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ENTIRE FUNCTIONS ON LOCALLY CONVEX SPACES AND CONVOLUTION OPERATORS

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Dedicated to the memory of Aldo Andreotti

Abstract

In this work we generalize the classical results on approximation and existence of solutions of convolution equations in $\mathcal{H}(\mathbb{C}^n)$. We introduce the spaces $\mathcal{H}_{SNb}(E)$ and $\mathcal{H}_{Nb}(E)$ of the nuclearly Silva entire functions of bounded type and of the nuclearly entire functions of bounded type in a complex locally convex space E. These spaces are endowed with natural locally convex topologies. Convolution equations are considered in these space and results of approximation for solutions of homogeneous convolution equations are proved for any E. Results of existence are demonstrated for a more restrictive class of locally convex spaces which includes the DF-spaces. These results generalize theorems of Gupta and Matos. We also introduce the spaces $\mathcal{H}_N(E)$ of the nuclearly entire functions and $\mathcal{H}_{SN}(E)$ of the nuclearly Silva entire functions. For these spaces we get results of approximation for solutions of homogeneous convolution equations, thus generalizing theorems of Gupta–Nachbin.

1. Introduction

In this work we generalize the classical results on approximation and existence of solutions for convolution equations in $\mathcal{H}(\mathbb{C}^n)$ (see [57]). We introduce the spaces $\mathcal{H}_{SNb}(E)$ and $\mathcal{H}_{Nb}(E)$ of the Silva nuclearly entire functions of bounded type and of nuclearly entire

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functions of bounded type in a complex locally convex space E. These spaces are endowed with natural locally convex topologies. Convolution equations are considered in these spaces and results of approximation for solutions of homogeneous convolution equations are proved for any E. Results of existence are demonstrated for a more restrictive class of locally convex spaces which includes the DF-spaces. These results generalize theorems of Gupta [48], [49] and Matos [51], [52]. We also introduce the spaces $\mathcal{H}_N(E)$ of the nuclearly entire functions and $\mathcal{H}_{SN}(E)$ of the nuclearly Silva entire functions. For these spaces we get results of approximation for solutions of homogeneous convolution equations, thus generalizing theorems of Gupta–Nachbin [50]. We remark that in the construction of the spaces $\mathcal{H}_{Nb}(E)$ and $\mathcal{H}_N(E)$ a fundamental role is played by the strong topology on the continuous dual E' of E. More precisely, a fundamental role is played by the Von Neumann bornology of E. A careful examination of all the results by the reader will convince himself that similar spaces and theorems may be obtained by taking any polar topology on E'. (More precisely, by considering any bornology in E). It is instructive to check which kind of "nuclear" entire functions arise from the consideration of these different polar topologies in E'.

Several authors have been working with topics closely related to the subject of this article. In the bibliography of this work we hope to have listed most of the research papers connected in some way with the infinite dimensional theory of convolution equations. We ask our excuses to those authors whose articles we might have left out of this list.

The idea of this paper is in our minds since 1970, but from postponement to postponement we have delayed for a long time the writing of this manuscript with details. The motivation behind this paper leads to the Silva-Holomorphy types (see [7] and [56]).

2. Silva nuclear and nuclear multilinear functions and polynomials

Let $E = \varinjlim_{i \in I} E_i$ be a bornological vector space over $\mathbb{C}$. We say that a subset B of E belongs to $\mathcal{B}_E$ is there is $i \in I$ such that B is a closed balanced bounded subset of the normed space E_i. For $M = 1, 2, \ldots$ we may consider the cartesian product $E^m = E \times \cdots \times E$ (m times) with the natural bornology induced by the bornology of E. In this case we may take the vector space $\mathcal{L}_b(^mE)$ of all m-linear complex mappings on E^m which are bounded over each element of $\mathcal{B}_{E^m}$. In $\mathcal{L}_b({}^mE)$ we consider the locally convex topology of the uniform convergence over the elements of $\mathcal{B}_{E^m}$. For $m=0$ we set $\mathcal{L}_b({}^0E)$ as the complex plane with its usual topology. We note that $E^* = \mathcal{L}_b({}^1E)$. If $m \in \mathbb{N}$ and $A \in \mathcal{L}_b({}^mE)$ we consider the function $\hat{A}: E \to \mathbb{C}$ given by $\hat{A}(x) = A(x, \ldots, x)$ (m times) for every x in E (for $m=0$ this function is the constant function $\hat{A}(x) = A$ for all $x \in E$). The vector space of all functions $\hat{A}$, as A varies in $\mathcal{L}_b({}^mE)$, is denoted by $\mathcal{P}_b({}^mE)$ and we consider on it the locally convex topology generated by the seminorms

$$\|\hat{A}\|_B = \sup\{|\hat{A}(x)|;\, x \in B\}$$

with B varying in $\mathcal{B}_E$. If $\mathcal{L}_{bs}({}^mE)$ denotes the vector subspace of $\mathcal{L}_b({}^mE)$ formed by all symmetric functions, then the natural mapping $A \mapsto \hat{A}$ gives an isomorphism between $\mathcal{L}_{bs}({}^mE)$ and $\mathcal{P}_b({}^mE)$ which is a homeomorphism if we consider the relative topology in $\mathcal{L}_{bs}({}^mE)$. If $m = 1, 2, \ldots$ and $\varphi_1, \ldots, \varphi_m \in E^*$ then $\varphi_1 \times \cdots \times \varphi_m$ denotes the element of $\mathcal{L}_b({}^mE)$ given by $\varphi_1 \times \cdots \times \varphi_m(x_1, \ldots, x_m) = \varphi_1(x_1) \ldots \varphi_m(x_m)$. If $\varphi_1 = \cdots = \varphi_m = \varphi$ we denote such function by φ^m. Let $\mathcal{L}_{bf}({}^mE)$ be the vector subspace of $\mathcal{L}_b({}^mE)$ generated by all functions $\varphi_1 \times \cdots \times \varphi_m$ with $\varphi_1, \ldots, \varphi_m \in E^*$. We set $\mathcal{L}_{bfs}({}^mE) = \mathcal{L}_{bs}({}^mE) \cap \mathcal{L}_{bf}({}^mE)$ and $\mathcal{L}_{bf}({}^0E) = \mathcal{L}_{bfs}({}^0E) = \mathbb{C}$. Let $\mathcal{P}_{bf}({}^mE)$ be the corresponding subspace of $\mathcal{P}_b({}^mE)$ which is isomorphic to $\mathcal{L}_{bfs}({}^mE)$, $m \in \mathbb{N}$. It is easy to show that $\mathcal{P}_{bf}({}^mE)$ is the set of elements P of $\mathcal{P}_b({}^mE)$ which can be written in the form $P = \sum_{j=1}^{n} \widehat{(\varphi_j)^m}$, $\varphi_j \in E^*$ for $j = 1, \ldots, n$. If $m = 1, 2, \ldots$ and $(E^*)^m$ denotes the topological cartesian product, we have the continuous m-linear mapping:

$$\alpha_m : (E^*)^m \to \mathcal{L}_b({}^mE)$$

$$(\varphi_1, \ldots, \varphi_m) \to \varphi_1 \times \cdots \times \varphi_m$$

Thus there is a unique continuous linear mapping β_m from the projective tensor product $E^* \otimes_\pi \cdots \otimes_\pi E^*$ (m times) into $\mathcal{L}_b({}^mE)$ such that $\alpha_m = \beta_m \circ \gamma_m$ where γ_m is the natural m-linear mapping from $(E^*)^m$ into $E^* \otimes_\pi \cdots \otimes_\pi E^*$. The mapping β_m is injective and its image is $\mathcal{L}_{bf}({}^mE)$. The nuclear topology in $\mathcal{L}_{bf}({}^mE)$ is the locally convex topology generated by all seminorms of the form

$$\|A\|_{N,B} = \inf\left\{\sum_{j=1}^{n} \|\varphi_{j1}\|_B \ldots \|\varphi_{jm}\|_B;\, A = \sum_{j=1}^{n} \varphi_{j1} \times \cdots \times \varphi_{jm},\right.$$
$$\left.\varphi_{ij} \in E^*,\, i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n\right\}$$

where $\|\varphi_{ij}\|_B = \sup\{|\varphi_{ij}(x)|; x \in B\}$ and $B \in \mathscr{B}_E$. The nuclear topology in $\mathscr{P}_{bf}(^mE)$ is the locally convex topology generated by all seminorms of the form:

$$\|P\|_{N,B} = \inf\left\{\sum_{j=1}^{n} \|\varphi_j\|_B^m; P = \sum_{j=1}^{n} (\varphi_j)^m, \varphi_j \in E^*, j = 1, \ldots, n\right\}$$

where $B \in \mathscr{B}_E$. It can be shown that

$$\|A\|_{N,B} \leq \|\hat{A}\|_{N,B} \leq m^m (m!)^{-1} \|A\|_{N,B} \tag{1}$$

for all $A \in \mathscr{L}_{bfs}(^mE)$ and $B \in \mathscr{B}_E$. The nuclear topology in $\mathscr{L}_{bf}(^mE)$ makes this space isomorphic and homeomorphic to $E^*\otimes_\pi \cdots \otimes_\pi E^*$ through the mapping β_m. The mapping β_m can be extended continuously to "the" completion $E^*\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E^*$ of $E^*\otimes_\pi \cdots \otimes_\pi E^*$ into $\mathscr{L}_b(^mE)$. This extension will be denoted by $\hat{\beta}_m$. We know that $\hat{\beta}_m$ is injective if and only if E^* has the approximation property. Let $\tilde{\beta}_m$ be the injective mapping from $E^*\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E^*/\ker \hat{\beta}_m$ into $\mathscr{L}_b(^mE)$. This mapping is continuous and agrees with β_m in $E^*\otimes_{\hat{\pi}} \cdots \otimes_\pi E^*$. If we consider in $E^*\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E^*/\ker \hat{\beta}_m$ the quotient topology and if we denote the image of $\hat{\beta}_m$ by $\mathscr{L}_{bN}(^mE)$, we may consider in $\mathscr{L}_{bN}(^mE)$ the locally convex topology transferred from the quotient through $\tilde{\beta}_m$. Thus $\mathscr{L}_{bN}(^mE)$ is "the" completion of $\mathscr{L}_{bf}(^mE)$ if this space is considered with the nuclear topology. We still denote by $\|\cdot\|_{N,B}$ the seminorm in $\mathscr{L}_{bN}(^mE)$ obtained by continuous extension of the seminorm $\|\cdot\|_{N,B}$ in $\mathscr{L}_{bf}(^mE)$. It can be proved that the image $\mathscr{P}_{bN}(^mE)$ of $\mathscr{L}_{bN}(^mE)$ through the natural mapping

$$A \in \mathscr{L}_b(^mE) \mapsto \hat{A} \in \mathscr{P}_b(^mE)$$

is isomorphic to "the" completion of $\mathscr{P}_{bf}(^mE)$ endowed with the nuclear topology. We still denote by $\|\cdot\|_{N,B}$ the continuous extension to $\mathscr{P}_{bN}(^mE)$ of the seminorm $\|\cdot\|_{N,B}$ in $\mathscr{P}_{bf}(^mE)$. If $\mathscr{L}_{bNs}(^mE) = \mathscr{L}_{bN}(^mE) \cap \mathscr{L}_{bs}(^mE)$ we have

$$\|A\|_{N,B} \leq \|\hat{A}\|_{N,B} \leq m^m (m!)^{-1} \|A\|_{N,B} \tag{2}$$

for all $B \in \mathscr{B}_E$ and $A \in \mathscr{L}_{bNs}(^mE)$. As usual we set $\mathscr{L}_{bN}(^0E) = \mathscr{L}_{bNs}(^0E) = \mathbb{C}$ and $\|A\|_{N,B} = |A|$ if $A \in \mathscr{L}_{bN}(^0E)$.

2.1. DEFINITION: If $m \in \mathbb{N}$, the elements of $\mathscr{L}_{bN}(^mE)$ are called m-linear functions of Silva-nuclear type and the elements of $\mathscr{P}_{bN}(^mE)$ are called m-homogeneous polynomials of Silva-nuclear type.

If we consider E a locally convex space and if we replace E^* by E' in the previous constructions we get the following spaces:

(1) The vector subspace $\mathscr{L}_f({}^mE)$ of $\mathscr{L}_{bf}({}^mE)$ formed by the continuous functions and $\mathscr{L}_{fs}({}^mE) = \mathscr{L}_f({}^mE) \cap \mathscr{L}_{bs}({}^mE)$.

(2) The vector subspace $\mathscr{P}_f({}^mE)$ of $\mathscr{P}_{bf}({}^mE)$ formed by the continuous functions.

(3) "The" completion $\mathscr{L}_N({}^mE)$ of $\mathscr{L}_f({}^mE)$ for the nuclear topology and $\mathscr{L}_{Ns}({}^mE) = \mathscr{L}_N({}^mE) \cap \mathscr{L}_{bs}({}^mE)$.

(4) "The" completion $\mathscr{P}_N({}^mE)$ of $\mathscr{P}_f({}^mE)$ for the nuclear topology.

2.2. Definition: The elements of $\mathscr{L}_N({}^mE)$ and $\mathscr{P}_N({}^mE)$ are called respectively m-linear functions of nuclear type and m-homogeneous polynomials of nuclear type.

2.3. Definition: We denote by $\mathscr{P}_{bN}(E)$ the algebraic direct sum of $(\mathscr{P}_{bN}({}^mE))_{m\in\mathbb{N}}$ and by $\mathscr{P}_N(E)$ the algebraic direct sum of $(\mathscr{P}_N({}^mE))_{m\in\mathbb{N}}$. The elements of $\mathscr{P}_{bN}(E)$ and $\mathscr{P}_N(E)$ are called respectively polynomials of Silva nuclear type and polynomials of nuclear type.

Throughout this article we consider E such that E^* has the approximation property. In the end of this paper we point out the modifications needed in order to prove similar results for any E.

3. Nuclearly entire functions and nuclearly entire functions of bounded type

If E is a bornological vector space $\mathscr{K}_e(E)$ denotes the family of all balanced strict compact subsets of E and $\mathscr{B}_E^c$ denotes the family of elements of $\mathscr{B}_E$ which are convex. If E is a locally convex space $\mathscr{K}(E)$ denotes the family of all balanced compact subsets of E.

Throughout this paper we consider E a locally convex space hence a bornological vector space relatively to the Von Neumann bornology. The reader will not have any difficulty in thinking how the theory would work for a general bornological vector space. For a study of Silva holomorphic mappings and Silva holomorphic types in this context see [56] and [7]. We use freely the notations and results of [56].

3.1. Definition: An element f of $\mathscr{H}_S(E)$ is called a nuclearly Silva entire function if:

(1) $\widehat{\delta^m f}(0) \in \mathscr{P}_{bN}({}^mE)$ for all $m \in \mathbb{N}$;

(2) For every $B \in \mathscr{B}_E^c$ and $K \in \mathscr{K}(E_B)$ there is $\epsilon > 0$ such that

$$\sum_{m=0}^{\infty} \left\| \frac{1}{m!} \widehat{\delta^m f}(0) \right\|_{N,K+\epsilon B} < +\infty$$

Here E_B denotes the vector subspace of E generated by B normed by the Minkowski functional $\|\cdot\|_B$ associated to B. We denote by $\mathcal{H}_{SN}(E)$ the vector space of all nuclearly Silva entire functions in E.

3.2. DEFINITION: An element f of $\mathcal{H}_S(E)$ is called a nuclearly entire function if $f \in \mathcal{H}_{SN}(E)$ and $\widehat{\delta^m f}(0) \in \mathcal{P}_N(^mE)$ for all $m \in \mathbb{N}$. We denote by $\mathcal{H}_N(E)$ the vector space of all nuclearly entire functions in E.

3.3. PROPOSITION: *If $P_m \in \mathcal{P}_{bN}(^mE)$ (respectively, $P_m \in \mathcal{P}_N(^mE)$) for $m \in \mathbb{N}$, then for every $B \in \mathcal{B}_E^c$ the following conditions are equivalent:*

(1) *For each $K \in \mathcal{K}(E_B)$ there is $\epsilon > 0$ such that*

$$\sum_{m=0}^{\infty} \|P_m\|_{N,K+\epsilon B} < +\infty$$

(2) *For each $K \in \mathcal{K}(E_B)$ and each $\rho > 0$ there is $\delta > 0$ such that*

$$\limsup_{m\to\infty} \|P_m\|_{N,K+\delta B}^{1/m} < \frac{1}{\rho}$$

PROOF: It is obvious that (2) implies (1). Now we prove that (1) implies (2). Let $K \in \mathcal{K}(E_B)$ and $\rho > 0$ be given. If $\lambda > 0$ then there is $\epsilon > 0$ such that

$$\sum_{m=0}^{\infty} \|P_m\|_{N,\lambda\rho K+\epsilon B} < +\infty$$

We applied (1) to $\lambda\rho K \in \mathcal{K}(E_B)$. Thus

$$M = \sup\{\|P_m\|_{N,\lambda\rho K+\epsilon B};\, m \in \mathbb{N}\} < +\infty$$

and

$$\|P_m\|_{N,K+(\epsilon/\lambda\rho)B} = \frac{1}{(\lambda\rho)^m}\|P_m\|_{N,\lambda\rho K+\epsilon B} \leq \frac{M}{(\lambda\rho)^m}$$

Hence, if we take $\lambda > 1$, we have

$$\limsup_{m\to\infty} \|P_M\|_{N,K+(\epsilon/\lambda\rho)B}^{1/m} \leq \frac{1}{\lambda\rho} < \frac{1}{\rho}$$

3.4. DEFINITION: An element f of $\mathcal{H}_S(E)$ is called a nuclearly Silva entire function of bounded type if:

(i) $\widehat{\delta^m f}(0)$, belongs to $\mathcal{P}_{bN}(^mE)$ for all $m \in \mathbb{N}$;

(ii) For each B in $\mathcal{B}_E^c$, $\lim_{m\to\infty} \left\| \frac{1}{m!} \widehat{\delta^m f}(0) \right\|_{N,B}^{1/m} = 0$. We denote by $\mathcal{H}_{SNb}(E)$ the vector space of all nuclearly Silva entire functions of bounded type in E.

3.5. DEFINITION: An element $f \in \mathcal{H}_{SNb}(E)$ is called a nuclearly entire function of bounded type if $\widehat{\delta^m f}(0)$ belongs to $\mathcal{P}_N(^mE)$ for all $m \in \mathbb{N}$. We denote by $\mathcal{H}_{Nb}(E)$ the vector space of all nuclearly entire functions of bounded type in E.

3.6. REMARK: When E is a Banach space the spaces $\mathcal{H}_{SN}(E)$ and $\mathcal{H}_N(E)$ coincide with the space $\mathcal{H}_N(E)$ introduced in Gupta–Nachbin [50]. Also the spaces $\mathcal{H}_{SNb}(E)$ and $\mathcal{H}_{Nb}(E)$ coincide with the space $\mathcal{H}_{Nb}(E)$ introduced in Gupta [48]. The space $\mathcal{H}_{Nb}(E)$ is the same space $\mathrm{H}_{Nb}(E)$ which appears in Matos [51] and [52]. In [48] Gupta gives an example of a Banach space E such that $\mathcal{H}_{Nb}(E) \underset{\neq}{\subset} \mathcal{H}_N(E)$.

In the definitions of $\mathcal{H}_{SN}(E)$, $\mathcal{H}_N(E)$, $\mathcal{H}_{SNb}(E)$ and $\mathcal{H}_{Nb}(E)$ the origin plays a very special role. We show that this fact can be avoided as follows.

3.7. PROPOSITION: *An element f of $\mathcal{H}_S(E)$ is nuclearly Silva entire (respectively, nuclearly entire) if and only if*

(i) $\widehat{\delta^m f}(x) \in \mathcal{P}_{bN}(^mE)$ *(respectively,* $\widehat{\delta^m f}(x) \in \mathcal{P}_N(^mE)$*) for all x in E and $m \in \mathbb{N}$.*

(ii) *For each $B \in \mathcal{B}_E^c$ and every $K, J \in \mathcal{K}(E_B)$ there is $\epsilon > 0$ such that*

$$\sum_{m=0}^{\infty} \sup_{x\in J} \left\| \frac{1}{m!} \widehat{\delta^m f}(x) \right\|_{N,K+\epsilon B} < +\infty$$

PROOF: It is clear that (i) and (ii) imply $f \in \mathcal{H}_{SN}(E)$ (respectively, $f \in \mathcal{H}_N(E)$). We prove the reverse implication. Let $B \in \mathcal{B}_E^c$ and K, $J \in \mathcal{K}(E_B)$. Let $\delta > 0$ be such that

$$\sum_{m=0}^{\infty} \frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,2[K\cup J]+\delta B} < +\infty.$$

If we take $\epsilon = \frac{1}{2}\delta$ we have:

$$\sum_{m=0}^{\infty} \sup_{t\in J} \left\| \frac{1}{m!} \hat{\delta}^m f(t) \right\|_{N,K+\epsilon B} =$$

$$= \sum_{m=0}^{\infty} \frac{1}{2^m} \sup_{t\in J} \left\| \frac{1}{m!} \hat{\delta}^m f(t) \right\|_{N,2K+\delta B}$$

$$\leq \sum_{m=0}^{\infty} \frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,2(K\cup J)+\delta B} < +\infty$$

where the last inequality follows from Lemma 3.8 with $r = s = \frac{1}{2}$.

3.8. Lemma: *If $f \in \mathscr{H}_{SN}(E)$ (respectively, $f \in \mathscr{H}_N(E)$) then $\hat{\delta}^m f(x) \in \mathscr{P}_{bN}(^mE)$ (respectively, $\hat{\delta}^m f(x) \in \mathscr{P}_N(^mE)$) for each $x \in E$ and $m \in \mathbb{N}$. Furthermore*:

$$\sum_{m=0}^{\infty} \frac{s^m}{m!} \sup_{x\in rB} \|\hat{\delta}^m f(x)\|_{N,B} \leq$$

$$\leq \sum_{m=0}^{\infty} (r+s)^m \frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,B}$$

if $s,r \in [0, 1)$, $r + s \leq 1$.

Proof: We know that the following equalities hold pointwise in E:

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{\delta}^m f(0)x$$

$$\hat{\delta}^m f(x) = \sum_{k=0}^{\infty} \hat{\delta}^m P_{m+k}(x)$$

where $P_{m+k} = [(m+k)!]^{-1} \hat{\delta}^{m+k} f(0)$.

Let $B \in \mathscr{B}_E^c$ and $x \in E_B$. Thus for each $\rho > 0$, $\rho < \frac{1}{2}$ there is $\delta > 0$ such that

$$\limsup_{m\to\infty} \|P_m\|_{N,\{\hat{x}\}+\delta B}^{1/m} < \frac{1}{\rho}.$$

Here $\{\hat{x}\}$ denotes the closed convex balanced hull of $\{x\}$. The inequality above and Lemma 3.9 imply

$$\sum_{k=0}^{\infty} \|\hat{\delta}^m P_{m+k}(x)\|_{N,\{\hat{x}\}+\delta B} \leq \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} \|P_{m+k}\|_{N,\{\hat{x}\}+\delta B} \|x\|_{\{\hat{x}\}+\delta B}^k \leq$$

$$\leq m! \sum_{k=0}^{\infty} 2^{m+k} \|P_{m+k}\|_{N,\{\hat{x}\}+\delta B} < +\infty$$

By Lemma 3.9 and by the fact that $B \in \mathcal{B}_E^c$ is arbitrary we conclude that $\{\sum_{k=0}^{N} \hat{\delta}^m P_{m+k}(x)\}_{N=0}^{\infty}$ is a Cauchy sequence in $\mathcal{P}_{bN}(^mE)$ (respectively, $\mathcal{P}_N(^mE)$). Hence its limit $\hat{\delta}^m f(x) \in \mathcal{P}_{bN}(^mE)$ (respectively, $\hat{\delta}^m f(x) \in \mathcal{P}_N(^mE)$). Now for $r,s \in [0, 1)$, $r + s \leq 1$, we have:

$$\sum_{m=0}^{\infty} \frac{s^m}{m!} \sup_{x \in rB} \|\hat{\delta}^m f(x)\|_{N,B} \leq$$

$$\leq \sum_{m=0}^{\infty} \frac{s^m}{m!} \sup_{x \in rB} \sum_{k=0}^{\infty} \|\hat{\delta}^m P_{m+k}(x)\|_{N,B} \leq$$

$$\leq \sum_{m=0}^{\infty} \frac{s^m}{m!} \sup_{x \in rB} \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} \|P_{m+k}\|_{N,B} \|x\|_B^k \leq$$

$$\leq \sum_{m=0}^{\infty} \frac{s^m}{m!} \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} r^k \|P_{m+k}\|_{N,B} =$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \frac{s^m r^{\ell-m} \ell!}{m!(\ell-m)!} \|P_\ell\|_{N,B} = \sum_{\ell=0}^{\infty} (r+s)^\ell \|P_\ell\|_{N,B}.$$

3.9. LEMMA: *For every $x \in E$ and $k \in \mathbb{N}$, $k \leq m$, the linear mapping $P \in \mathcal{P}_{bN}(^mE) \mapsto \hat{\delta}^k P(x) \in \mathcal{P}_{bN}(^kE)$ (respectively, $P \in \mathcal{P}_N(^mE) \mapsto \hat{\delta}^k P(x) \in \mathcal{P}_N(^kE)$) is continuous and for all $B \in \mathcal{B}_E^c$ with $x \in E_B$*

$$\|\hat{\delta}^k P(x)\|_{N,B} \leq \frac{m!}{(m-k)!} \|P\|_{N,B} \|x\|_B^{m-k}$$

PROOF: For $P \in \mathcal{P}_{bf}(^mE)$ (respectively, $P \in \mathcal{P}_f(^mE)$) with $P = \sum_{j=1}^{n} \widehat{\varphi_j^m}$, $\delta_j \in E^*$ (respectively, $\varphi_j \in E'$), $j = 1, \ldots, n$, we have

$$\hat{\delta}^k P(x) = \frac{m!}{(m-k)!} \sum_{j=1}^{n} [\varphi_j(x)]^{m-k} \widehat{\varphi_j^k} \in \mathcal{P}_{bf}(^kE)$$

(respectively, $\hat{\delta}^k P(x) \in \mathcal{P}_f(^kE)$). Hence for each $B \in \mathcal{B}_E^c$ with $x \in E_B$ it follows that

$$\|\hat{\delta}^k P(x)\|_{N,B} \leq \frac{m!}{(m-k)!} \left[\sum_{j=1}^{n} \|\varphi_j\|_B^m\right] \|x\|_B^{m-k}.$$

Therefore

$$\|\hat{\delta}^k P(x)\|_{N,B} \leq \frac{m!}{(m-k)!} \|P\|_{N,B} \|x\|_B^{m-k}$$

The result follows from the density of $\mathcal{P}_{bf}({}^mE)$ in $\mathcal{P}_{bN}({}^mE)$ (respectively, of $\mathcal{P}_f({}^mE)$ in $\mathcal{P}_N({}^mE)$).

3.10. PROPOSITION: *Let $f \in \mathcal{H}_S(E)$. Then $f \in \mathcal{H}_{SNb}(E)$ (respectively, $f \in \mathcal{H}_{Nb}(E)$) if and only if*

(i) *$\hat{\delta}^m f(x) \in \mathcal{P}_{bN}({}^mE)$ (respectively, $\hat{\delta}^m f(x) \in \mathcal{P}_N({}^mE)$) for all $x \in E$ and $m \in \mathbb{N}$.*
(ii) *For each $B \in \mathcal{B}_E^c$ and $x \in E_B$*

$$\lim_{m\to\infty}\left[\frac{1}{m!}\|\hat{\delta}^m f(x)\|_{N,B}\right]^{1/m} = 0$$

PROOF: It is clear that (i) and (ii) imply $f \in \mathcal{H}_{SNb}(E)$ (respectively, $f \in \mathcal{H}_{Nb}(E)$). Now we prove the converse. If $f \in \mathcal{H}_{SNb}(E)$ (respectively, $f \in \mathcal{H}_{Nb}(E)$), for each $B \in \mathcal{B}_E^c$ and each $x \in E_B$ we choose $\epsilon > 0$ such that $\epsilon\|x\|_B \le 1$. Thus there is $C > 0$ such that

$$\frac{1}{m!}\|\hat{\delta}^m f(0)\|_{N,B} \le C\left(\frac{\epsilon}{2}\right)^m$$

for all $m \in \mathbb{N}$. By Lemma 3.8 and the the fact that $\mathcal{H}_{SNb}(E) \subset \mathcal{H}_{SN}(E)$ (respectively, $\mathcal{H}_{Nb}(E) \subset \mathcal{H}_N(E)$) we get $\frac{1}{m!}\hat{\delta}^m f(x) \in \mathcal{P}_{bN}({}^mE)$ (respectively, $\frac{1}{m!}\hat{\delta}^m f(x) \in \mathcal{P}_N({}^mE)$). If we use Lemma 3.9 we may write

$$\begin{aligned}\left\|\frac{1}{m!}\hat{\delta}^m f(x)\right\|_{N,B} &\le \sum_{k=0}^{\infty}\frac{1}{m!}\left\|\hat{\delta}^m\left[\frac{1}{(k+m)!}\hat{\delta}^{m+k}f(0)\right](x)\right\|_{N,B} \le \\ &\le \sum_{k=0}^{\infty}\frac{1}{m!}\frac{(k+m)!}{k!}\frac{1}{(k+m)!}\|\hat{\delta}^m f(0)\|_{N,B}\|x\|_B^k \le \\ &\le \sum_{k=0}^{\infty}2^{k+m}C\left(\frac{\epsilon}{2}\right)^{k+m}\|x\|_B^k = C\frac{\epsilon^m}{1-\epsilon\|x\|_B} < +\infty\end{aligned}$$

Hence

$$\limsup_{m\to\infty}\left[\frac{1}{m!}\|\hat{\delta}^m f(x)\|_{N,B}\right]^{1/m} \le \epsilon$$

As ϵ goes to 0 we get

$$\lim_{m\to\infty}\left[\frac{1}{m!}\|\hat{\delta}^m f(x)\|_{N,B}\right]^{1/m} = 0.$$

3.11. PROPOSITION: *Let f be in $\mathcal{H}_S(E)$. Then $f \in \mathcal{H}_{SNb}(E)$ (respectively, $f \in \mathcal{H}_{Nb}(E)$) if and only if $\hat{\delta}^m f(0) \in \mathcal{P}_{bN}({}^m E)$ (respectively, $\hat{\delta}^m f(0) \in \mathcal{P}_N({}^m E)$) for each $m \in \mathbb{N}$ and*

$$\sum_{m=0}^{\infty} \left\| \frac{1}{m!} \hat{\delta}^m f(0) \right\|_{N,B} < +\infty \tag{*}$$

for each $B \in \mathcal{B}_E^c$.

PROOF: We observe that

$$\lim_{m\to\infty} \left[\frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,B} \right]^{1/m} = 0$$

implies that

$$\sum_{m=0}^{\infty} \frac{\rho^m}{m!} \|\hat{\delta}^m f(0)\|_{N,B} < +\infty$$

for all $\rho \in \mathbb{R}$, $\rho > 0$. Hence this holds for $\rho = 1$. Thus we have proved that $f \in \mathcal{H}_{SNb}(E)$ (respectively, $f \in \mathcal{H}_{Nb}(E)$) implies condition (*) and $\hat{\delta}^m f(0) \in \mathcal{P}_{bN}({}^m E)$ (respectively, $\hat{\delta}^m f(0) \in \mathcal{P}_N({}^m E)$) for all $m \in \mathbb{N}$. In order to prove the converse we observe that

$$\rho^m \left\| \frac{1}{m!} \hat{\delta}^m f(0) \right\|_{N,B} = \left\| \frac{1}{m!} \hat{\delta}^m f(0) \right\|_{N,\rho B}$$

for all $\rho > 0$ and $B \in \mathcal{B}_E^c$. Hence

$$\sum_{m=0}^{\infty} \frac{\rho^m}{m!} \|\hat{\delta}^m f(0)\|_{N,B} = \sum_{m=0}^{\infty} \frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,\rho B} < +\infty$$

for all $\rho > 0$ and $B \in \mathcal{B}_E^c$. It follows that

$$\lim_{m\to\infty} \left[\frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,B} \right]^{1/m} = 0$$

for each $B \in \mathcal{B}_E^c$.

4. Topologies in $\mathcal{H}_{SNb}(E)$ and in $\mathcal{H}_N(E)$

4.1. DEFINITION: The natural topology in $\mathcal{H}_{SNb}(E)$ (respectively, $\mathcal{H}_{Nb}(E)$) is the locally convex topology generated by the seminorms

$$\|f\|_{N,B} = \sum_{m=0}^{\infty} \frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,B}$$

for $B \in \mathcal{B}_E^c$, $f \in \mathcal{H}_{SNb}(E)$ (respectively, $f \in \mathcal{H}_{Nb}(E)$).

4.2. Proposition: *The spaces $\mathcal{H}_{SNb}(E)$ and $\mathcal{H}_{Nb}(E)$ are complete under their natural topologies.*

Proof: We prove the result for $\mathcal{H}_{SNb}(E)$. The proof for $\mathcal{H}_{Nb}(E)$ is similar. Let $(f_\alpha)_{\alpha\in A}$ be a Cauchy net in $\mathcal{H}_{SNb}(E)$ for the natural topology. Hence for each $\epsilon > 0$, $B \in \mathcal{B}_E^c$, there is $\alpha_\epsilon \in A$ such that $\|f_\alpha - f_\beta\|_{N,B} < \epsilon$ for $\alpha \geq \alpha_\epsilon$ and $\beta \geq \alpha_\epsilon$. Hence for each $m \in \mathbb{N}$ we have $\frac{1}{m!}\|\hat{\delta}^m f_\alpha(0) - \hat{\delta}^m f_\beta(0)\|_{N,B} < \epsilon$ for $\alpha \geq \alpha_\epsilon$ and $\beta \geq \alpha_\epsilon$. Thus $(\hat{\delta}^m f_\alpha(0))_{\alpha\in A}$ is a Cauchy net in $\mathcal{P}_{bN}(^mE)$, which is complete. Therefore we have $P_m \in \mathcal{P}_{bN}(^mE)$ such that $\frac{1}{m!}\|\hat{\delta}^m f_\alpha(0) - P_m\|_{N,B} \leq \epsilon$ for $\alpha \geq \alpha_\epsilon$. In particular, if we call $f_k = f_{\alpha_{(1/k)}}$, we have

$$\frac{1}{m!}\|\hat{\delta}^m f_k(0) - P_m\|_{N,B} \leq \frac{1}{k}$$

for all $k \in \mathbb{N} - \{0\}$, and

$$\frac{1}{m!}\|\hat{\delta}^m f_k(0) - \hat{\delta}^m f_\ell(0)\|_{N,B} < \frac{1}{k}$$

if $\ell \geq k$ and we take $\frac{\alpha_1}{\ell} \geq \frac{\alpha_1}{k}$ for $\ell \geq k$ (this is possible by induction). Hence there is $M \geq 0$ such that $\frac{1}{m!}\|\hat{\delta}^m f_k(0)\|_{N,B} \leq M$ for all $k \in \mathbb{N} - \{0\}$. Now we may write for all $m \in \mathbb{N}$ and $k \in \mathbb{N} - \{0\}$

$$\frac{1}{m!}\|P_m\|_{N,B} \leq \frac{1}{k} + M$$

Therefore

$$\sum_{m=0}^{\infty} \frac{1}{m!}\|P_m\|_{N,B} < +\infty$$

It follows that

$$\limsup_{m\to\infty} \left[\frac{1}{m!}\|P_m\|_{N,B}\right]^{1/m} \leq 1$$

for all $B \in \mathcal{B}_E^c$. Hence if $B \in \mathcal{B}_E^c$ and $\delta > 1$

$$\limsup_{m\to\infty} \left[\frac{1}{m!}\|P_m\|_{N,\delta B}\right]^{1/m} = \limsup_{m\to\infty} \left[\frac{\delta^m}{m!}\|P_m\|_{N,B}\right]^{1/m} \leq 1$$

and

$$\limsup_{m\to\infty}\left[\frac{1}{m!}\|P_m\|_{N,B}\right]^{1/m}\le\frac{1}{\delta}<1$$

for all $B\in\mathcal{B}_E^c$. Now, from $\|f_\alpha-f_\beta\|_{N,B}<\epsilon$ for $\alpha\ge\alpha_\epsilon$, $\beta\ge\alpha_\epsilon$ we get

$$\sum_{m=0}^{\infty}\frac{1}{m!}\|\hat{\delta}^m f_\alpha(0)-P_m\|_{N,B}\le\epsilon$$

for $\alpha\ge\alpha_\epsilon$. If we prove that

$$f(x)=\sum_{m=0}^{\infty}\frac{1}{m!}P_m(x)\quad(x\in E)$$

defines an element of $\mathcal{H}_S(E)$, we get that $f\in\mathcal{H}_{SNb}(E)$ and $\lim_{\alpha\in A} f_\alpha=f$ for the natural topology of $\mathcal{H}_{SNb}(E)$. First we prove that f is finitely holomorphic in E. If $x\in E$, let B_x be the closed absolutely convex hull of $\{x\}$. Hence $B_x\in\mathcal{B}_E^c$ and

$$\left|\sum_{m=0}^{\infty}\frac{1}{m!}P_m(x)\right|\le\sum_{m=0}^{\infty}\frac{1}{m!}\|P_m\|_{B_x}\le\sum_{m=0}^{\infty}\frac{1}{m!}\|P_m\|_{N,B_x}<+\infty$$

Now, in order to prove that $f\in\mathcal{H}_S(U)$, we must show that f is bounded over each $K\in\mathcal{K}(E_B)$, as B varies over $\mathcal{B}_E^c$. Let B_K be the closed absolutely convex hull of K. Thus $B_K'\in\mathcal{B}_E^c$, and

$$\sup_{x\in K}|f(x)|\le\sum_{m=0}^{\infty}\sup_{x\in K}\left|\frac{1}{m!}P_m(x)\right|\le$$

$$\le\sum_{m=0}^{\infty}\frac{1}{m!}\|P_m\|_{B_K}\le\sum_{m=0}^{\infty}\frac{1}{m!}\|P_m\|_{N,B_K}<+\infty.$$

4.3. COROLLARY: *If E has a countable fundamental system of bounded subsets, then $\mathcal{H}_{SNb}(E)$ and $\mathcal{H}_{Nb}(E)$ are Fréchet spaces under their natural topologies.*

4.4. PROPOSITION: *If $f\in\mathcal{H}_{SNb}(E)$ (respectively, $f\in\mathcal{H}_{Nb}(E)$) its Taylor series at 0 converges to f for the natural topology.*

PROOF: It is immediate from the following fact which holds for all $B\in\mathcal{B}_E^c$

$$\left\|f-\sum_{k=0}^{n}\frac{1}{k!}\hat{\delta}^k f(0)\right\|_{N,B}=\sum_{k=n+1}^{\infty}\frac{1}{k!}\|\hat{\delta}^k f(0)\|_{N,B}\to 0$$

as $n\to\infty$.

4.5. PROPOSITION: *The vector subspace $\mathcal{S}$ of $\mathcal{H}_{SNb}(E)$ (respectively, $\mathcal{H}_{Nb}(E)$) generated by $\{e^{\varphi}; \varphi \in E^*\}$ (respectively, $\{e^{\varphi}; \varphi \in E'\}$) is dense in $\mathcal{H}_{SNb}(E)$ (respectively, in $\mathcal{H}_{Nb}(E)$) for the natural topology.*

PROOF: We prove the result for $\mathcal{H}_{SNb}(E)$. The proof for $\mathcal{H}_{Nb}(E)$ is similar. By Proposition 4.4. it is enough to show that $P \in \bar{\mathcal{S}}$ for all $P \in \mathcal{P}_{bN}(^mE)$, $m \in \mathbb{N}$. Since the natural topology of $\mathcal{H}_{SNb}(E)$ induces in each $\mathcal{P}_{bN}(^mE)$ the nuclear topology it is enough to prove that $P \in \bar{\mathcal{S}}$ for all $P \in \mathcal{P}_{bf}(^mE)$, $m \in \mathbb{N}$. We have

$$e^{\lambda\varphi} = \sum_{n=0}^{\infty} \frac{\lambda^n \varphi^n}{n!}$$

for every $\varphi \in E^*$ and $\lambda \in \mathbb{C}$ in the sense of the natural topology of $\mathcal{H}_{SNb}(E)$. Thus for each $B \in \mathcal{B}_E^c$

$$\left\| \frac{e^{\lambda\varphi} - 1}{\lambda} - \varphi \right\|_{N,B} = |\lambda| \sum_{n=2}^{\infty} \frac{|\lambda|^{n-2} \|\varphi\|_B^n}{n!} \to 0$$

ad $|\lambda| \to 0$ for all $\varphi \in E^*$. Hence $\varphi \in \bar{\mathcal{S}}$ for all $\varphi \in E^*$. Now we suppose that $\widehat{\varphi^i} \in \bar{\mathcal{S}}$ for $i = 1, 2, \ldots, n-1$ and $\varphi \in E^*$. Then

$$\left\| \frac{e^{\lambda\varphi} - \sum_{i=1}^{n-1} \lambda^i \widehat{\varphi^i}}{\lambda^n} - \widehat{\varphi^n} \right\|_{N,B} = |\lambda| \sum_{i=n+1}^{\infty} \frac{|\lambda|^{i-n}}{i!} \|\sigma\|_B^i \to 0$$

as $|\lambda| \to 0$ for each $\varphi \in E^*$ and $B \in \mathcal{B}_E^c$. Hence, by induction, we have proved that $\mathcal{P}_{bf}(^nE) \subset \bar{\mathcal{S}}$ for $n \in \mathbb{N}$.

4.6. REMARK: $\mathcal{H}_{Nb}(E)$ is "the" completion of its vector subspace generated by the continuous functions.

5. Topologies in $\mathcal{H}_{SN}(E)$ and in $\mathcal{H}_N(E)$

5.1. DEFINITION: Let $B \in \mathcal{B}_E^c$ and $K \in \mathcal{K}(E_B)$. A seminorm p in $\mathcal{H}_{SN}(E)$ (respectively, in $\mathcal{H}_N(E)$) is said to be N-ported by (K, B) if for each $\epsilon > 0$ there is $C(\epsilon) > 0$ such that

$$p(f) \leq C(\epsilon) \sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n f(0)\|_{N,K+\epsilon B}$$

for all $f \in \mathcal{H}_{SN}(E)$ (respectively, $f \in \mathcal{H}_N(E)$). The natural topology in $\mathcal{H}_{SN}(E)$ (respectively, in $\mathcal{H}_N(E)$) is the locally convex topology generated by all seminorms which are N-ported by (K, B), where $B \in \mathcal{B}_E^c$ and $K \in \mathcal{K}(E_B)$

5.2. Proposition: *For each $f \in \mathcal{H}_{SN}(E)$ (respectively, $f \in \mathcal{H}_N(E)$) its Taylor series at 0 coverges to f in the natural topology.*

Proof: We prove the result for $f \in \mathcal{H}_{SN}(E)$. If $f \in \mathcal{H}_N(E)$ the proof is similar. Let p be a seminorm in $\mathcal{H}_{SN}(E)$ N-ported by (K, B). Since $B \in \mathcal{B}_E^c$ and $K \in \mathcal{K}(E_B)$ we know that there is $\epsilon > 0$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n f(0)\|_{N,K+\epsilon B} < +\infty$$

Hence there is $C(\epsilon) > 0$ such that

$$p(g) \leq C(\epsilon) \sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n g(0)\|_{N,K+\epsilon B}$$

for all $g \in \mathcal{H}_{SN}(E)$. Hence

$$p\left(f - \sum_{k=0}^{n} \frac{1}{k!} \hat{\delta}^k f(0)\right) \leq C(\epsilon) \sum_{m=n+1}^{\infty} \frac{1}{m!} \|\hat{\delta}^m f(0)\|_{N,K+\epsilon B}$$

and this tends to 0 as n tends to $+\infty$.

5.3. Proposition: *The vector subspace of $\mathcal{H}_{SN}(E)$ (respectively, $\mathcal{H}_N(E)$) generated by $\{e^{\varphi}; \varphi \in E^*\}$ (respectively, $\{e^{\varphi}; \varphi \in E'\}$) is dense in $\mathcal{H}_{SN}(E)$ (respectively, $\mathcal{H}_N(E)$) for the natural topology.*

Proof: It follows the pattern of the proof of Proposition 5.4.

5.4. Remark: For more information about the topology of the space $\mathcal{H}_{SN}(E)$ see Bianchini [7].

6. Translations and directional derivatives

6.1. Definition: If f is a complex function defined in E and $a \in E$, we define the translation $\tau_a f$ of f by a in the following way:

$$(\tau_a f)(x) = f(x - a) \quad (\forall x \in E).$$

6.2. PROPOSITION: *Let* $a \in E$ *and* $f \in \mathcal{H}_{SNb}(E)$ *(respectively,* $f \in \mathcal{H}_{Nb}(E)$*). Then:*

(i) $\hat{\delta}^n f(\cdot)\, a \in \mathcal{H}_{SNb}(E)$ *(respectively,* $\hat{\delta}^n f(\cdot) a \in \mathcal{H}_{Nb}(E)$*) and*

$$\hat{\delta}^n f(\cdot)a = \sum_{i=0}^{\infty} \frac{1}{i!} \widehat{\delta^{i+n} f(0) \cdot {}^i}(a)$$

in the sense of the natural topology of the space, for all $n \in \mathbb{N}$.

(ii) $\tau_a f \in \mathcal{H}_{SNb}(E)$ *(respectively,* $\tau_a f \in \mathcal{H}_{Nb}(E)$*) and*

$$\tau_{-a} f = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\delta}^n f(\cdot)a$$

in the sense of the natural topology of the space.

PROOF: We prove the results for $f \in \mathcal{H}_{SNb}(E)$. The proof for $f \in \mathcal{H}_{Nb}(E)$ is similar.

(i) By Proposition 3.36 of Matos–Nachbin [56] we have

$$\hat{\delta}^i f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\delta^{n+i} f(0) x^n}$$

for each $x \in E$, the series converging in the natural topology of $\mathcal{P}_b({}^iE)$. Hence

$$\hat{\delta}^i f(x)(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\delta^{n-i} f(0) x}^n(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\delta^{n+i} f(0) a}^i(x)$$

for every $x \in E$. It is easy to see that $\widehat{\delta^{i+n} f(0) a^i} \in \mathcal{P}_{bN}({}^nE)$ and, for every $B \in \mathcal{B}_E^c$ with $a \in E_B$, we have:

$$\|\widehat{\delta^{i+n} f(0) a^i}\|_{N,B} \le \|\hat{\delta}^{i+n} f(0)\|_{N,B} \|a\|_B^i$$

Hence

$$\lim_{n\to\infty} \left[\frac{1}{n!} \|\widehat{\delta^{i+n} f(0) a^i}\|_{N,B}\right]^{1/n} \le \lim_{n\to\infty} \left[\frac{1}{n!} \|\hat{\delta}^{i+n} f(0)\|_{N,B} \|a\|_B^i\right]^{1/n} = 0$$

for every $i \in \mathbb{N}$. Thus $\hat{\delta}^i f(\cdot)a \in \mathcal{H}_{SNb}(E)$ for each $i \in \mathbb{N}$. We have:

$$\left\|\hat{\delta}^i f(\cdot)a - \sum_{n=0}^{k} \frac{1}{n!} \widehat{\delta^{i+n} f(0) \cdot {}^n}(a)\right\|_{N,B} = \sum_{n=k+1}^{\infty} \frac{1}{n!} \|\widehat{\delta^{i+n} f(0) a^i}\|_{N,B} \le$$

$$\le \sum_{n=k+1}^{\infty} \frac{\|a\|_B^i}{n!} \|\hat{\delta}^{i+n} f(0)\|_{N,B} \le i!\|a\|_B^i \sum_{n=k+1}^{\infty} \frac{2^{i+n}}{(i+n)!} \|\hat{\delta}^{i+n} f(0)\|_{N,B} =$$

$$= i!\|a\|_B^i \sum_{n=k+1}^{\infty} \frac{1}{(i+n)!} \|\hat{\delta}^{i+n} f(0)\|_{N,2B} \to 0$$

as $k \to \infty$, for all $B \in \mathcal{B}_E^c$ with $a \in E_B$.

(ii) We consider

$$\left\|\tau_{-a}f - \sum_{n=0}^{k} \frac{1}{n!}\hat{\delta}^n f(\cdot)(a)\right\|_{N,B} =$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\left\|\hat{\delta}^i(\tau_{-a}f)(0) - \sum_{n=0}^{k} \widehat{\delta^{i+n}f(0)a^n}\right\|_{N,B} =$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\left\|\sum_{n=k+1}^{\infty} \frac{1}{n!}\widehat{\delta^{i+n}f(0)a^n}\right\|_{N,B} \leq$$

$$\leq \sum_{i=0}^{\infty}\sum_{n=k+1}^{\infty} \frac{(n+i)!}{i!n!}\frac{1}{(n+i)!}\|\hat{\delta}^{i+n}f(0)\|_{N,B}\|a\|_B^n \leq$$

$$\leq \sum_{i=0}^{\infty}\sum_{n=k+1}^{\infty} 2^{n+1}\|a\|_B^n \frac{1}{(n+i)!}\|\hat{\delta}^{i+n}f(0)\|_{N,2B} = \otimes$$

Now given $\epsilon > 0$, with $\epsilon\|a\|_B < 1$, $\epsilon < 1$, there is $c(\epsilon) > 0$ such that $[(i+n)!]^{-1}\|\hat{\delta}^{i+n}f(0)\|_{N,2B} \leq c(\epsilon)\epsilon^{i+n}$ for all $i \in \mathbb{N}$. Therefore

$$\otimes \leq \sum_{i=0}^{\infty}\sum_{n=k+1}^{\infty} \|a\|_B^n \epsilon^n c(\epsilon)\epsilon^i = c(\epsilon)\left[\sum_{i=0}^{\infty}\epsilon^i\right]\left[\sum_{n=k+1}^{\infty}(\epsilon\|a\|_B)^n\right]$$

which tends to 0 as $k \to \infty$

6.3. Proposition: *Let $a \in E$ and $f \in \mathcal{H}_{SN}(E)$ (respectively, $f \in \mathcal{H}_N(E)$). Then:*

(i) *$\hat{\delta}^n f(\cdot)a \in \mathcal{H}_{SN}(E)$ (respectively, $\hat{\delta}^n f(\cdot)a \in \mathcal{H}_N(E)$) and*

$$\hat{\delta}^n f(\cdot)a = \sum_{i=0}^{\infty}\frac{1}{i!}\widehat{\delta^{i+n}f(0)\cdot{}^i}(a)$$

in the sense of the natural topology, for all $n \in \mathbb{N}$.

(ii)

$$\tau_{-a}f = \sum_{n=0}^{\infty}\frac{1}{n!}\hat{\delta}^n f(\cdot)a$$

in the sense of the natural topology.

Proof: We prove the results for $f \in \mathcal{H}_{SN}(E)$. The proof for $f \in \mathcal{H}_N(E)$ is similar.

(i) If $f \in \mathcal{H}_{SN}(E)$ then $\hat{\delta}^n f(\cdot)a \in \mathcal{H}_S(E)$ for every $a \in E$. (See Matos–Nachbin [56]). Let $B \in \mathcal{B}_E^c$ with $a \in E_B$ and $K \in \mathcal{K}(E_B)$. Thus there is $\epsilon > 0$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n f(0)\|_{N,2K+\epsilon B} < +\infty$$

We have

$$\sum_{m=0}^{\infty} \frac{1}{m!} \|\hat{\delta}^m (\hat{\delta}^n f(\,\cdot\,)a)(0)\|_{N,K+\epsilon B} = \sum_{m=0}^{\infty} \frac{1}{m!} \|\widehat{\delta^{n+m} f(0) a^n}\|_{N,K+\epsilon B} = \otimes$$

We note that

$$\|\widehat{\delta^{n+m} f(0) a^n}\|_{N,K+\epsilon B} \le \|\hat{\delta}^{n+m} f(0)\|_{N,K+\epsilon B} \cdot [\alpha(a)]^n$$

where $\alpha(a) > 0$ is such that $[\alpha(a)]^{-1} a \in K + \epsilon B$. Hence

$$\begin{aligned}
\otimes &\le \sum_{m=0}^{\infty} \frac{[\alpha(a)]^n}{m!} \|\hat{\delta}^{n+m} f(0)\|_{N,K+\epsilon B} \le \\
&\le [\alpha(a)]^n n! \sum_{m=0}^{\infty} \frac{2^{n+m}}{(n+m)!} \|\hat{\delta}^{n+m} f(0)\|_{N,K+\epsilon B} = \\
&= [\alpha(a)]^n n! \sum_{m=0}^{\infty} \frac{1}{(n+m)!} \|\hat{\delta}^{n+m} f(0)\|_{N,2K+2\epsilon B} < +\infty
\end{aligned}$$

Hence $\hat{\delta}^n f(\,\cdot\,)a \in \mathcal{H}_{SN}(E)$ for all $n \in \mathbb{N}$. Since

$$\sum_{i=0}^{\infty} \frac{1}{i!} \widehat{\delta^{i+n} f(0) \cdot {}^{i}}(a) = \sum_{i=0}^{\infty} \frac{1}{i!} \widehat{\delta^{i+n} f(0) a^n}$$

is the Taylor series of $\hat{\delta}^n f(\,\cdot\,)a$ at 0, the result follows from 5.2.

(ii) Let p be a continuous seminorm in $\mathcal{H}_{SN}(E)$ N-ported by (K, B) with $B \in \mathcal{B}_E^c$, $K \in \mathcal{K}(E_B)$, $a \in E_B$. Let K_1 be the balanced hull of $K \cup \{a\}$. Hence $K_1 \in \mathcal{K}(E_B)$. If $\rho > 2$ is given there is $\epsilon > 0$ such that

$$\limsup_{n\to\infty} \left[\frac{1}{n!} \|\hat{\delta}^n f(0)\|_{N,K_1+\epsilon B}\right]^{1/n} < \frac{1}{\rho}$$

(See 3.3 part (2)). Thus there is $c > 0$ such that

$$\|\hat{\delta}^n f(0)\|_{N,K_1+\epsilon B} \le \rho^{-n} c[n!]$$

for all $n \in \mathbb{N}$. We also find $c(\epsilon) > 0$ such that

$$p(g) \le C(\epsilon) \sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n g(0)\|_{N,K_1+\epsilon B}$$

for all $g \in \mathcal{H}_{SN}(E)$. Now

$$
\begin{aligned}
&p\left(\tau_{-a}f - \sum_{n=0}^{k} \frac{1}{n!}\hat{\delta}^n f(\cdot)a\right) \leq \\
&\leq C(\epsilon) \sum_{m=0}^{\infty} \frac{1}{m!}\left\|\hat{\delta}^m\left(\tau_{-a}f - \sum_{n=0}^{k}\frac{1}{n!}\hat{\delta}^n f(\cdot)a\right)(0)\right\|_{N,K_1+\epsilon B} \leq \\
&\leq C(\epsilon) \sum_{m=0}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{m!}\frac{1}{n!}\|\widehat{\sigma^{n+m}f(0)a^n}\|_{N,K_1+\epsilon B} \leq \\
&\leq C(\epsilon) \sum_{m=0}^{\infty} \sum_{n=k+1}^{\infty} \frac{2^{n+m}}{(n+m)!}\|\hat{\delta}^{n+m}f(0)\|_{N,K_1-\epsilon B}[\alpha(a)]^n \leq \\
&\leq C \cdot C(\epsilon) \sum_{m=0}^{\infty} \sum_{n=k+1}^{\infty} 2^{n+m}\rho^{-(n+m)} = \otimes
\end{aligned}
$$

We note that as in part (i) there is $\alpha(a) > 0$ such that $a \in \alpha(a)[K_1 + \epsilon B]$. In this case we may take $\alpha(a) = 1$ since $a \in K_1$.

$$\otimes = C \cdot C(\epsilon) \sum_{m=0}^{\infty} \left(\frac{2}{\rho}\right)^m \sum_{n=k+1}^{\infty} \left(\frac{2}{\rho}\right)^n \to 0 \quad \text{as} \quad k \to \infty.$$

7. Convolution operators and convolution products

From now on every time we write $\mathcal{H}_{SNb}(E)$, $\mathcal{H}_{Nb}(E)$, $\mathcal{H}_{SN}(E)$ and $\mathcal{H}_N(E)$ we consider these spaces endowed with their natural topologies.

7.1. Definition: A mapping $\mathcal{O}: \mathcal{H}_{SNb}(E) \to \mathcal{H}_{SNb}(E)$ (respectively, $\mathcal{O}: \mathcal{H}_{Nb}(E) \to \mathcal{H}_{Nb}(E)$, $\mathcal{O}: \mathcal{H}_{SN}(E) \mapsto \mathcal{H}_{SN}(E)$, $\mathcal{O}: \mathcal{H}_N(E) \to \mathcal{H}_N(E)$) is called a convolution operator in $\mathcal{H}_{SNb}(E)$ (respectively, $\mathcal{H}_{Nb}(E)$, $\mathcal{H}_{SN}(E)$, $\mathcal{H}_N(E)$) if it is linear continuous and translation invariant (i.e., $\mathcal{O} \circ \tau_a = \tau_a \circ \mathcal{O}$ for all $a \in E$). The vector space of all such convolution operators is denoted by $\mathcal{A}_{SNb}$ (respectively, $\mathcal{A}_{Nb}$, $\mathcal{A}_{SN}$, $\mathcal{A}_N$). It is obviously an algebra with unity under composition of mappings as multiplication.

7.2. Definition: If $T \in \mathcal{H}'_{SNb}(E)$ (respectively, $\mathcal{H}'_{Nb}(E)$, $\mathcal{H}'_{SN}(E)$, $\mathcal{H}'_N(E)$) and $f \in \mathcal{H}_{SNb}(E)$ (respectively, $\mathcal{H}_{Nb}(E)$, $\mathcal{H}_{SN}(E)$, $\mathcal{H}_N(E)$) we define the function

$$
\begin{aligned}
T * f: E &\to \mathbb{C} \\
x &\mapsto (T * f)(x) = T(\tau_{-x}f)
\end{aligned}
$$

and we call $T * f$ the convolution product of T and f.

7.3. Proposition: *If* $T \in \mathcal{H}'_{SNb}(E)$ *(respectively,* $\mathcal{H}'_{Nb}(E)$*) then* $T * f \in \mathcal{H}_{SNb}(E)$ *(respectively,* $\mathcal{H}_{Nb}(E)$*) for all* $f \in \mathcal{H}_{SNb}(E)$ *(respectively,* $\mathcal{H}_{Nb}(E)$*). Moreover* $T* = \mathcal{O} \in \mathcal{A}_{SNb}$ *(respectively,* $\mathcal{A}_{Nb}$*).*

We need a lemma for the proof of this proposition.

7.4. Lemma: *Let* $T \in \mathcal{H}'_{SNb}(E)$ *(respectively,* $T \in \mathcal{H}'_{Nb}(E)$*) so that there is* $B \in \mathcal{B}^c_E$ *and* $C > 0$ *such that* $|T(f)| \leq C\|f\|_{N,B}$ *for every* $f \in \mathcal{H}_{SNb}(E)$ *(respectively,* $f \in \mathcal{H}_{Nb}(E)$*). Then for each* $P \in \mathcal{P}_{bN}({}^nE)$ *(respectively,* $\mathcal{P}_N({}^nE)$*) with* A *in* $\mathcal{L}_{bNs}({}^nE)$ *(respectively,* $\mathcal{L}_{Ns}({}^nE)$*) such that* $P = \hat{A}$*, the polynomial* $y \in E \mapsto T(A \cdot {}^k y^{n-k}) \in \mathbb{C}$ *denoted by* $T\widehat{(A \cdot {}^k)}$ *is in* $\mathcal{P}_{bN}({}^{n-k}E)$ *(respectively,* $\mathcal{P}_N({}^{n-k}E)$*) for every* $k \leq n$*. Further* $\|T\widehat{(A \cdot {}^k)}\|_{N,B} \leq C\|P\|_{N,B}$

Proof: We suppose first that $P \in \mathcal{P}_{bf}({}^nE)$ and $A \in \mathcal{L}_{bfs}({}^nE)$ with $\hat{A} = P$. If $P = \sum_{j=1}^m \widehat{\varphi_j^m}$ with $\varphi_i \in E^*$ for $j = 1, \ldots, m$, we have $T(A \cdot {}^k)(y) = T(A \cdot {}^k y^{n-k}) = \sum_{j=1}^m T(\widehat{\varphi_j^k})[\varphi_j(y)]^{n-k}$ for every $y \in E$, so that

$$T\widehat{(A \cdot {}^k)} = \sum_{j=1}^m T(\widehat{\varphi_j^k})\widehat{\varphi_j^{n-k}} \in \mathcal{P}_{bf}({}^{n-k}E).$$

We also have

$$|T(\widehat{\varphi_j^k})| \leq C\|\widehat{\varphi_j^k}\|_{N,B} = C\|\varphi_j\|_B^k$$

Thus

$$\|T\widehat{(A \cdot {}^k)}\|_{N,B} \leq \sum_{j=1}^m |T(\widehat{\varphi_j^k})|\, \|\varphi_j\|_B^{n-k} \leq$$

$$\leq C \sum_{j=1}^m \|\varphi_j\|_B^n$$

This gives that $\|T\widehat{(A \cdot {}^k)}\|_{N,B} \leq C\|P\|_{N,B}$ for every $P \in \mathcal{P}_{bf}({}^nE)$ and $k \leq n$. The result for arbitrary $P \in \mathcal{P}_{bN}({}^nE)$ follows from the density of $\mathcal{P}_{bf}({}^nE)$ in $\mathcal{P}_{bN}({}^nE)$. The result for the other case has a similar Proof.

Proof of 7.3: By 6.2 we have:

$$(T * f)(x) = T(\tau_{-x}f) = \sum_{n=0}^{\infty} \frac{1}{n!} T(\hat{\delta}^n f(\cdot)x) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{\infty} \frac{1}{i!} T(\widehat{\delta^{i+n} f(0) \cdot {}^i})(x)$$

Now by Lemma 7.4, $T(\widehat{\delta^{i+n}f(0)\cdot{}^{i}})\in\mathcal{P}_{bN}({}^{n}E)$ for every n and $\|T(\widehat{\delta^{i+n}f(0)\cdot{}^{i}})\|_{N,B}\le C\|\hat{\delta}^{i+n}f(0)\|_{N,B}$ where $B\in\mathcal{B}_E^c$ and $C>0$ are such that $|T(f)|\le C\|f\|_{N,B}$ for all f in $\mathcal{H}_{SNb}(E)$. Thus for each $\rho>1$ we have

$$\begin{aligned}&\sum_{i=0}^{\infty}\frac{1}{i!}\|T(\widehat{\delta^{i+n}f(0)\cdot{}^{i}})\|_{N,B}\le\\&\le\sum_{i=0}^{\infty}\frac{C}{i!}\|\hat{\delta}^{i+n}f(0)\|_{N,B}\le C\sum_{i=0}^{\infty}\frac{\rho^{i}}{i!}\|\hat{\delta}^{i+n}f(0)\|_{N,B}\le\\&\le\frac{Cn!}{\rho^{n}}\|f\|_{N,\rho B}\end{aligned}$$

Hence the series $\sum_{i=0}^{\infty}\frac{1}{i!}T(\widehat{\delta^{i+n}f(0)\cdot{}^{i}})$ converges in $\mathcal{P}_{bN}({}^{n}E)$ to an element $P_m\in\mathcal{P}_{bN}({}^{n}E)$. (The above inequality holds for B' replacing B if $B'\supset B$, $B'\in\mathcal{B}_E^c$). Also for all $\rho>1$, and $B'\in\mathcal{B}_E^c$, $B'\supset B$,

$$\|P_n\|_{N,B'}\le\frac{Cn!}{\rho^{n}}\|f\|_{N,\rho B'}\tag{1}$$

This implies

$$\limsup_{n\to\infty}\left[\frac{1}{n!}\|P_n\|_{N,B'}\right]^{1/n}\le\frac{1}{\rho}$$

for all $\rho>1$ and $B'\in\mathcal{B}_E^c$, $B'\supset B$. Hence

$$\lim_{n\to\infty}\left[\frac{1}{n!}\|P_n\|_{N,B'}\right]^{1/n}=0$$

for all $B'\in\mathcal{B}_E^c$ and $T*f=\sum_{n=0}^{\infty}\frac{1}{n!}P_n\in\mathcal{H}_{SNb}(E)$.

In order to prove that the mapping

$$T*:f\in\mathcal{H}_{SNb}(E)\mapsto T*f\in\mathcal{H}_{SNb}(E)$$

is continuous, given $B_1\in\mathcal{B}_E^c$ we have from (1)

$$\|P_n\|_{N,B+B_1}\le\frac{Cn!}{\rho^{n}}\|f\|_{N,\rho(B+B_1)}$$

for all $\rho>1$. Hence

$$\begin{aligned}\|T*f\|_{N,B_1}&=\sum_{n=0}^{\infty}\frac{1}{n!}\|P_n\|_{N,B_1}\le\sum_{n=0}^{\infty}\frac{1}{n!}\|P_n\|_{N,B+B_1}\le\\&\le\sum_{n=0}^{\infty}\frac{C}{\rho^{n}}\|F\|_{N,\rho(B+B_1)}=\frac{C\rho}{\rho-1}\|f\|_{N,\rho(B+B_1)}\end{aligned}$$

It is very easy to show that $T * \in \mathscr{A}_{SNb}$. The proof for the other result is similar.

7.5. PROPOSITION: *If* $T \in \mathscr{H}'_{SN}(E)$ *(respectively,* $\mathscr{H}'_N(E)$*) then* $T * f \in \mathscr{H}_{SN}(E)$ *(respectively,* $\mathscr{H}_N(E)$*) for all* $f \in \mathscr{H}_{SN}(E)$ *(respectively,* $\mathscr{H}_N(E)$*). Moreover* $\mathscr{O} = T * \in \mathscr{A}_{SN}$ *(respectively,* $\mathscr{A}_N$*).*

PROOF: $T \in \mathscr{H}'_{SN}(E)$ implies that there are $B \in \mathscr{B}^c_E$ and $K \in \mathscr{K}(E_B)$ such that for each $\epsilon > 0$ we find $C(\epsilon) > 0$ satisfying

$$(*) \qquad |T(f)| \le C(\epsilon) \sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n f(0)\|_{N,K+\epsilon B}$$

for all $f \in \mathscr{H}_{SN}(E)$.

We fix $f \in \mathscr{H}_{SN}(E)$. Let $D \in \mathscr{B}^c_E$ and $J \in \mathscr{K}(E_D)$ be given. Let $\rho > 2$ be considered. By Proposition 3.3, there are $C > 0$, $\delta > 0$ such that

$$\|\hat{\delta}^n f(0)\|_{N,J \cup K + \delta\Gamma(D \cup B)} \le \frac{Cn!}{\rho^n}$$

for all $n \in \mathbb{N}$, where $\Gamma(D \cup B)$ is the closed bounded balanced convex hull of $D \cup B$. We note that (*) holds for K replaced by $K \cup J$ and B replaced by $\Gamma(D \cup B)$. From previous results we know that

$$(T * f)(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} T(\widehat{\delta^{n+m} f(0) \cdot {}^m})(x)$$

We have $T(\widehat{\delta^{n+m} f(0) \cdot {}^m}) \in \mathscr{P}_{bN}({}^nE)$ for all $n \in \mathbb{N}$ and

$$\|T(\widehat{\delta^{n+m} f(0) \cdot {}^m})\|_{N,K \cup J + \delta\Gamma(D \cup B)} \le C(\delta) \|\hat{\delta}^{n+m} f(0)\|_{N,K \cup J + \delta\Gamma(D \cup B)}$$

(The proof goes like the analogous result of 7.4). Hence

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n (T * f)(0)\|_{N,K \cup J + \delta\Gamma(D \cup B)} \le$$

$$\le \sum_{n=0}^{\infty} \frac{1}{n!} \left\| \sum_{m=0}^{\infty} \frac{1}{m!} T(\widehat{\delta^{n+m} f(0) \cdot {}^m}) \right\|_{N,K \cup J + \delta\Gamma(D \cup B)} \le$$

$$\le \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} \|T(\widehat{\delta^{n+m} f(0) \cdot {}^m})\|_{N,K \cup J + \delta\Gamma(D \cup B)} \le$$

$$\le C(\delta) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} \|\hat{\delta}^{n+m} f(0)\|_{N,K \cup J + \delta\Gamma(D \cup B)} \le$$

$$\leq C(\delta)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2^{n+m}}{(n+m)!}\|\hat{\delta}^{n+m}f(0)\|_{N,K\cup J+\delta\Gamma(D\cup B)}\leq$$

$$\leq C(\delta)\cdot C\cdot\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{2^{n+m}}{\rho^{n+m}}=C(\delta)\cdot C\left[\sum_{n=0}^{\infty}\left(\frac{2}{\rho}\right)^{n}\right]^{2}<+\infty$$

Hence $T * f \in \mathcal{H}_{SN}(E)$.

It is easy to see that $T*$ is linear and translation invariant in $\mathcal{H}_{SN}(E)$. In order to show that $T*$ is continuous we consider a seminorm q in $\mathcal{H}_{SN}(E)$ N-ported by (J, D) with $D \in \mathcal{B}_E^c$ and $J \in \mathcal{K}(E_D)$. Hence for each $\epsilon > 0$ there is $d(\epsilon) > 0$ such that

$$q(g)\leq d(\epsilon)\sum_{n=0}^{\infty}\frac{1}{n!}\|\hat{\delta}^{n}g(0)\|_{N,J+\epsilon D} \tag{**}$$

for all $g \in \mathcal{H}_{SN}(E)$. We want to show that $p = q \circ T*$ is a continuous seminorm in $\mathcal{H}_{SN}(E)$. We have:

$$q(T*f)\leq\sum_{n=0}^{\infty}\frac{1}{n!}q\left(\sum_{m=0}^{\infty}\frac{1}{m!}T(\widehat{\delta^{n+m}f(0)\cdot{}^{m}})\right)\leq$$

$$\leq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!m!}q(T(\widehat{\delta^{n+m}f(0)\cdot{}^{m}}))\leq$$

$$\leq\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{d(\epsilon)}{n!m!}C(\epsilon)\|\hat{\delta}^{n+m}f(0)\|_{N,K\cup J+\epsilon\Gamma(B\cup D)}=$$

$$=d(\epsilon)C(\epsilon)\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{1}{n!}\frac{1}{(m-n)!}\|\hat{\delta}^{m}f(0)\|_{N,K\cup J+\epsilon\Gamma(B\cup D)}=$$

$$=d(\epsilon)C(\epsilon)\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}\|\hat{\delta}^{m}f(0)\|_{N,J\cup K+\epsilon\Gamma(D\cup B)}=$$

$$=d(\epsilon)C(\epsilon)\sum_{m=0}^{\infty}\frac{1}{m!}2^{m}\|\hat{\delta}^{m}f(0)\|_{N,J\cup K+\epsilon\Gamma(D\cup B)}=$$

$$=d(\epsilon)C(\epsilon)\sum_{m=0}^{\infty}\frac{1}{m!}\|\hat{\delta}^{m}f(0)\|_{N,2(J\cup K)+2\epsilon\Gamma(D\cup B)}$$

for all $\epsilon > 0$ and $f \in \mathcal{H}_{SN}(E)$. It follows that $q \circ T*$ is a seminorm in $\mathcal{H}_{SN}(E)$ N-ported by $(2(J \cup K), \Gamma(D \cup B))$. The proof for the other part is similar.

7.6. Definition: The mapping γ_{SNb} is defined by

$$\gamma_{SNb}:\mathcal{A}_{SNb}\to\mathcal{H}'_{SNb}(E)$$
$$\mathcal{O}\mapsto\gamma_{SNb}\mathcal{O}$$

where $(\gamma_{SNb}\mathcal{O})(f) = (\mathcal{O}f)(0)$ for all $f \in \mathcal{H}_{SNb}(E)$. We define similarly the mappings

$$\gamma_{Nb} : \mathcal{A}_{Nb} \to \mathcal{H}'_{Nb}(E)$$
$$\gamma_{SN} : \mathcal{A}_{SN} \to \mathcal{H}'_{SN}(E)$$
$$\gamma_{N} : \mathcal{A}_{N} \to \mathcal{H}'_{N}(E)$$

7.7. PROPOSITION: *The mappings γ_{SNb}, γ_{Nb}, γ_{SN} and γ_N are linear bijections.*

PROOF: We just show that γ_{SNb} is a linear bijection. The proof for the other cases are similar. We consider the mapping $\gamma'_{SNb} : \mathcal{H}'_{SNb}(E) \mapsto \mathcal{A}_{SNb}$ defined by $\gamma'_{SNb}(T) = T\,*$ for all $T \in \mathcal{H}'_{SNb}(E)$. We have

$$\begin{aligned}
[(\gamma'_{SNb} \circ \gamma_{SNb})\mathcal{O}](f) &= [\gamma'_{SNb}(\gamma_{SNb}\mathcal{O})](f) = \\
&= (\gamma_{SNb}\mathcal{O}) * f = \mathcal{O}(f) \quad \text{because} \quad [(\gamma_{SNb}\mathcal{O}) * f](x) = \\
&= (\gamma_{SNb}\mathcal{O})(\tau_{-x}f) = [\mathcal{O}(\tau_{-x}f)](0) = [\tau_{-x}(\mathcal{O}f)](0) = \\
&= (\mathcal{O}f)(x) \quad \text{for all } x \in E, f \in \mathcal{H}_{SNb}(E) \quad \text{and } \mathcal{O} \in \mathcal{A}_{SNb}.
\end{aligned}$$

Hence $\gamma'_{SNb} \circ \gamma_{SNb} =$ identity in $\mathcal{A}_{SNb}$. Also for every $T \in \mathcal{H}'_{SNb}(E)$ and $f \in \mathcal{H}_{SNb}(E)$ we have

$$\begin{aligned}
[(\gamma_{SNb} \circ \gamma'_{SNb})T](f) &= [\gamma_{SNb}(\gamma'_{SNb}T)](f) = \\
&= [(\gamma'_{SNb}T)(f)](0) = (T * f)(0) = T(f).
\end{aligned}$$

Hence $\gamma_{SNb} \circ \gamma'_{SNb} =$ identity in $\mathcal{H}'_{SNb}(E)$.

7.8. DEFINITION: For $T_i \in \mathcal{H}'_{SNb}(E)$ (respectively, $\mathcal{H}'_{Nb}(E)$, $\mathcal{H}'_{SN}(E)$, $\mathcal{H}'_{N}(E)$), $\mathcal{O}_i = T_i\,*$, $i = 1, 2$ we define $T_1 * T_2 = \gamma_{SNb}(\mathcal{O}_1 \circ \mathcal{O}_2)$ (respectively, $T_1 * T_2 = \gamma_{Nb}(\mathcal{O}_1 \circ \mathcal{O}_2)$, $T_1 * T_2 = \gamma_{SN}(\mathcal{O}_1 \circ \mathcal{O}_2)$, $T_1 * T_2 = \gamma_N(\mathcal{O}_1 \circ \mathcal{O}_2)$) which is an element of $\mathcal{H}'_{SNb}(E)$ (respectively, $\mathcal{H}'_{Nb}(E)$, $\mathcal{H}'_{SN}(E)$, $\mathcal{H}'_{N}(E)$). We say that $T_1 * T_2$ is the convolution product or simply the convolution of T_1 and T_2.

7.9. REMARK: We note that $(T_1 * T_2) * f = T_1 * (T_2 * f)$ if T_1, $T_2 \in \mathcal{H}'_{SNb}(E)$ (respectively, $\mathcal{H}'_{Nb}(E)$, $\mathcal{H}'_{SN}(E)$, $\mathcal{H}'_{N}(E)$) and $f \in \mathcal{H}_{SNb}(E)$ (respectively, $\mathcal{H}_{Nb}(E)$, $\mathcal{H}_{SN}(E)$, $\mathcal{H}_{N}(E)$). We also note that the mappings γ_{SNb}, γ_{Nb}, γ_N, γ_{SN} preserve convolution products. In all spaces considered the convolution product is associative and have a unity δ defined by $\delta(f) = f(0)$. Hence $\mathcal{H}'_{SNb}(E)$, $\mathcal{H}'_{Nb}(E)$, $\mathcal{H}'_{SN}(E)$ and $\mathcal{H}'_{N}(E)$

are algebras with unity under the usual vector spaces operations and the convolution product as multiplication. They are called convolution algebras.

8. The Borel transformations

8.1. DEFINITION: The Borel transformation $\hat{T}$ of $T \in \mathcal{H}'_{SNb}(E)$ (respectively, $\mathcal{H}'_{SN}(E)$) is defined by

$$\hat{T} : \varphi \in E^* \mapsto \hat{T}(\varphi) = T(e^{\varphi}) \in \mathbb{C}$$

The Borel transformation $\hat{S}$ of $S \in \mathcal{H}'_{Nb}(E)$ (respectively, $\mathcal{H}'_N(E)$) is given by

$$\hat{S} : \varphi \in E' \mapsto \hat{S}(\varphi) = S(e^{\varphi}) \in \mathbb{C}$$

We shall use later the following result

8.2. PROPOSITION: (1) *The mapping $\beta_b : \mathcal{P}'_{bN}(^nE) \mapsto \mathcal{P}(^nE^*)$ defined by $[\beta_b(T)](\varphi) = T(\widehat{\varphi^n})$ for every $\varphi \in E^*$ and $T \in \mathcal{P}'_{bN}(^nE)$ establishes an isomorphism between the two spaces such that for each $B \in \mathcal{B}_E$, $T \in \mathcal{P}'_{bN}(^nE)$.*

$$\begin{aligned} \|T\|_{N,B} &= \sup\{|T(\widehat{\varphi^n})|;\ \varphi \in E^*, \|\varphi\|_B \le 1\} = \|\beta_b(T)\|_{B^\bullet} \qquad (*) \\ &= \sup\{|T(P)|;\ \|P\|_{N,B} \le 1, P \in \mathcal{P}_{bN}(^nE)\} \end{aligned}$$

(2) *The mapping $\beta : \mathcal{P}'_N(^nE) \mapsto \mathcal{P}(^nE')$ defined by $[\beta(T)](\varphi) = T(\widehat{\varphi^n})$ for all $\varphi \in E'$ and $T \in \mathcal{P}'_N(^nE)$ establishes an isomorphism between the spaces such that for every $B \in \mathcal{B}_E$, $T \in \mathcal{P}'_N(^nE)$,*

$$\begin{aligned} \|T\|_{N,B} &= \sup\{|T(\widehat{\varphi^n});\ \|\varphi\|_B \le 1, \varphi \in E'\} = \|\beta T\|_{B^\circ} \\ &= \sup\{|T(P)|;\ \|P\|_{N,B} \le 1, P \in \mathcal{P}_{bN}(^nE)\} \end{aligned}$$

Here $\qquad B^\circ = \{\varphi \in E';\ |\varphi(x)| \le 1 \forall x \in B\}$ *and*

$$B^\bullet = \{\varphi \in E^*;\ |\varphi(x)| \le 1\ \forall x \in B\}.$$

PROOF: We prove part (1). The proof of part (2) is similar. It is clear that β_b is linear and well-defined. For each $\varphi \in E^*$ we have $|[\beta_b(T)](\varphi)| = |T(\widehat{\varphi^n})| \le \|T\|_{N,B}\|\varphi\|_B^n$. (Here $\|T\|_{N,B}$ may be $+\infty$ and we know that for each $T \in \mathcal{P}'_{bn}(^nE)$ there is a $B \in \mathcal{B}_E$ which makes

$\|T\|_{N,B} < +\infty$). Hence $\|\beta_b(T)\|_{B^\bullet} \leq \|T\|_{N,B}$. Now let $P \in \mathcal{P}_{bf}(^nE^*)$. If $P = \sum_{i=1}^m \varphi_i^n$ we have

$$|T(P)| = \left| T\left(\sum_{i=1}^m \widehat{\varphi_i^n}\right)\right| \leq$$

$$\leq \sum_{i=1}^m |T(\widehat{\varphi_i^n})| = \sum_{i=1}^m |[\beta_b(T)](\varphi_i)| \leq \|\beta_b(T)\|_{B^\bullet} \sum_{i=1}^m \|\varphi_i\|_B^n$$

Hence $|T(P)| \leq \|\beta_b(T)\|_{B^\bullet}\|P\|_{N,B}$ and $\|\beta_b(T)\|_{B^\bullet} \geq \|T\|_{N,B}$. Thus for every $T \in \mathcal{P}'_{bN}(^nE)$ and $B \in \mathcal{B}_E \|T\|_{N,B} = \|\beta_b(T)\|_{B^\bullet}$ (Here when one of the sides is finite the other is also finite). Hence β is injective. Now, for $P' \in \mathcal{P}(^nE^*)$ define

$$T_{p'}: P \in \mathcal{P}_{bf}(^nE) \mapsto T_{P'}(P) = \sum_{i=1}^m P'(\varphi_i) \in \mathbb{C}$$

where $P = \sum_{i=1}^m \widehat{\varphi_i^n} \in \mathcal{P}_{bf}(^nE)$. We have

$$|T_{p'}(P)| = \left|\sum_{i=1}^m P'(\varphi_i)\right| \leq \|P'\|_{B^\bullet} \sum_{i=1}^m \|\varphi_i\|_B^n$$

with $\|P'\|_{B^\bullet} < +\infty$ for some $B \in \mathcal{B}_E$. Hence $|T_{p'}(P)| \leq \|P'\|_{B^\bullet}\|P\|_{N,B}$ for all $P \in \mathcal{P}_{bf}(^nE)$. Thus $T_{p'}$ is linear and continuous for the nuclear topology in $\mathcal{P}_{bf}(^nE)$. Since $\mathcal{P}_{bf}(^nE)$ is dense in $\mathcal{P}_{bN}(^nE)$ we can extend $T_{p'}$ continuously to $\mathcal{P}_{bN}(^nE)$. We also have $T_{p'}(\widehat{\varphi^n}) = P'(\varphi)$ for all $\varphi \in E^*$. Thus $\beta_b(T_{P'}) = P'$ and β_b is surjective.

8.3. Definition: An entire function $f \in \mathcal{H}(E^*)$ is said to be of exponential-type on E^* if there are $B \in \mathcal{B}_E^c$ and $C > 0$ such that $|f(\varphi)| \leq Ce^{\|\varphi\|_B}$ for all $\varphi \in E^*$. We denote by Exp E^*, the algebra, under usual vector-space operations and pointwise multiplication, of entire functions of exponential type on E^*. Similar definition for Exp E'.

8.4. Proposition: (1) *For each* $T \in \mathcal{H}'_{SNb}(E)$ *the Borel transformation* $\hat{T} \in \operatorname{Exp} E^*$ *and the mapping* $T \in \mathcal{H}'_{SNb}(E) \mapsto \hat{T} \in \operatorname{Exp} E^*$ *is an algebra isomorphism between the two spaces.*

(2) *For every* $T \in \mathcal{H}'_{Nb}(E)$, *we have* $\hat{T} \in \operatorname{Exp} E'$ *and the mapping* $T \in \mathcal{H}'_{Nb}(E) \mapsto \hat{T} \in \operatorname{Exp} E'$ *is an algebra isomorphism between the two spaces.*

Proof: We prove part (1). The proof of part (2) is similar. Since for each $\varphi \in E^*$ we have $e^\varphi = \sum_{n=0}^\infty \frac{1}{n!}\widehat{\varphi^n}$ in the sense of $\mathcal{H}_{SNb}(E)$, we can write $\hat{T}(\varphi)=\sum_{n=0}^{\infty}\frac{1}{n!}T(\widehat{\varphi^n})$. As we know $\mathcal{P}_{bN}(^nE)$ is closed subspace of $\mathcal{H}_{SNb}(E)$. We set $T_n = T \mid \mathcal{P}_{bN}(^nE)\in\mathcal{P}'_{bN}(\underline{E})$. By Proposition 8.2, there is a unique $P'_n\in\mathcal{P}(^nE^*)$ such that $T_n(\widehat{\varphi^n})=P'_n(\varphi)$ for all $\varphi\in E^*$ and $\|T_n\|_{N,B}=\|P'_n\|_{B^\bullet}$ for all $n\in\mathbb{N}$ and all $B\in\mathcal{B}_E^c$. Since T is continuous in $\mathcal{H}_{SNb}(E)$ there is $D\in\mathcal{B}_E^c$ and $C>0$ such that $|T(f)|\leq C\|f\|_{N,D}$ for all $f\in\mathcal{H}_{SNb}(E)$. This gives $\|T_n\|_{N,D}=\|P'_n\|_{D^\bullet}\leq C$ and

$$\limsup_{n\to\infty}\|P'_n\|_{D^\bullet}^{1/n}\leq 1.$$

Hence

$$\hat{T}(\varphi)=\sum_{n=0}^{\infty}\frac{1}{n!}T(\widehat{\varphi^n})=\sum_{n=0}^{\infty}\frac{1}{n!}P'_n(\varphi)$$

defines an entire function of exponential type on E^*. Thus $T\in\mathcal{H}'_{SNb}(E)\mapsto\hat{T}\in\operatorname{Exp}D^*$ is a well defined linear mapping and it is injective by Proposition 4.5. If, now, $H=\sum_{n=0}^{\infty}\frac{1}{n!}P'_n\in\operatorname{Exp}E^*$, then there is $D\in\mathcal{B}_E^c$ such that $\{\|P'_n\|_{D^\bullet}^{1/n}\}$ is a bounded sequence. Hence there are $C>0$ and $\rho>0$ such that $\|P'_n\|_{D^\bullet}\leq C\rho^n$ for all $n\in\mathbb{N}$. By 8.2, there is a unique $H_n\in\mathcal{P}'_{bN}(^nE)$ such that $H_n(\widehat{\varphi^n})=P'_n(\varphi)$ for all $\varphi\in E^*$ and $\|H_n\|_{N,D}=\|P'_n\|_{D^\bullet}$. For every $f\in\mathcal{H}_{SNb}(E)$, $f=\sum_{n=0}^{\infty}P_n$ we set $T(f)=\sum_{n=0}^{\infty}H_n(P_n)$. Now

$$\sum_{n=0}^{\infty}|H_n(P_n)|\leq\sum_{n=0}^{\infty}\|H_n\|_{N,D}\|P_n\|_{N,D}\leq C\sum_{n=0}^{\infty}\rho^n\|P_n\|_{N,D}=C\|f\|_{N,\rho D}.$$

Thus $T\in\mathcal{H}'_{SNb}(E)$ and

$$\hat{T}(\varphi)=T(e^\varphi)=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(\widehat{\varphi^n})=\sum_{n=0}^{\infty}\frac{1}{n!}P'_n(\varphi)=H(\varphi)$$

for all $\varphi\in E^*$. Thus $\hat{T}=H$ and the Borel transformation $T\in\mathcal{H}'_{SNb}(E)\mapsto\hat{T}\in\operatorname{Exp}E^*$ is surjective. Now, using the fact that $T*e^\varphi=e^\varphi T(e^\varphi)$ for all $T\in\mathcal{H}'_{SNb}(E)$ and $\varphi\in E^*$ we prove easily that $T_1*T_2=\hat{T}_1\cdot\hat{T}_2$ for all $T_1, T_2\in\mathcal{H}'_{SNb}(E)$.

8.5. Definition: A function $f\in\mathcal{H}(E^*)$ (respectively, $\mathcal{H}(E')$) is said to be of compact exponential type if there are $B\in\mathcal{B}_E^c$, $K\in\mathcal{K}(E_B)$ such that for every $\epsilon>0$ there is $c(\epsilon)>0$ satisfying

$$|f(\varphi)|\leq c(\epsilon)e^{\|\varphi\|_K+\epsilon\|\varphi\|_B}$$

for all $\varphi\in E^*$ (respectively, $\varphi\in E'$). Note:

$$\|\varphi\|_K+\epsilon\|\varphi\|_B=\|\varphi\|_{K+\epsilon B}$$

We denote by $\mathrm{Exp}_c E^*$ (respectively, $\mathrm{Exp}_c E'$) the set of all functions in $\mathscr{H}(E^*)$ (respectively, $\mathscr{H}(E')$) which are of compact exponential type. $\mathrm{Exp}_c E^*$ (respectively, $\mathrm{Exp}_c E'$) is an algebra under the usual vector space operations and the pointwise multiplication.

8.6. PROPOSITION: *For each $T \in \mathscr{H}'_{SN}(E)$ (respectively, $\mathscr{H}'_N(E)$) its Borel transform $\hat{T} \in \mathrm{Exp}_c E^*$ (respectively, $\mathrm{Exp}_c E'$). The mapping $T \in \mathscr{H}'_{SN}(E) \mapsto \hat{T} \in \mathrm{Exp}_c E^*$ (respectively, $T \in \mathscr{H}'_N(E) \mapsto \hat{T} \in \mathrm{Exp}_c E'$) is an algebra isomorphism.*

PROOF: We prove the result for $T \in \mathscr{H}'_{SN}(E)$. The proof for the other case is similar. For each $\varphi \in E^*$ we have $e^\varphi = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\varphi^n}$ in the sense of the topology of $\mathscr{H}_{SN}(E)$. Since $\mathscr{P}_{bN}({}^nE)$ is a closed vector subspace of $\mathscr{H}_{SN}(E)$, if we set $T_n = T \mid \mathscr{P}_{bN}({}^nE)$ we have $T_n \in \mathscr{P}'_{bN}({}^nE)$ and there is $P'_n \in \mathscr{P}({}^nE^*)$ such that $T_n(\widehat{\varphi^n}) = P_n(\varphi)$ for all $\varphi \in E^*$. Moreover $\|T_n\|_{N,B} = \|P'_n\|_{B^\bullet}$ for all $B \in \mathscr{B}_E$. Let $T \in \mathscr{H}'_{SN}(E)$. Thus there are $D \in \mathscr{B}^c_B$ and $K \in \mathscr{K}(E_D)$ such that for all $\epsilon > 0$ there exists $c(\epsilon) > 0$ satisfying

$$|T(f)| \leq c(\epsilon) \sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{d}^n f(0)\|_{N,K+\epsilon D}$$

for each $f \in \mathscr{H}_{SN}(E)$. It follows that

$$\limsup_{n\to\infty} \|T_n\|_{N,K+\epsilon D}^{1/n} \leq 1.$$

Hence, $\hat{T}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} T(\widehat{\varphi^n}) = \sum_{n=0}^{\infty} \frac{1}{n!} P'_n(\varphi)$ is an entire function on E^*. We take $f = e^\varphi$ with $\varphi \in E^*$. Hence $\hat{d}^n f(0) = \widehat{\varphi^n}$ and $\|\hat{\delta}^n f(0)\|_{N,K+\epsilon D} = \|\varphi\|^n_{K+\epsilon D} = [\|\varphi\|_K + \epsilon\|\varphi\|_D]^n$. It follows that

$$|\hat{T}(\varphi)| \leq c(\epsilon) e^{\|\varphi\|_K + \epsilon\|\varphi\|_D}$$

for all $\varphi \in E^*$. Thus $\hat{T} \in \mathrm{Exp}_c E^*$.

Let $F \in \mathrm{Exp}_c E^*$ with $|F(\varphi)| \leq c(\epsilon) e^{\|\varphi\|_K + \epsilon\|\varphi\|_D}$ for all $\varphi \in E^*$. Here D is a fixed element of $\mathscr{B}^c_E$ and $K \in \mathscr{K}(E_D)$ also fixed. If $F = \sum_{n=0}^{\infty} \frac{1}{n!} P'_n$ is the Taylor's series of F at 0 we have $\|P'_n\|_{(K+\epsilon D)^\bullet} \leq c(\epsilon) \left(\frac{e}{n}\right)^n n!$ for all $n \in \mathbb{N}$ (Use Cauchy's inequality). Thus there is $T_n \in \mathscr{P}'_{bN}({}^nE^*)$ such that $T_n(\widehat{\varphi^n}) = P'_n(\varphi)$ for all $\varphi \in E^*$ and $\|T_n\|_{N,K+\epsilon D} = \|P'_n\|_{(K+\epsilon D)^\bullet}$. If

$f \in \mathcal{H}_{SN}(E)$ $f = \sum_{n=0}^{\infty} P_n$ we define $T(f) = \sum_{n=0}^{\infty} T_n(P_n)$. We have

$$\begin{aligned}\sum_{n=0}^{\infty} |T_n(P_n)| \leq \sum_{n=0}^{\infty} \|T_n\|_{N,K+\epsilon D} \|P_n\|_{N,K+\epsilon D} \leq \\ \leq c(\epsilon) \sum_{n=0}^{\infty} \left(\frac{e}{n}\right)^n n! \|P_n\|_{N,K+\epsilon D} \leq \\ \leq c_1(\epsilon) \sum_{n=0}^{\infty} \|P_n\|_{N,K+\epsilon D} < +\infty\end{aligned}$$

provided $\epsilon > 0$ is small enough. Hence

$$|T(f)| \leq c_1(\epsilon) \sum_{n=0}^{\infty} \frac{1}{n!} \|\hat{\delta}^n f(0)\|_{N,K+\epsilon D}$$

for all $f \in \mathcal{H}_{SN}(E)$. Thus $T \in \mathcal{H}'_{SN}(E)$ and it is clear that $\hat{T} = F$. By 5.3 we see that the Borel transformation is injective. It is surjective by the above reasoning and the rest of the proof is straightforward.

9. Malgrange's theorems for convolution operators

In this section we prove generalizations of the results of B. Malgrange (see [57]) on approximation and existence of solutions for convolution equations in $\mathcal{H}_{SNb}(E)$, $\mathcal{H}_{Nb}(E)$, $\mathcal{H}_{SN}(E)$ and $\mathcal{H}_N(E)$. See Gupta [48], [49] and Gupta–Nachbin [50] for these results when E is a Banach space.

We shall need some division results.

9.1. Proposition: *Let f_1, f_2, $f_3 \in \mathcal{H}(E^*)$ (respectively, $\mathcal{H}(E')$) such that $f_1 = f_2 \cdot f_3$ and f_1, $f_2 \in Exp\ E^*$ (respectively, $Exp\ E'$), $f_2 \not\equiv 0$. Then $f_3 \in Exp\ E^*$ (respectively, $Exp\ E'$).*

9.2. Lemma (See [57]): *If $c_1 \geq 0, C_1 \geq 0, c_2 \geq 0, C_2 \geq 0$ are given, there exist $c_3 \geq 0$, $C_3 \geq 0$ such that for f_1, f_2, $f_3 \in \mathcal{H}(\mathbf{C})$, $f_1 = f_2 \cdot f_3$, $f_1(0) = f_2(0) = 1$ and $|f_1(z)| \leq C_1 e^{c_1|z|}$, $f_2(z) \leq C_2 e^{c_2|z|}$ for all $z \in \mathbf{C}$, then we have $|f_3(z)| \leq C_3 e^{c_3|z|}$ for every $z \in \mathbf{C}$.*

Proof of 9.1: Since $f_2 \not\equiv 0$ and $f_1 = f_2 \cdot f_3$ we have $f_3 \equiv 0$ if $f_1 \equiv 0$. Hence the result is trivially true if $f_1 \equiv 0$. If $f_1 \not\equiv 0$ we may suppose $f_1(0) = f_2(0) = 1$ by making a translation in E and by multiplying f_1, f_2, f_3 by suitable constants, if necessary. Since f_1, $f_2 \in \text{Exp}\ E^*$ and

$F_3 \in \mathscr{H}(E^*)$ we find $B \in \mathscr{B}_E^c$, $C_1 \geq 0$, $C_2 \geq 0$, $M \geq 0$ such that

$$\text{(1)} \ |f_i(\varphi)| \leq C_i e^{\|\varphi\|_B} \quad \text{for } i = 1, 2 \quad \text{and } \varphi \in E^*$$
$$\text{(2)} \ |f_2(\varphi)| \leq M \quad \text{if } \varphi \in E^* \quad \text{and } \|\varphi\|_B \leq 1.$$

If $\|\varphi\|_B \neq 0$ we define $g_i \in \mathscr{H}(\mathbb{C})$ by $g_i(z) = f_i\left(z \frac{\varphi}{\|\varphi\|_B}\right)$ for all $z \in \mathbb{C}$, $i = 1, 2, 3$. We have $g_1(0) = g_2(0) = 1$, $g_1 = g_2 \cdot g_3$, $|g_1(z)| \leq C_1 e^{|z|}$, $|g_2(z)| \leq C_2 e^{|z|}$ for all $z \in \mathbb{C}$. By 9.2, there are $C_3 \geq 0$, $c_3 \geq 0$ (independent of φ) such that $|g_3(z)| \leq C_3 e^{c_3|z|}$ for all $z \in \mathbb{C}$. If we take $z = \|\varphi\|_B$ this inequality becomes $|f_3(\varphi)| \leq C_3 e^{c_3\|\varphi\|_B}$. We take $D = c_3 B$. Hence $|f_3(\varphi)| \leq C_3 e^{\|\varphi\|_D}$ for all $\varphi \in E^*$, $\|\varphi\|_B \neq 0$. If $\|\varphi\|_B = 0$, by (2) we have $|f_3(\varphi)| \leq M$. Hence $|f_3(\varphi)| \leq C e^{\|\varphi\|_D}$ for all $\varphi \in E^*$ with $C = \max\{C_3, M\}$. Thus $f_3 \in \operatorname{Exp} E^*$. The proof of the other part is similar.

9.3. Proposition: *Let f_1, f_2, $f_3 \in \mathscr{H}(E^*)$ (respectively, $\mathscr{H}(E')$) such that $f_1 = f_2 \cdot f_3$, $f_i \in Exp_c E^*$ (respectively, $Exp_c E'$) $i = 1, 2$, $\varphi_2 \neq 0$. Then $f_3 \in Exp_c E^*$ (respectively, $Exp_c E'$).*

9.4. Lemma (See [65]): *If $f, g \in \mathscr{H}(\mathbb{C})$ with $g(0) \neq 0$* and $\frac{f}{g} \in \mathscr{H}(\mathbb{C})$, *then* $M\left(r, \frac{f}{g}\right) \leq \frac{1}{|g(0)|^3}[1 + M(2r, f)]^3 [1 + M(2r, g)]^3$ *for all $r > 0$. Here*

$$M(r, f) = \sup\{|f(z)|; |z| \leq r\}.$$

Proof of 9.3: Since f_1, $f_2 \in Exp_c E^*$ and $f_3 \in \mathscr{H}(E^*)$ we can find $B \in \mathscr{B}_E^c$, $K \in \mathscr{K}(E_B)$ such that

(1) For all $\epsilon > 0$ there are $C_1(\epsilon) \geq 0$, $C_2(\epsilon) \geq 0$ satisfying

$$|f_i(\varphi)| \leq C_i(\epsilon) e^{\|\varphi\|_{K+\epsilon B}} \quad \text{for } i = 1, 2, \ \varphi \in E^*$$

(2) For all $\epsilon > 0$, $\epsilon < 1$, there is $M(\epsilon) \geq 0$ such that

$$|f_3(\varphi)| \leq M(\epsilon) \quad \text{if } \|\varphi\|_{\epsilon B} \leq 1, \ \varphi \in E^*$$

With no loss of generality we still may suppose that $f_3(0) \neq 0$. If $\varphi \in E^*$ and $\|\varphi\|_{K+\epsilon B} \neq 0$ we set $g_i(z) = f_i\left(z \frac{\varphi}{\|\varphi\|_{K+\epsilon B}}\right)$ $i = 1, 2, 3$ and $z \in \mathbb{C}$. We have $g_3 = \frac{g_1}{g_2} \in \mathscr{H}(\mathbb{C})$, $g_2(0) \neq 0$. By 9.4, we have for all

$r > 0$

$$
\begin{aligned}
M(r, g_3) &\leq \frac{1}{|g_2(0)|^3} [1 + M(2r, g_1)]^3 [1 + M(2r, g_2)]^3 \leq \\
&\leq \frac{1}{|g_2(0)|^3} [1 + C_1(\epsilon) e^{2r}]^3 [1 + C_2(\epsilon) e^{2r}]^3 \leq \\
&\leq \frac{[1 + C_1(\epsilon)]^3 [1 + C_2(\epsilon)]^3}{|g_2(0)|^3} e^{12r}
\end{aligned}
$$

If we take $r = \|\varphi\|_{K+\epsilon B}$ we get

$$
|f_3(\varphi)| \leq \frac{1}{|f_2(0)|^3} [1 + C_1(\epsilon)]^3 [1 + C_2(\epsilon)]^3 \, e^{12\|\varphi\|_{K+\epsilon B}} = d(\epsilon) \, e^{\|\varphi\|_{12K+\epsilon 12B}}
$$

with $k(\epsilon) = \max\{M(\epsilon), d(\epsilon)\}$. Thus $f^3 \in \mathrm{Exp}^c E^*$.

9.5. Proposition: *Let U be a non empty open connected subset of a complex locally convex space F. Let f,g be elements of $\mathcal{H}(U)$ g not identically zero, such that for every affine subspace S of F, of dimension one, and for any connected component S' of $S \cap U$, where g is not identically zero, $f \mid S'$ is divisible by $g \mid S'$ with the quotient holomorphic in S'. Then f is divisible by g and the quotient belongs to $\mathcal{H}(U)$.*

Proof: It is enough to prove the result locally. If $x_0 \in U$, there is $y \in F$ such that $g(x_0 + y) \neq 0$ and $x_0 + \lambda y \in U$ for all $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. Since the zeros of a holomorphic function of one complex variable are isolated there is $0 < r < 1$ such that $|g(x_0 + \lambda y)| > 0$ for all $\lambda \in \mathbb{C}$, $|\lambda| = r$. Since g is continuous in U and $\{x_0 + \lambda y; |\lambda| = r, \lambda \in \mathbb{C}\}$ is compact there is a neighborhood V of x_0, $V + \{\lambda y; |\lambda| = r\} \subset U$ such that $|g(x + \lambda y)| \geq \delta > 0$ for all $x \in V$ and $\lambda \in \mathbb{C}$, $|\lambda| = r$. Now we define

$$
h(x) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{1}{\lambda} \frac{f(x + \lambda y)}{g(x + \lambda y)} \, d\lambda
$$

for all $x \in V$. It is easy to prove that h is locally bounded in V. By our hypothesis there is a holomorphic function φ_x in $\{\lambda \in \mathbb{C}, |\lambda| < 1\}$ such that $f(x + ty) = g(x + ty)\varphi_x(t)$ for all $t \in \mathbb{C}$, $|t| < 1$. Hence

$$
\varphi_x(0) = \frac{1}{2\pi i} \int_{|t|=r} \varphi_x(t) \frac{dt}{t} = \frac{dt}{2\pi i} \int_{|t|=r} \frac{1}{t} \frac{f(x + ty)}{g(x + ty)} \, dt = h(x)
$$

for every x in V. Thus $f(x) = g(x)h(x)$ for all $x \in V$. Since h is finitely holomorphic in V and locally bounded in V, h is holomorphic in V.

9.6. PROPOSITION: (1) *Suppose* $T_1, T_2 \in \mathcal{H}'_{SNb}(E)$ *(respectively,* $\mathcal{H}'_{Nb}(E)$*)* $T_2 \neq 0$, *such that if* $p \in \mathcal{P}_{bN}(^nE)$ *(respectively,* $P \in \mathcal{P}_N(^nE)$*)* $\varphi \in E^*$ *(respectively,* $\varphi \in E'$*),* $T_2 * Pe^{\varphi} = 0$ *implies* $T_1(Pe^{\varphi}) = 0$. *Then* $\hat{T}_1$ *is divisible by* $\hat{T}_2$ *and the quotient is in* Exp E^* *(respectively,* Exp E'*).*

(2) *Suppose* $T_1, T_2 \in \mathcal{H}'_{SN}(E)$ *(respectively,* $\mathcal{H}'_N(E)$*),* $T_2 \neq 0$, *such that if* $P \in \mathcal{P}_{bN}(^nE)$ *(respectively,* $P \in \mathcal{P}_N(^nE)$*),* $\varphi \in E^*$ *(respectively,* $\varphi \in E'$*),* $T_2 * Pe^{\varphi} = 0$ *implies* $T_1(Pe^{\varphi}) = 0$. *Then* $\hat{T}_1$ *is divisible by* $\hat{T}_2$ *and the quotient belongs to* $\mathrm{Exp}_c\, E^*$ *(respectively,* $\mathrm{Exp}_c\, E'$*).*

PROOF: Let S be an affine subspace of dimension 1 of E^*, so that there are $\varphi_1, \varphi_2 \in E^*$ such that $S = \{\varphi_1 + t\varphi_2; t \in \mathbb{C}\}$. If t_0 is a zero of order k of $g(t) = \hat{T}_2(\varphi_1 + t\varphi_s) = T_2(e^{\varphi_1 + t\varphi_2})$ we have $T_2(\widehat{\varphi_2^i e^{\varphi_1 + t_0\varphi_2}}) = 0$ for $i > k$. This gives

$$T_2 * \widehat{\varphi_2^i e}^{\varphi_1 + t_0\varphi_2} = \sum_{j=0}^{i} \binom{i}{j} \widehat{\varphi_2^{i-j}}\, e^{\varphi_1 + t_0\varphi_2} T_2(\widehat{\varphi_2^j e}^{\varphi_1 + t_0\varphi_2}) = 0$$

for all $i < k$. Hence $T_1(\widehat{\varphi_2^i e}^{\varphi_2 t_0 \varphi_2}) = 0$ for every $i < k$. This gives t_0 a zero or order $\geq k$ or $f(t) = \hat{T}_1(\varphi_1 + t\varphi_2)$. Hence $\hat{T}_1 \mid S$ is divisible by $\hat{T}_2 \mid S$ with the quotient holomorphic in S. Thus, by 9.5, there is $h \in \mathcal{H}(E^*)$ such that $\hat{T}_1 = \hat{T}_2 h$. By 9.1 $h \in \mathrm{Exp}\, E^*$. The proofs of the other parts from the same reasoning. Eventually we used 9.3, instead of 9.1.

9.7. THEOREM: *Let* $\mathcal{O}$ *be an* $\mathcal{A}_{SNb}$ *(respectively,* $\mathcal{A}_{Nb}$, $\mathcal{A}_{SN}$, $\mathcal{A}_N$*). Then the vector subspace of* $\mathcal{H}_{SNb}(E)$ *(respectively* $\mathcal{H}_{Nb}(E)$, $\mathcal{H}_{SN}(E)$, $\mathcal{H}_N(E)$*) generated by*

$$\{P \cdot e^{\varphi}; P \in \mathcal{P}_{bN}(^nE), \varphi \in E^*, n \in \mathbb{N}, \mathcal{O}(Pe^{\varphi}) = 0\} \qquad (*)$$

(respectively,

$$\{P \cdot e^{\varphi}; P \in \mathcal{P}_N(^nE), \varphi \in E', n \in \mathbb{N}, \mathcal{O}(Pe^{\varphi}) = 0\} \qquad (**),$$

(*), (**)) *is dense for the natural topology in the closed vector subspace* $\mathcal{K} = \{f \in \mathcal{H}_{SNb}(E); \mathcal{O}f = 0\}$ *(respectively,* $\{f \in \mathcal{H}_{Nb}(E); \mathcal{O}f = 0\}$, $\{f \in \mathcal{H}_{SN}(E); \mathcal{O}f = 0\}$, $\{f \in \mathcal{H}_N(E); \mathcal{O}f = 0\}$*).*

PROOF: We prove one part. The proofs of the other parts are similar. By 4.5, the result holds when $\mathcal{O} = 0$. Now we suppose $\mathcal{O} \neq 0$.

By 7.3, there is $T \in \mathcal{H}'_{SNb}(E)$ such that $\mathcal{O} = T\,*$. Let $X \in \mathcal{H}'_{SNb}(E)$ be such that if $P \in \mathcal{P}_{bN}(^nE)$, $\varphi \in E^*$, $T * Pe^{\varphi} = 0$ implies $X(Pe^{\varphi}) = 0$. Then, by 9.6, there is $h \in \operatorname{Exp} E^*$ such that $\hat{X} = h\hat{T}$. By 8.4, we have $h = \hat{S}$ for some $S \in \mathcal{H}'_{SNb}(E)$. Hence $\hat{X} = \hat{S}\hat{T} = S * T$. Thus $X = S * T$. If $f \in \mathcal{K}$ we have $X * f = S * (T * f) = 0$. Hence $X(f) = (X * f)(0) = 0$. By the Hahn–Banach theorem the result follows.

The preceding theorem is known as the Approximation Theorem for solutions of homogeneous convolution equations. Next theorem is a very important step in order to obtain an Existance Theorem for convolution equations.

9.8. THEOREM: *Let* $\mathcal{O} \in \mathcal{A}_{SNb}$, $\mathcal{O} \neq 0$. *Then* ${}^t\mathcal{O}[\mathcal{H}'_{SNb}(E)] = \{f \in \mathcal{H}_{SNb}(E); \mathcal{O}f = 0\}^{\perp}$ *(the orthogonal of* $\{f \in \mathcal{H}_{SNb}(E); \mathcal{O}f = 0\}$ *in* $\mathcal{H}'_{SNb}(E)$*) and* ${}^t\mathcal{O}[\mathcal{H}'_{SNb}(E)]$ *is closed in the weak topology of* $\mathcal{H}'_{SNb}(E)$ *defined by* $\mathcal{H}_{SNb}(E)$*. Similar results hold when* $\mathcal{O} \neq 0$ *is in* $\mathcal{A}_{Nb}$, $\mathcal{A}_{SN}$, $\mathcal{A}_{N}$.

PROOF: Let $T \in \mathcal{H}'_{SNb}(E)$ be such that $\mathcal{O} = T*$. Let $\mathcal{K}$ be the set $\{f \in \mathcal{H}_{SNb}(E); \mathcal{O}f = 0\}$. For $X \in {}^t\mathcal{O}[\mathcal{H}'_{SNb}(E)]$, *we have* $X = {}^t\mathcal{O}(S)$ for some $S \in \mathcal{H}'_{SNb}(E)$. Hence $X(f) = [{}^t\mathcal{O}(S)](f) = S(\mathcal{O}f) = 0$ for every $f \in \mathcal{K}$. Thus ${}^t\mathcal{O}[\mathcal{H}'_{SNB}(E)] \subset \mathcal{K}^{\perp}$. Now we take $X \in \mathcal{K}^{\perp}$. As in the proof of the preceding result we know that there is $S \in \mathcal{H}'_{SNb}(E)$ such that $X = S * T$. Thus, if $f \in \mathcal{H}_{SNb}(E)$, we get

$$X(f) = (S * T)(f) = [(S * T) * f](0) = [S * (T * f)](0) = \\ = S(T * f) = S(\mathcal{O}f) = [{}^t\mathcal{O}S)(f)$$

Hence $X = {}^t\mathcal{O}S \in {}^t\mathcal{O}[\mathcal{H}'_{SNb}(E)]$ and $\mathcal{K}^{\perp} \subset {}^t\mathcal{O}[\mathcal{H}'_{SNb}(E)]$. Further

$$\mathcal{K}^{\perp} = \cap \{T \in \mathcal{H}'_{SNb}(E); T(f) = 0, \forall f \in \mathcal{K}\}$$

is the intersection of closed subspaces of $\mathcal{H}'_{SNb}(E)$ for the weak topology in $\mathcal{H}'_{SNb}(E)$ defined by $\mathcal{H}_{SNb}(E)$.

9.9. THEOREM: *If* $\mathcal{O} \in \mathcal{A}_{SNb}$ *(respectively,* $\mathcal{A}_{Nb}$*),* $\mathcal{O} \neq 0$, *then* $\mathcal{O}[\mathcal{H}_{SNb}(E)] = \mathcal{H}_{SNb}(E)$ *(respectively,* $\mathcal{O}[\mathcal{H}_{Nb}(E)] = \mathcal{H}_{Nb}(E)$*) if* E *has countable fundamental system of closed balanced bounded convex sets. (This holds if* E *is a* $\mathcal{OF}$*-space).*

PROOF: When E has a countable fundamental system of closed balanced convex bounded sets then $\mathcal{H}_{SNb}(E)$ and $\mathcal{H}_{Nb}(E)$ are Fréchet spaces. By a result of Dieudonné–Schwartz [66] in order to prove that $\mathcal{O}$ is surjective it is enough to show that ${}^t\mathcal{O}: \mathcal{H}'_{SNb}(E) \to \mathcal{H}'_{SNb}(E)$ is injective and ${}^t\mathcal{O}[\mathcal{H}'_{SNb}(E)]$ is closed for the weak topology of $\mathcal{H}'_{SNb}(E)$ defined by $\mathcal{H}_{SNb}(E)$. The later condition is true by 9.8. We prove the former condition: Let $T \in \mathcal{H}'_{SNb}(E)$ such that $\mathcal{O} = T *$. As before we find for every $S \in \mathcal{H}'_{SNb}(E)$ ${}^t\mathcal{O}(S) = S * T$. If ${}^t\mathcal{O}(S) = 0$ for some $S \in \mathcal{H}'_{SNb}(E)$, then $0 = \widehat{S * T} = \hat{S}\hat{T}$. Since $T \neq 0$, $\hat{T} \neq 0$ and $S = 0$. Hence $S = 0$ and ${}^t\mathcal{O}$ is injective. The proof for the other case is similar.

9.10. Remark: If the reader looks at the Appendix of [49] he will have no difficulties to see the modifications which should be made in this article in order to get similar results when E^* does not have the approximation property.

9.11. Remark: The readers must be warned that there are existence theorems for convolution equations in some other spaces of entire functions, for which the countability condition of 9.9 is not needed. See [18], [25], [26], [27], [54] and [55]. Except for the very special case of [54], all the other results appeared after this article was conceived in final form. On the other hand, the mentioned references introduce new spaces of entire functions which, in general, are not the same as the spaces of the nuclearly Silva entire functions of bounded type and of the nuclearly entire functions of bounded type with which we deal. In [27] and in a forthcoming paper by Colombeau and Matos to appear in Functional Analysis, Holomorphy and Approximation Theory (Editor: J.A. Barroso), Birkhäuser Boston, USA, connections among these spaces are studied. We plan to come back to this point in subsequent writings, and simply be brief at this final remark.

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