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THE LAPLACIAN ON ASYMPTOTICALLY FLAT MANIFOLDS AND THE SPECIFICATION OF SCALAR CURVATURE

Murray Cantor* and Dieter Brill*

Abstract

It is shown that the Laplacian on an asymptotically flat manifold is an isomorphism between certain weighted Sobolev spaces. This is used to find a necessary and sufficient condition for an asymptotically flat metric to be conformally equivalent to one with vanishing scalar curvature. This in turn is used to give an example of a metric which cannot be conformally deformed within the class of asymptotically flat metrics to one with zero scalar curvature.

Introduction

The problem of conformally deforming a Riemannian metric to achieve a specified scalar curvature has received much attention (see Kazden and Warner [12, 13] and references therein). In this paper a limited case of this problem is considered. We restrict ourselves to asymptotically flat metrics (in a sense made precise below) where the specified scalar curvature is the zero function. We find the situation is quite different than the compact case (where it is easily shown the sign of the scalar curvature is a conformal invariant) or the general open case where one has a great amount of freedom in specifying scalar curvature.

In particular, in section 2 a necessary and sufficient condition is given for when an asymptotically flat metric is conformally equivalent

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to an asymptotically flat metric with vanishing scalar curvature. This criterion is used in section 3 to give an example of a metric on $\mathbb{R}^3$ which is not deformable in the class of asymptotically flat metrics to one with vanishing curvature.

On the other hand, in section 4, it is shown that the deformation is possible as long as the scalar curvature of g is "not too negative".

The method used in this study depends on a study of the differential equation $(4(n-1)/n-2)\Delta_g\varphi - R(g)\varphi = 0$ where Δ_g is the Laplacian operator $\Delta_g\varphi = g^{ij}\varphi_{|i|j}$ and $R(g)$ is the scalar curvature of the metric g. One needs a positive solution φ which approaches 1 sufficiently rapidly at infinity. In section 1, the necessary existence and uniqueness theorems for Δ_g are established. It is also shown using perturbation techniques that it is possible to have the scalar curvature change sign (Corollary 1.8 below).

Our interest in these metrics (beyond pure mathematical) arises from various problems in general relativity. One of us (Cantor [6]) has shown that the solvability of the Lichnerowicz–York equation for asymptotically flat maximal slices is equivalent to the given metric being conformally deformable to one with vanishing scalar curvature. Our definition of asymptotically flat is consistent (when $n = 3$) with the one used in general relativity. Also, it should be noted that the example given in section 3 gives a precise proof of the physics result that one cannot have an open axi-symmetric maximal slice of a vacuum spacetime with "too much" gravitational energy (see Brill [1], Wheeler [15], Eppley [8]).

Also, Hawking [10] in his formulation of the action conjecture for Euclidean theories of gravity implicitly restricts his attention to those conformal equivalence classes of asymptotically flat metrics on $\mathbb{R}^4$ which have a scalar flat representative. It easily follows from the example given in section 3 that this does not include every conformal class of asymptotically flat metrics.

Throughout this paper we assume $n \geq 3$. Also $W^{p,s}$ is the usual space of functions whose first s partial derivatives lie in L^p.

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1. Weighted Sobolev Spaces

DEFINITION (1.1): A n-manifold M is said to have *ends* if the complement of a compact set N_0 in M may be written as the disjoint union $M - N_0 = \cup_{i=1}^{m} N_i$ where each N_i, $1 \leq i \leq n$, diffeomorphic under

φ_i to the complement of a unit ball in $\mathbb{R}^n$. Each N_i for $1 \leq i \leq m$ is called an *end.*

DEFINITION (1.2): *Let $p \geq 1$, $s \in \mathbb{N}$, $\delta \in \mathbb{R}$ and $\sigma(x) = (1+|x|^2)^{1/2}$. We say a tensor field V is of class $M^p_{s,\delta}$ if it is locally of class $W^{p,s}$ and over each end, N_ℓ, we have*

$$\sum_{|\alpha| \leq s} |D^\alpha V_{i_1,\ldots,i_j} \sigma^{\delta+|\alpha|}|_{L^p(N_\ell)} < \infty$$

where the coordinates are taken with respect to φ_ℓ.

DEFINITION (1.3): *Let p, s, δ, and σ be as in Definition (1.1). We say a metric g is of class $R^p_{s,\delta}$ if g is locally of class $W^{p,s}$ and in the coordinates given over each end*

$$\sum_{|\alpha| \leq s} |D^\alpha (g_{ij} - \delta_{ij}) \sigma^{\delta+|\alpha|}|_{L^p(N_\ell)} < \infty.$$

We can make each $M^p_{s,\delta}$ space a Banach space by setting

$$|V|_{p,s,\delta} = \sum_{\ell=0}^{n} \sum_{|\alpha| \leq s} |\bar{\sigma}^{\delta+|\alpha|} D^\alpha V|_{L^p(N_\ell)}$$

where $\bar{\sigma}$ is taken to be a C^∞ positive function which restricts to $(1+|x|^2)^{1/2}$ on each end. It is routine to check that $|\ |_{p,s,\delta}$ is a norm on $M^p_{s,\delta}$ and that the space is complete with respect to this norm.

There are various multiplicative properties of these $M^p_{s,\delta}$ spaces which we will require. These are summarized below:

LEMMA (1.4): *Let $M^p_{s,\delta}(1) = \{f : M \to \mathbb{R} : f - 1 \in M^p_{s,\delta}\}$ be given the topology such that the map $f \mapsto f+1$ is continuous. Then for $p > 1$, $s > n/p$, $0 \leq t \leq s$ pointwise multiplication induces smooth maps*

$$M^p_{s,\delta} \times M^p_{s-t,\delta+t} \to M^p_{s-t,\delta+t}$$

$$M^p_{s,\delta}(1) \times M^p_{s-t,\delta+t} \to M^p_{s-t,\delta+t}$$

$$M^p_{s,\delta}(1) \times M^p_{s-t,\delta+t}(1) \to M^p_{s-t,\delta+t}(1).$$

PROOF: These theorems are well known when $M = \mathbb{R}^n$ and $\delta \geq 0$ (see Cantor [4, 5, 7]). The extension to more general M and δ is an easy exercise. (Also see McOwen [14].) Q.E.D.

In fact, a stronger version of this lemma exists (see Cantor [7]).

LEMMA (1.5): *Let $p>1$, $s>n/p$, $\ell\geq 0$, and $\alpha>-n/p$. Then pointwise multiplication induces a continuous map*

$$M^p_{s,\delta_1}\times M^p_{\ell,\delta_2}\to M^p_{\ell,\delta_1+\delta_2+\alpha}.$$

The following theorem is essential to our analysis of scalar curvature.

THEOREM (1.6): *Let $s>n/p+2$, $p>1$, $-n/p<\delta<-2+n(1-1/p)$ and $g\in R^p_{s,\delta}$. Then $\Delta_g: M^p_{s,\delta}(M,\mathbb{R})\to M^p_{s-2,\delta+2}(M,\mathbb{R})$ is an isomorphism. Also if $R\in M^p_{s-2,\delta+2}\Delta_g+R$ has closed range.* ·

PROOF:

Step 1. Δ_g is an injection on $M^p_{0,\alpha}$ for $\alpha>-n/p$.

From distribution theory and standard regularity arguments we see if $u\in M^p_{0,\alpha}$ and $\Delta_g u=0$ then $u\in C^\alpha$. We wish to show $u(x)\to 0$ as $x\to\infty$ on each end.

Using spherical coordinates (r,θ) on each end we find $r\mapsto u^p(r,\theta)r^{p\alpha+n-1}$ must be in L^1 as a function of r. Now $p\alpha+n-1>-1$. It follows that $r\mapsto u^p(r,\theta)$ must be integrable. Since u is C^1 it follows u must vanish as $r\to\infty$ on each end. Thus we can conclude $\lim_{x\to\infty}u(x)=0$ on each end.

Suppose $\Delta_g u=0$, $u\in M^p_{0,\alpha}$ and $u\neq 0$. Then we may assume u takes an absolute maximum or minimum at some $x_0\in M$, where $|u(x_0)|>0$. However the maximum principle applies to Δ_g and thus we can conclude $u(x)=u(x_0)$ on all of M. This contradicts the fact u vanishes at the infinities. Thus Δ_g is an injection on $M_{0,\alpha}$.

Step 2. Δ_g+R has closed range.

Let $A=\Delta_g+R$. We establish the following inequality.

(1) $$|u|_{p,s',\delta}\leq C|Au|_{p,s'-2,\delta+2}$$

where $2\leq s'\leq s$ and C does not depend on u. To prove this note first that by Theorem (1.4) of Cantor [5] as extended by McOwen [14] A has closed range when restricted to functions over an end. (Also see Cantor [7].) Thus for $\ell=1,\ldots,n$ there is a C_ℓ such that if $\mathrm{supp}(u)\subset N_\ell$ we have

(2) $$|u|_{p,s',\delta}\leq C_\ell|Au|_{p,s'-2,\delta+2}.$$

Also let $\psi_0, \psi_1, \ldots, \psi_m$ be a partition of unity of M such that ψ_ℓ is supported on N_ℓ and $\psi_\ell = 1$ on all but a compact set in N_ℓ. Also we may assume ψ_0 has compact support.

Suppose there is no C so that (1) holds. Then there is a sequence $\{u_i\} \subset M^p_{s',\delta}$ such that $|u_i|_{p,s',\delta} = 1$ and $Au_i \to 0$. For each i, we have, using (2) and standard elliptic estimates

$$\begin{aligned}
|u_i|_{p,s',\delta} &\leq \sum_{\ell=0}^{n} |\psi_\ell u_i|_{p,s',\delta} \\
&\leq |\psi_0 u_i|_{p,s',\delta} + \sum_{\ell=1}^{n} |\psi_\ell u_i|_{p,s',\delta} \\
&\leq C_1 |\psi_0 u_i|_{W^{p,s'}} + \sum_{\ell=1}^{n} C_\ell |A\psi_\ell u_i|_{p,s'-2,\delta+2} \\
&\leq C_2(|A\psi_0 u_i|_{W^{p,s'-2}} + |\psi_0 u_i|_{L^p}) \\
&\quad + C_3 \sum_{\ell=1}^{n} |A\psi_\ell u_i|_{p,s'-2,\delta+2} \\
&\leq C_4(|\psi_0 A u_i|_{W^{p,s'-2}} + |A\psi_0 u_i - \psi_0 A u_i|_{W^{p,s'-2}} \\
&\quad + |\psi_0 u_i|_{L^p} + \sum_{\ell=1}^{n} (|\psi_\ell A u_i|_{p,s'-2,\delta+2} \\
&\quad + |A\psi_\ell u_i - \psi_\ell A u_i|_{p,s'-2,\delta+2})).
\end{aligned}$$

Now $\psi_0 u_i$ is bounded in $W^{p,s'}(\operatorname{supp} \psi_0)$ with $s' \geq 1$, and so by the Rellich compactness theorem we may assume by passing to a subsequence that $|\psi_0 u_i|_{L^p}$ is Cauchy. Similarly $\{A\psi_0 u_i - \psi_0 A u_i\}$ is a bounded sequence in $W^{p,s'-1}(\operatorname{supp} \psi_0)$ (the highest derivatives vanish) and so may be taken to be Cauchy in $W^{p,s'-2}$. For each ℓ, the sequence $\{A\psi_\ell u_i - \psi_\ell A u_i\}$ consists of functions all having the same compact support $B_\ell \subset M$ and so by the previous argument may be taken to be Cauchy in $W^{p,s-2}(B_\ell)$. Hence $\{u_i\}$ has a Cauchy subsequence in $M^p_{s-2,\delta+2}$. Finally since $\psi_i \in C^\infty$ it is clear that $|\psi_0 A u_i|$ and $|\psi_\ell A u_i|$ go to zero and hence are Cauchy. It follows that by passing to a subsequence we may assume $\{u_i\}$ is Cauchy in $M^p_{s,\delta}$ and so $u_i \to u \in M^p_{s,\delta}$. By continuity $A\bar{u} = 0$, but $|\bar{u}|_{p,s,\delta} = 1$. This contradicts uniqueness and the inequality is established.

It follows immediately that for $2 \leq s' \leq s$, $A: M^p_{s',\delta} \to M^p_{s'-2,\delta+2}$ has closed range. We now use the following well-known result on operators between Banach spaces (see Kato [11], Theorem 5.13).

LEMMA: *Let E and F be reflexive Banach spaces and $L: E \to F$ a bounded operator with closed range. Let $L^*: F^* \to E^*$ be the adjoint. Then L is onto iff L^* is injective.*

Step 3. $\Delta_g : M^p_{s,\delta} \to M^p_{s-2,\delta+2}$ is onto.

We first will show $\Delta_g : M^p_{2,\delta} \to M^p_{0,\delta+2}$ is onto. We already know the range of Δ_g is closed in $M^p_{0,\delta+2}$.

Using the coordinate formula for $d\mu_g$, the volume form determined by g, we see that $\mu \in M^p_{0,\delta}$ if and only if

$$\int_M |f(p)\sigma^\delta|^p \, d\mu_g < \infty.$$

It is easily shown that $(M^p_{0,\delta+2})^* = M^{p'}_{0,-(\delta+2)}$ where $1/p + 1/p' = 1$. Now since for $v, w \in M^p_{2,\delta}$

$$\int_M w\Delta_g v \, d\mu_g = \int_M v\Delta_g w \, d\mu_g$$

we have $(\Delta_g)^* = \Delta_g$. Thus Δ_g is the surjection onto $M^p_{0,\delta+2}$ if and only if it is an injection on $M^{p'}_{0,-(\delta+2)}$. Thus, from step 1, we need check that $-(\delta+2) > -n/p'$. This follows easily from the assumption $\delta < -2 + n(1 - 1/p)$.

To conclude let $f \in M^p_{s-2,\delta+2} \subset M^p_{0,\delta+2}$. We know there is a $u \in M^p_{2,\delta}$ with $\Delta_g u = f$. We need to check $u \in M^p_{s,\delta}$. This follows from inequality (1). Q.E.D.

The following result is a modest extension of one of Fischer and Marsden [9]. They establish the theorem for $M = \mathbb{R}^n$.

THEOREM (1.7): *Let p, s, δ and g be as in Theorem* (1.6). *Suppose further that $R(g) = 0$. Then there is an $\epsilon > 0$ such that if $|\bar{R}|_{p,s-2,\delta+2} < \epsilon$ than g is conformally equivalent to an asymptotically flat metric $\bar{g}$ with scalar curvature $\bar{R}$.*

PROOF: Setting $\bar{g} = \varphi^{4/(n-2)}g$. We need to solve

$$(3) \quad \begin{cases} \dfrac{4(n-1)}{n-2}\Delta_g\varphi + \bar{R}\varphi^{(n+2)/(n-2)} = 0 \\ \varphi - 1 \in M^p_{s,\delta} \\ \varphi > 0. \end{cases}$$

We do this using the implicit function theorem. Set

$$\Phi : M^p_{s,\delta}(1) \times M^p_{s-2,\delta+2} \to M^p_{s-2,\delta+2}$$

$$\Phi(\varphi, \bar{R}) = \frac{4(n-1)}{n-2}\Delta_g\varphi + \bar{R}\varphi^{(n+2)/(n-2)}.$$

Using lemmata 4 and 5 we see Φ is smooth. Also note $\Phi(1,0)=0$. By the implicit function theorem the results follows if

$$D_1\Phi(1,0): M^p_{s,\delta} \to M^p_{s-2,\delta+2}$$

is an isomorphism. However $D_1\Phi(1,0) = 4(n-1)/(n-2)\,\Delta_g$ which is an isomorphism by the previous theorem.

The fact φ is positive follows from continuity of the solution and the Sobolev embedding theorem. Q.E.D.

COROLLARY (1.8): *Let p, s, and δ be as Theorem* (1.6). *Suppose M has a metric $g \in R^p_{s,\delta}$ such that $R(g)=0$. Then there exist two conformally equivalent asymptotically flat metrics,* g_1 and g_2, on M such that $R(g_1)>0$, $R(g_2)<0$.

PROOF: Let $\epsilon>0$ be given as in Theorem (1.7). Let $R_1>0$ with $|R_1|_{p,s-2,\delta+2}<\epsilon$ and $R_2<0$ with $|R_2|_{p,s-2,\delta+2}<\epsilon$. Let φ_i be the solutions of $4(n-1)/(n-2)\Delta_g\varphi + R_i\varphi^{(n+2)/(n-1)}=0$ guaranteed by the proof of Theorem (1.7). Set $g_i = \varphi_i^{4/(n-2)}g$. Q.E.D.

2. Conformal deformation to zero scalar curvature

In this section we present a necessary and sufficient condition for an asymptotically flat metric to be conformally equivalent to another asymptotically flat metric with zero scalar curvature. In what follows $R(g)$ is the scalar curvature of g, and $\langle\ ,\ \rangle_g$ is the inner product given by g.

THEOREM (2.1): *Let* $1<p<2n/(n-2)$, $-n/p<\delta<-2+n(1-1/p)$, $s>n/p+2$, *and* $g\in R^p_{s,\delta}$. *The following are equivalent:*

(I) *For all* $f\in C^\infty_0$, $f\neq 0$, *we have*

$$-\int R(g)f^2\,d\mu_g < \frac{4(n-1)}{n-2}\int_M \langle\nabla_g f, \nabla_g f\rangle_g\,d\mu_g.$$

(II) *There is a* $\bar g \in R^p_{s,\delta}$ such that $\bar g$ is conformally equivalent to g and $R(\bar g)=0$.

PROOF: We will write $\bar g = \varphi^{4/(n-2)}g$. It is well known that statement (II) is equivalent to finding a solution of the following problem (see

Kazden and Warner [12]):

$$
(4)\qquad \begin{cases} \dfrac{4(n-1)}{n-2}\Delta_g\varphi - R(g)\varphi = 0 \\ \varphi > 0 \\ \varphi - 1 \in M^p_{s,\delta} \end{cases}
$$

Using Lemma (1.4) one can show that $\varphi - 1 \in M^p_{s,\delta}$ insures that $\bar{g} \in M^p_{s,\delta}$. Thus we need show that (4) has a solution if and only if (I) is satisfied.

Let us assume (4) has a solution φ. Let $f \in C_0^\infty$. Since $\varphi > 0$ we may write $f = \varphi u$ where $u \in C_0^\infty$. Now

$$\nabla_g f = (\nabla_g\varphi)u + \varphi\nabla_g u$$

and

$$
\begin{aligned}
\langle\nabla_g f, \nabla_g f\rangle_g &= u^2\langle\nabla_g\varphi, \nabla_g\varphi\rangle_g + 2\varphi u\langle\nabla_g\varphi, \nabla_g u\rangle_g + \varphi^2\langle\nabla_g u, \nabla_g u\rangle \\
&= u^2\langle\nabla_g\varphi, \nabla_g\varphi\rangle_g + \langle\varphi\nabla_g\varphi, \nabla_g u^2\rangle + \varphi^2\langle\nabla_g u, \nabla_g u\rangle_g.
\end{aligned}
$$

Thus, integrating by parts, we find

$$
\begin{aligned}
\int_M \langle\nabla_g f, \nabla_g f\rangle_g \, d\mu_g &= \int_M (u^2\langle\nabla_g\varphi, \nabla_g\varphi\rangle - \operatorname{div}_g(\varphi\nabla_g\varphi)u^2 \\
&\quad + \cdots + \varphi^2\langle\nabla_g u, \nabla_g u\rangle_g) \, d\mu_g \\
&= \int_M \Big(u^2\langle\nabla_g\varphi, \nabla_g\varphi\rangle - u^2\langle\nabla_g\varphi, \nabla_g\varphi\rangle \\
&\quad - \int u^2\varphi\Delta_g\varphi + \varphi^2\langle\nabla_g u, \nabla_g u\rangle_g) \, d\mu_g.
\end{aligned}
$$

Hence, since $\varphi > 0$, if $f \neq 0$ we have

$$
(5)\qquad \int_M \langle\nabla_g f, \nabla_g f\rangle_g \, d\mu_g > -\int_M u^2\varphi\Delta_g\varphi \, d\mu_g.
$$

Using the fact that φ satisfies (4) we get

$$
\begin{aligned}
\frac{4(n-1)}{n-2}\int_M \langle\nabla_g f, \nabla_g f\rangle_g \, d\mu_g &> -\int_M u^2\varphi^2 R(g) \, d\mu_g \\
&> -\int_M R(g) f^2 \, d\mu_g.
\end{aligned}
$$

We now show that (I) implies (II). Let $A_\lambda = \Delta_g - \lambda R$. We will first show that for $\lambda \in [0, 1]$, if (I) holds then A_λ is an injection on $M^p_{s',\alpha}$ with $s' \geq s$ and $\alpha > -n/p$. Note since $p < 2n/(n-1)$, then if $u \in M^p_{s',\alpha}$, $\nabla_g u \in L^2$ (see Cantor [5, 6]).

Suppose $u \in M^p_{s,\alpha}$ and $A_\lambda u = 0$. Let $\{u_i\} \subset C_0^\infty$ with $u_i \to u$ in $M^p_{s,\alpha}$. Then for each i,

$$\int_M \left(\frac{4(n-1)}{(n-2)} u_i \Delta_g u - \lambda R(g) u_i u\right) \mathrm{d}u_g = 0.$$

Integration by parts yields

$$-\frac{4(n-1)}{n-2} \int \langle \nabla_g u_i, \nabla_g u \rangle \,\mathrm{d}\mu_g = \lambda \int R(g) u_i u \,\mathrm{d}\mu_g. \tag{6}$$

Now since $u_i \to u$ in $M^p_{s,\delta}$ we have $\nabla_g u_i \to \nabla_g u$ in L^2, and using standard Sobolev theorems $u_i \to u$ uniformly. Also we may assume for each $x \in M$ $|u_i(x)| < 2|u(x)|$. Since $R(g) \in M^p_{s-2,\delta+2}$ one may show that $R(g)u^2 \in L^1$. Thus using the dominated convergence theorem we may take the limit on both sides of (6) and conclude that

$$\frac{4(n-1)}{n-2} \int_M \langle \nabla_g u, \nabla_g u \rangle \,\mathrm{d}\mu_g = -\lambda \int_M R(g) u^2 \,\mathrm{d}\mu_g. \tag{7}$$

If $\lambda = 0$ we see $\int_M \langle \nabla_g u, \nabla_g u \rangle \,\mathrm{d}\mu_g = 0$. It follows $u = 0$.

If $\lambda \neq 0$ one may show using (I) and (7) that

$$\begin{aligned} &\frac{4(n-1)}{(n-2)} \left(\frac{1}{\lambda}\right) \int_M \langle \nabla_g u, \nabla_g u \rangle \,\mathrm{d}\mu_g \\ &= -\int_M R(g) u^2 \,\mathrm{d}\mu_g < \frac{4(n-1)}{n-2} \int_M \langle \nabla_g u, \nabla_g u \rangle \,\mathrm{d}\mu_g. \end{aligned} \tag{8}$$

If $u \neq 0$ it follows from (8) that $\lambda > 1$, which contradicts our assumption that $\lambda \leq 1$ and so in all cases $u = 0$.

We wish to show A_1 is an onto map. To do this we use the following well known lemma, usually called the continuity method. (See Cantor [5].)

LEMMA: *Let E, F be Banach spaces and for $\lambda \in [0, 1]$, $\lambda \to L_\lambda$ is a continuous family of bounded linear operators from E to F. Suppose L_0 is an isomorphism and that for each $\lambda \in [0, 1]$, L_λ is an injection with closed range. Then L_λ is an isomorphism for all $\lambda \in [0, 1]$.*

We apply this lemma to $A_\lambda : M^p_{s,\delta} \to M^p_{s-2,\delta+2}$. From Theorem (1.6) we see the hypotheses of the lemma are satisfied.

Now to solve (4), write $\varphi = \bar{\varphi} + 1$. To find $\bar{\varphi}$ we have to solve

$$\frac{4(n-1)}{n-2}\Delta_g\bar{\varphi} - R(g)(\bar{\varphi}) = R(g) \qquad \bar{\varphi} \in M^p_{s,\delta}.$$

However since $R(g) \in M^p_{s-2,\delta+2}$, it follows from the above result that $\bar{\varphi}$ exists and is unique.

To finish we need show our solution φ to (4) is positive. We know, in fact, for $\lambda \in [0, 1]$ there is a φ_λ such that $4(n-1)/(n-2)\Delta_g\varphi_\lambda - \lambda R\varphi_\lambda = 0$ and $\varphi_\lambda - 1 \in M^p_{s,\delta}$. Also φ_λ depends continuously on λ in the C^0 topology. Note that $\varphi_0 = 1$ and so if any φ_λ has a zero there is a $\lambda_0 \in [0, 1]$ such that $\varphi_{\lambda_0} \geq 0$ and there is an $x_0 \in M$ such that $\varphi_{\lambda_0}(x_0) = 0$. At this point we have $\Delta_g\varphi_{\lambda_0}(x_0) = \lambda_0 R(g)\varphi_{\lambda_0}(x_0) = 0$. It is well known that this is impossible (see, for example, Cantor [6]). Thus $\varphi_\lambda > 0$ for all λ and in particular $\varphi = \varphi_1$ is positive. Q.E.D.

3. An example

In this section we present examples of asymptotically flat metrics which are not conformally deformable to zero curvature. Let $M = R^3$ be described in cylindrical coordinates r, z, θ, and let the metric have the axially symmetric form

$$g = e^{2Aq}(dr^2 + dz^2) + r^2\, d\theta^2. \tag{9}$$

Here A is a constant and q is an arbitrary function of r and z, except for the following conditions: For regularity at the origin we require $q = 0 = q_r$ at the origin. For simplicity we further assume that $q \in C^\infty_0$ and has compact support. In this case the scalar curvature is given by (Brill [1959])

$$R = 2A\, e^{-2Aq}(q_{rr} + q_{zz}).$$

Now let $\varphi = Aq$. Then

$$\langle \nabla_g\varphi, \nabla_g\varphi \rangle = e^{-2Aq}A^2(q_r^2 + q_z^2).$$

If condition (I) of Theorem (2.1) holds, then the inequality would hold in particular for $f = \varphi$. But we wish to show that for A sufficiently large,

$$-\int R\varphi^2\,\mathrm{d}\mu_g > 8\int \langle\nabla_g\varphi, \nabla_g\varphi\rangle\,\mathrm{d}\mu_g,$$

that is

$$0 > \int (8\langle\nabla_g\varphi, \nabla_g\varphi\rangle + R\varphi^2)\,\mathrm{d}\mu_g \equiv I.$$

Now we compute

$$I = \int [8A^2\,\mathrm{e}^{-2Aq}(q_r^2 + q_z^2) + 2A^3\,\mathrm{e}^{-2Aq}(q_{rr} + q_{zz})q^2]\,\mathrm{e}^{2Aq} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta.$$

Integration by parts of the second term in the bracket gives

$$I/2\pi = \int A^2(q_r^2 + q_z^2)(8 - 4A^3 q)r\,\mathrm{d}r\,\mathrm{d}z - \int 2A^3 q^2 q_r\,\mathrm{d}r\,\mathrm{d}z.$$

The last integral vanishes because the integrand is the (two-dimensional) divergence of $(2/3)A^2q^3\partial/\partial r$, and q has compact support. The first integral can clearly be made negative by choosing A sufficiently large. It then follows that $I < 0$. Thus we may conclude that for A sufficiently large, metrics of form (9) may not be conformally deformed to an asymptotically flat metric with zero scalar curvature.

Also note that violation of condition (1) of Theorem (2.1) is an open condition in $R^p_{s,\delta}$. Hence there is a non-empty open set of asymptotically flat metrics on $\mathbb{R}^n$ which are not deformable to asymptotically flat scalar flat metrics.

4. The $|R(g)|^{n/2}$ criterion

It follows immediately from Theorem (2.1) that if g is asymptotically flat and $R(g) \geq 0$ then g is conformally equivalent to an asymptotically flat metric with zero scalar curvature. This result is already known (see Cantor [6]). However, it also follows from Theorem (2.1) that the condition that $R(g) \geq 0$ may be weakened to allow some negative scalar curvature. We first prove a Sobolev-type lemma.

LEMMA (4.1): *Let M be a manifold with ends. Let g be a Riemannian metric on M such that if g_{ij} is the coordinate expression g, $g_{ij} - \delta_{ij}$*

is bounded on each end and for each end there is a $B > 0$ *such that*

$$\sum g_{ij}\xi^i\xi^j \geq B \sum (\xi^i)^2$$

for all $\xi \in \mathbb{R}^n$. *Then if* $q > 1$ *and* $1/p = 1/q - 1/n$, *there is a* $C > 0$ *such that for all* $f \in C_0^\infty(M, \mathbb{R})$, *we have*

$$\left(\int_M |f|^p \, d\mu_g\right)^{1/p} \leq C \left(\int_M \langle \nabla_g f, \nabla_g f \rangle^{q/2} \, d\mu_g\right)^{1/q}. \tag{10}$$

PROOF: For convenience, denote the left hand side of (10) by $|f|_p$ and the right hand side by $|\nabla_g f|_g$. These are both norm functions on C_0^∞. Also we will denote all large constants by C.

Now let $\psi_0, \ldots, \psi_m$ be the partition of unity defined in the proof of Theorem (1.6). As shown in Cantor [2] there is a C such that for each i

$$|\psi_i f|_p \leq C|\nabla_g \psi_i f|_q.$$

Thus for $f \in C_0^\infty$,

$$|f|_p \leq \sum |\psi_i f|_p \leq C \sum |\nabla_g \psi_i f|_q$$

$$\leq C \sum (|\nabla_g \psi_i) f|_q + |\psi_i \nabla_g f|_q).$$

Now, we may use the Poincaré inequality on the compact set containing the union of the supports of the $\nabla_g \psi_i$'s to establish

$$\sum |\nabla_g \psi_i f|_q \leq mC|\nabla_g f|_g.$$

Also

$$|\psi_i \nabla_g f|_q \leq |\nabla_g f|_q.$$

Thus

$$|f|_p \leq C|\nabla_g f|_q. \qquad \text{Q.E.D.}$$

THEOREM (4.2): *Let g be an asymptotically flat metric on M satisfying the hypotheses of Theorem* (2.1). *Then if C is the constant of Lemma* (4.1), *and if*

$$\left(\int_{R(g) \leq 0} |R(g)|^{n/2} \, d\mu_g\right)^{2/n} \leq \frac{4(n-1)}{C(n-2)},$$

g is conformally equivalent to an asymptotically flat metric with zero scalar curvature.

PROOF: We show that under the hypothesis of this theorem that (I) of Theorem (2.1) is satisfied. Let $f \in C_0^\infty$, then using Hölder's inequality

$$-\int_M R(g)f^2 \,d\mu_g \le -\int_{R(g)\le 0} Rf^2 \,d\mu_g \le \left(\int_{R(g)\le 0} |R(g)|^{n/2} \,d\mu_g\right)^{2/n} \times \left(\int_M f^{2n/(n-2)} \,d\mu_g\right)^{(n-2)/n}.$$

Note we have used the fact that the integral over $\{x \in M : R(g)(x) \le 0\}$ of $|f|^{2n/n-2}$ is dominated by the integral over M. We now apply Lemma (4.1) to conclude

$$-\int_M R(g)f^2 \le \left(\int_{R(g)\le 0} |R(g)|^{n/2} \,d\mu_g\right)^{2/n} C \int_M \langle \nabla_g f, \nabla_g f\rangle \,d\mu_g.$$

Thus condition (I) of Theorem (2.1) is satisfied if

$$C\left(\int |R(g)|^{n/2} \,d\mu_g\right)^{2/n} \le \frac{4(n-1)}{n-2}$$

which holds by hypothesis. Q.E.D.

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