COMPOSITIO MATHEMATICA

REINHARD SCHULTZ

Correction to the paper "Compact fiberings of homogeneous spaces. I"

Compositio Mathematica, tome 43, nº 3 (1981), p. 419-421

<http://www.numdam.org/item?id=CM_1981__43_3_419_0>

COMPOSITIO MATHEMATICA, Vol. 43, Fasc. 3, 1981, pag. 419–421.
Sijthoff & Noordhoff International Publishers–Alphen aan den Rijn
Printed in the Netherlands

CORRECTION TO THE PAPER

"COMPACT FIBERINGS OF HOMOGENEOUS SPACES. I"

Reinhard Schultz

It was pointed out by R. Stong that the methods of [I] do not apply to the oriented Grassmann manifolds $G^+_{8,2}(\mathbb{R})$ and $G^+_{12,2}(\mathbb{R})$, and in fact $G^+_{8,2}(\mathbb{R})$ fibers over S^6 with fiber $\mathbb{C}P^3$; there is an analogous fibering of the unoriented Grassmann manifold $G_{8,2}(\mathbb{R})$ over $\mathbb{R}P^6$ with the same fiber. The fallacy in [I] is that condition (6.6) requires $n \neq 3,5$ (compare the statement of Bertrand's Hypothesis in [I, §5]). Upon reflection it is apparent that these fiberings are consistent with the principal conjecture in [I]; specifically, they come from the fact that Spin_7 acts transitively on the Stiefel manifold $V_{8,2}(\mathbb{R})$ via the spinor representation $\mathrm{Spin}_7 \to SO_8$ (in fact, the induced action on $V_{8,3}(\mathbb{R})$ is also transitive). Stong has indicated a more direct description of this fibering.

In contrast, the manifold $G^+_{12,2}(\mathbb{R})$ is indeed connectedwise prime, and we shall verify this here. We adopt notation from [I] as needed. The proof of [I, Theorem 6.1] implies that the question reduces to considering compact fiberings $F \to G^+_{12,2}(\mathbb{R}) \to B$ with B a 1-connected $\mathbb{Z}\,[6^{-1}]$ cohomology 10-sphere. The idea is to construct an associated compact fibering of $V_{12,2}(\mathbb{R})$ over B. Specifically, if $\hat{F}$ is the principal SO_2-bundle over F classified by the composite $F \to G^+_{12,2}(\mathbb{R}) \to BSO_2$, then the sequence

$$\hat{F} \longrightarrow V_{12,2}(\mathbb{R}) \longrightarrow B \tag{1}$$

is exact.

0010-437X/81/060419-03$00.20/0

The first step in providing $G^+_{12,2}(\mathbb{R})$ is connectedwise prime is to show that B *is actually a* $\mathbb{Z}_{(3)}$*-homology sphere*. Given this, it is not difficult to modify the argument for $n = 5$.

The analysis of B begins with the observation that the boundary homomorphism $\partial_3: \pi_3(B) \to \pi_2(F)$ is zero by a result of S. Weingram [55, §3]. But ∂_3 is an isomorphism since $V_{12,2}(\mathbb{R})$ is highly connected, and therefore B and $\hat{F}$ are 2- and 3-connected respectively (by [I, 4.2] we already knew that they were 1- and 2-connected).

To shorten notation, set $V_i = H^i(\hat{F}; \mathbb{Z}_3)$ and $W_j = H^j(B; \mathbb{Z}_3)$. Then $V_i \otimes W_j = E_2^{i,j}$ in the $\mathbb{Z}_3$ Serre spectral sequence for (1). Our connectivity assumptions and Poincaré duality yield the following information:

$$W_0 = V_0 = W_{10} = V_{11} = \mathbb{Z}_3,$$

$$V_1 = V_2 = V_9 = V_{10} = 0,$$

$$W_1 = W_2 = W_3 = W_7 = W_8 = W_9 = 0,$$

$$V_3 = V_8,\ V_4 = V_7,\ V_5 = V_6,$$

$$W_4 = W_6,\ \dim W_5 \equiv 0(2).$$

Thus there are only five unknown dimensions. From the connectivity conditions and the Serre spectral sequence we have $V_3 = W_4$, $V_4 = W_5$, $V_5 = W_6$. Further inspection of the Serre spectral sequence shows $V_6 = V_3 \otimes W_4$ and $V_7 = (V_3 \otimes W_5) \oplus (V_4 \oplus W_4)$; the latter requires an observation that $d_2^{4,4} = 0$ by the multiplicative properties of the Serre spectral sequence. If we combine all this information, we obtain the following equation:

$$\dim V_4 = 2 \dim V_4 \dim V_3. \tag{2}$$

This has an integral solution only if $0 = \dim V_4 = \dim W_5$. But B is a rational homology sphere. Therefore, if W_4 were the first nonzero $\mathbb{Z}_3$ cohomology group in positive degree (we know nothing lower is), Bockstein considerations would imply $W_5 \neq 0$ also. This means $0 = W_4 = W_5 = W_6$, or B is a $\mathbb{Z}_3$ (hence $\mathbb{Z}_{(3)}$) homology 10-sphere. From our formulas it also follows that $\hat{F}$ is a $\mathbb{Z}_{(3)}$ homology 11-sphere.

This brings us to the final step. Let $S^{11}_{(3)} \to E' \to S^{10}_{(3)}$ be the localization of (1) at 3. It is immediate from obstruction theory that this fibration has a cross section. Hence the localized fibration $F_{(3)} \to G^+_{12,2}(\mathbb{R})_{(3)} \to B_{(3)}$ also has a cross section. If one uses this 3-local cross section in place of the transfer and sets $p = 3$, then the argument in the last paragraph of the proof of [I, 6.1] goes through word for word. ■

I am grateful to R. Stong for pointing out my mistake.

REFERENCES

[1] R. SCHULTZ: Compact fiberings of homogeneous spaces. I. *Comp. Math. 43* (1981) 181–215.

[55] S. WEINGRAM, On the incompressibility of certain maps. *Ann. of Math. 93* (1971) 476–485.