \leq n+2$ points of $T(z_0)$, therefore all the points of $T(z_0)$ are the vertices of a nondegenerated simplex and their number $\mu(z) \leq n+1$. ∎

Now for $z \in C(i)$ with $\mu(z) \geq 2$ we can define the natural partition $T(z) = \{y_1\} \cup \ldots \cup \{y_{\mu(z)}\}$, satisfying conditions of theorem 1 and the global part $C_g(z)$ with respect to this partition. Thus we have

Theorem 2: *Let $i: M \to R^n$ be the generic embedding, $z \in C(i)$ and $\mu(z) = \mu \geq 2$. Let $C_g(z)$ be the global part of $C(i)$ at z. Then there exists a neighborhood U of z such that the triple $(U, C_g(z) \cap u, z)$ is L-homeomorphic to the triple $(\mathring{\Delta}^{\mu-1} \times I^{n-\mu+1}, C^\mu \times I^{n-\mu+1}, c \times 0)$.* ∎

5. The points of type C^μ_p

Let $\psi: (R^m, x) \to (R, c)$ be the germ of a smooth function. This germ has a minimum of type A_1 (A_3) at x is in some coordinate system

$(u_1, \ldots, u_m)$ at $x \in R^m$ the function ψ can be written as $\psi(u_1, \ldots, u_m) = c + u_1^2 + u_2^2 + \cdots + u_m^2$ $(\psi(u_1, \ldots, u_m) = c + u_1^4 + u_2^2 + \cdots + u_m^2$, respectively).

The minimum A_1 (or nondegenerate) of $\rho(z, \cdot)$ at $x_0 \in M$ corresponds to the first order tangency of $i(M)$ and $B(z)$ at $y_0 = i(x_0)$, and the A_3-minimum to the tangency, which is of the third order in one direction.

Let us fix some $r \geq 4$ and let W_p^s be the submanifold of ${}_sJ^r(M, R)$ consisting of multijets $(\eta_1, \ldots, \eta_s)$ such that $\bar{\eta}_1 = \cdots = \bar{\eta}_s$, some p of jets $\eta_1, \ldots, \eta_s$ are the minima of type A_3 and the rest of type A_1.

The set of embeddings $i: M \to R^n$ with $\hat{\rho}_s^i$ transversal to W_p^s, $1 \leq s \leq n+2$, $0 \leq p \leq s$, is residual, and these embeddings will be called generic. Since $W_p^s \subset W^s$, they will be generic also in the sense of section 4.

Now let us define the subset C_p^μ of $I^{2p} \times \mathring{\Delta}^{\mu-1}$ as follows: consider R^{2p} with the coordinates $x_1, x_2, \ldots, x_{2p-1}, x_{2p}$ and the cube I^{2p}: $|x_j| < \frac{1}{2}$, $j = 1, \ldots, 2p$.

Let $C^\mu \subset \mathring{\Delta}^{\mu-1}$ be as above. The complement $\mathring{\Delta}^{\mu-1} \setminus C^\mu$ is the disjoint union $\bigcup_{\ell=1}^{\mu} U_\ell$, where U_ℓ is defined in coordinates $u_1, \ldots, u_\mu$ on $\mathring{\Delta}^{\mu-1}$ by $u_\ell < \min_{j \neq \ell} u_j$. Let C_p^μ be the subset

$$I^{2p} \times C^\mu \cup \bigcup_{j=1}^{p} \{x_{2j-1} = 0, x_{2j} \geq 0\} \times U_j$$

in $I^{2p} \times \mathring{\Delta}^{\mu-1}$.

For example, $C_0^\mu = C^\mu$, C_1^2 is the plane with the attached rectangle in I^3, C_1^1 is the half-segment, and so on (Figure 1).

THEOREM 3: *Let $i: M \to R^n$ be the generic embedding. Let $z_0 \in C(i)$, $\mu = \mu(z_0)$, $T(z_0) = \{y_1^0, \ldots, y_\mu^0\}$, $y_\ell^0 = i(x_\ell^0)$. Suppose that the function $\rho^i(z_0, \cdot)$ has the A_3-minimum at $x_1^0, \ldots, x_p^0$ and the A_1-minimum at $x_{p+1}^0, \ldots, x_\mu^0$. Then $2p + \mu - 1 \leq n$ and for some neighborhood U of z_0 the quadruple $(U, C(i) \cap U, C_g(z_0) \cap U, z_0)$ is L-homeomorphic to $(I^{2p} \times \mathring{\Delta}^{\mu-1} \times I^q, C_p^\mu \times I^q, I^{2p} \times C^\mu \times I^q, 0 \times c \times o)$, where $q = n - 2p - \mu + 1$.*

PROOF: Let $\ell = 1, \ldots, p$. Consider the function $\rho^i(z, x)$ for z near z_0 and x near x_ℓ^0 as the deformation of the germ of $\rho^i(z_0, \cdot)$ at x_ℓ^0. This deformation can be induced from the universal one. Precisely this means the following (see e.g. [1], §14): let $f_{t_1, t_2}: (R^k, 0) \to (R, 0)$ be

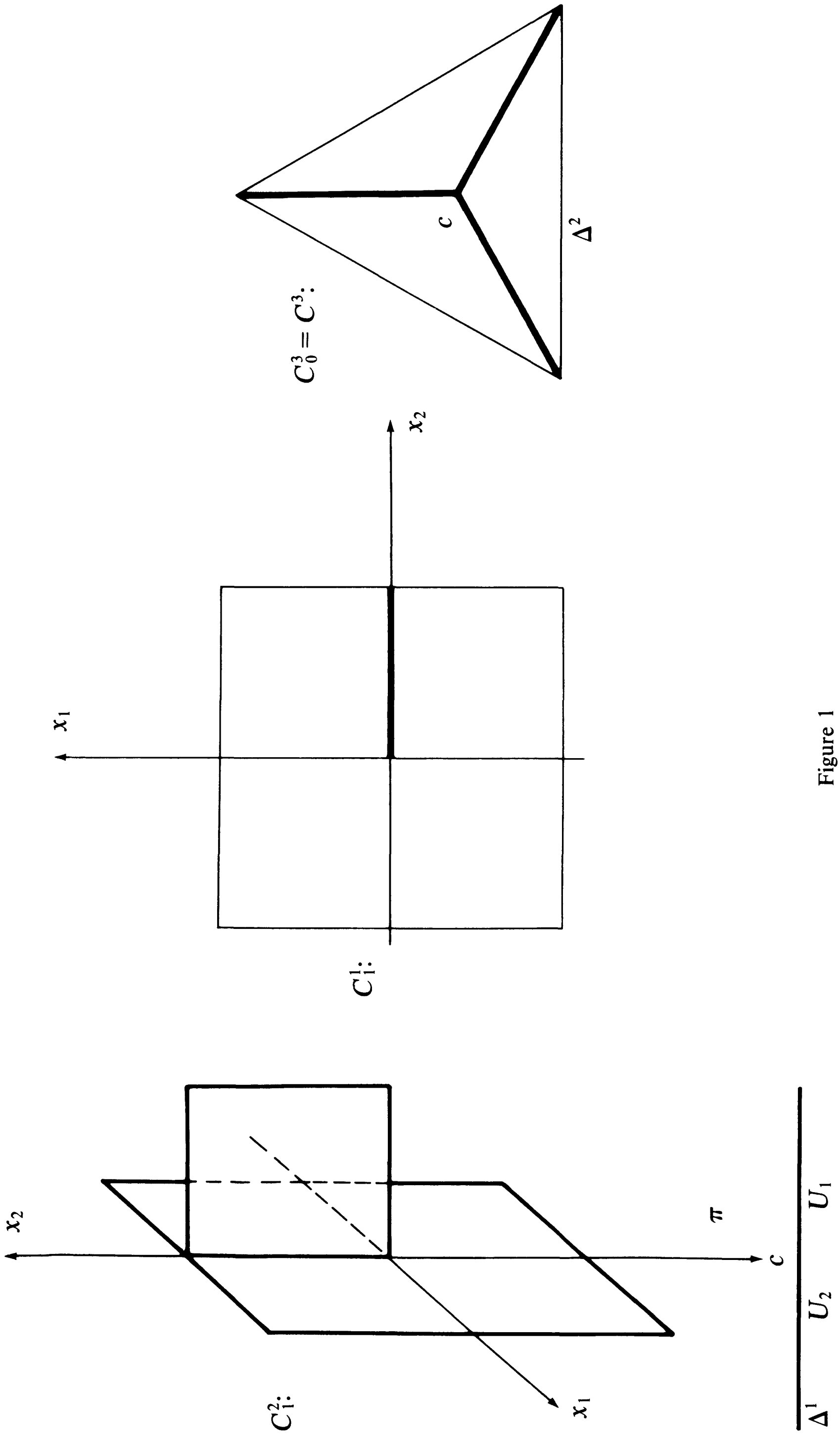

Figure 1

defined by

$$f_{t_1,t_2}(u_1, \ldots, u_k) = u_1^4 - t_2 u_1^2 + t_1 u_1 + u_2^2 + \cdots + u_k^2.$$

Then there exist

(i) two smooth functions $t_{\mathcal{U}-1}(z)$ and $t_{\mathcal{U}}(z)$ defined in some neighborhood N_ℓ of z_0, $t_{\mathcal{U}-1}(z_0) = t_{\mathcal{U}}(z_0) = 0$,

(ii) the smooth family of diffeomorphisms φ_z, $z \in N_\ell$, of some neighborhood V_ℓ of x_ℓ^0 in M on a neighborhood of $0 \in R^k$,

(iii) the smooth function $\alpha(z)$, defined in N_ℓ, such that for any $z \in N_\ell$

$$\rho^i(z, x) = f_{t_{\mathcal{U}-1}(z), t_{\mathcal{U}}(z)}(\varphi(x)) + \alpha(z), x \in V_\ell. \tag{3}$$

Choosing this representation for each $\ell = 1, \ldots, p$, we define in some neighborhood U_0 of z_0 the smooth functions $t_1, t_2, \ldots, y_{2p-1}, t_{2p}$.

Let $\tilde{h} : (R^n, z_0) \to (R^{2p} \times R^\mu)$ be defined by

$$\tilde{h}(t) = (t_1, t_2, \ldots, t_{2p-1}, t_{2p}) \times (\rho^i(z, x_1^0), \ldots, \rho^i(z, x_\mu{}^0)).$$

As in the proof of proposition 3 above it can be easily shown that the transversality of $\hat{\rho}_\mu^i$ to W_p^μ at $(x_1^0, \ldots, x_\mu^0) \times z_0$ is equivalent to the transversality of $\tilde{h}$ at z_0 to the submanifold $0 \times D \subset R^{2p} \times R^\mu$.

Now let $\delta_1(z), \ldots, \delta_\mu(z)$ be, as in section 3, the distance functions with respect to the partition $T(z_0) = \{y_1^0\} \cup \ldots \cup \{y_\mu^0\} : \delta_\ell(z) = \min_{x \in V_\ell} \rho^i(z, x)$, where V_ℓ is a sufficiently small neighborhood of x_ℓ^0 in M.

LEMMA 4: *In a neighborhood of z_0 the central set $C(i)$ consists of z, satisfying at least one of the following relations*:

$$(a_{\ell j}) : \delta_\ell(z) = \delta_j(z) \leq \min_{q \neq \ell, j} \delta_q(z), \quad \ell < j \leq \mu.$$

$$(b_\ell) : \delta_\ell(z) < \min_{j \neq \ell} \delta_j(z), \quad t_{\mathcal{U}-1}(z) = 0, \quad t_{\mathcal{U}-1}(z) \geq 0, \ell = 1, \ldots, p.$$

PROOF: Let z be near z_0. $z \in C_g(z_0) \subset C(i)$ if and only if it satisfies one of the conditions $(a_{\ell j})$. If $z \notin C_g(z_0)$ then, for some $\ell, \delta_\ell(z) < \min_{j \neq \ell} \delta_j(z)$. By lemma 1, $\min_{x \in M} \rho^1(z, x) = \delta_\ell(z)$ is attained in a neighborhood of x_ℓ^0. Now if $\ell > p$, the minimum of $\rho^i(z_0, \cdot)$ at x_ℓ^0 is non-degenerate, hence also the minimum of $\rho^i(z, \cdot)$ near x_ℓ^0 is unique and nondegenerate, i.e. $z \notin C(i)$. If $\ell \leq p$, from the representation (3) of $\rho^i(z, x)$ we see that $z \in C(i)$ if and only if the function $f_{t_{\mathscr{U}-1}, t_{\mathscr{U}}}$ has either two equal minima or one degenerated minimum, e.g. $t_{\mathscr{U}-1} = 0$, $t_{\mathscr{U}} \geq 0$. ■

Now define the Lipschitzian mapping $H:(R^n, z_0) \to (R^{2p} \times \Delta^{\mu-1}, 0 \times c)$ by

$$H(z) = (t_1, t_2, \ldots, t_{2p-1}, t_{2p}) \times \frac{1}{\sum_{\ell=1}^{\mu} \delta_\ell(z)} (\delta_1(z), \ldots, \delta_\mu(z)).$$

LEMMA 5:

(i) $H^{-1}(C_p^\mu) = C(i)$

(ii) *the generalized differential $\partial H(z_0)$ is of maximal rank.*

PROOF: (i) follows from the definition of C_p^μ and lemma 4. To prove (ii) note that as in the proof of theorem 1, $H = h_2 \circ h$, where

$$h:(R^N, z_0) \to (R^{2p} \times R^\mu, 0 \times d),$$

$$h(z) = (t_1, t_2, \ldots, t_{2p-1}, t_{2p}) \times (\delta_1(z), \ldots, \delta_\mu(z)),$$

$$h_2:(R^{2p} \times R^\mu, 0 \times d) \to (R^{2p} \times \Delta^{\mu-1}, 0 \times c),$$

$h_2 = Id \times h_1$, where h_1 as in lemma 3.

Now the minimum of $\rho^i(z_0, \cdot)$ in a neighborhood of x_ℓ^0 is attained at only point x_ℓ^0, then by proposition 2, the generalized gradient $\partial\delta_\ell(z_0) = 2(z_0 - y_\ell^0) = d_z\,\rho^i(z, x_\ell^0)(z_0)$. Then the generalized differential $\partial h(z_0)$ consists of the only linear mapping ℓ equal to the differential of $\tilde{h}$ at z_0. Since $\tilde{h}$ is transversal to $0 \times D$ at z_0, (ii) follows from lemma 3 and proposition 1. ■

The rest of the proof of theorem 3 follows as in theorem 1. ■

REMARK: Actually the L-homeomorphism constructed in theorem 3 is once differentiable at z_0 (but not at near points).

COROLLARY: *If for $z \in C(i)$ the tangency of $B(z)$ and $i(M)$ at each point of $T(z)$ is of the first order, then $C(i)$ is a neighborhood of z coincides with its global part $C_g(z) \cong C^{\mu(z)} \times I^{n-\mu(z)+1}$.*

Since the singularity C_p^{μ} of $C(i)$ arises generically only for $n \geq 2p + \mu - 1$, we can describe all the possible generic local types of $C(i)$ for small dimension n.

THEOREM 4: *For $n = 2, 3, 4$ and $i: M \to R^n$ a generic embedding, the central set $C(i)$ at any $z \in C(i)$ has one of the following types (in the sense of theorem* 3):

$$n = 2 \qquad C_0^2, C_0^3, C_1^1$$

$$n = 3 \qquad C_0^2, C_0^3, C_0^4, C_1^1, C_1^2$$

$$n = 4 \qquad C_0^2, C_0^3, C_0^4, C_0^5, C_1^1, C_1^2, C_1^3, \tilde{C}.$$

Here $\tilde{C}$ corresponds to the only global minimum of $\rho^i(z, \cdot)$ of type A_5. At such a point $C(i)$ coincides with the Maxwell set in the base of the universal deformation of the singularity A_5 (see e.g. [1], §17).

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