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PLURICANONICAL EMBEDDINGS

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Dedicated to the memory of a friend and teacher, Aldo Andreotti

1. Introduction

In this article we continue our study of embeddings begun in [8, 9]. An embedding $i : Y \to X$ of a compact complex manifold Y in a complex manifold X is *pluricanonical* if the normal bundle to Y in X is some (kth) symmetric power of the tangent bundle to Y. We are mainly interested in *canonical* embeddings ($k = 1$); in particular, the *diagonal embedding* Δ of Y in $Y \times Y$ and the embedding z_T as the zero section of the tangent bundle. In section 2 we recall the machinery of finite equivalence of embeddings, and the Nirenberg–Spencer [9] obstruction $\delta(\alpha_k)$ to extending a kth order equivalence α_k to an α_{k+1}. Δ and z_T are naturally first order equivalent; this equivalence extends to a second order equivalence if and only if Y admits a holomorphic connection. In this case Δ and z_T are actually analytically equivalent in the sense that they have biholomorphic neighborhoods carrying Δ to z_T.

We say that Y is an HNR in X if there is a holomorphic fibration of a neighborhood of Y in X transversal to Y. Suppose Y is of codimension 1 in X. In section 3 we construct a model, up to second order equivalence, for all such embeddings as follows. Let $b \in H^1(Y, N^{-1})$ be the obstruction to extending the natural first order equivalence of i with z_N. b determines a plane-bundle extension V of N:

$$0 \to \mathcal{O} \to V \to N \to 0,$$

and $\mathbb{P}(V) \to Y$ is a $\mathbb{P}^1$-bundle with a distinguished section σ; i is second order equivalent to σ. We show further that for C a curve and Δ the diagonal embedding, $N^{-1} = K$ and b is the Chern class of C, and

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Δ is actually analytically equivalent to σ in $X_K(b)$. This is completely transcendental if C is of genus more than 1. In this case Δ and σ can be blown down and we show (in section 5) that the blown-down $C\times C$ is an algebraic variety, whereas the blown-down $X_k(b)$ is not. In section 5 we also show that the resulting singular point has a Gorenstein structure ring.

In section 3 we consider also the diagonal embedding Δ_s of C in the symmetric product $C^{(2)}$. The normal bundle is $(TC)^2$ but Δ_s is not even first order equivalent with $(TC)^2$, for Δ_s is not HNR.

In section 4 we describe all pluricanonical embeddings of a curve of genus $g>1$. If the normal bundle is $(TC)^k$, $k>2$, there is only $z_{(TC)^k}$. If the normal bundle is $(TC)^2$, there are only $z_{(TC)^2}$ and Δ_s. For canonical embeddings, the space ξ of embeddings (up to analytic equivalent) is a fiber space over $\mathbb{P}(H^1(K))\cup\{0\}$; the fiber over 0 is a $\mathbb{P}^1$, and other fibers consist in just two points. This construction is completely local along C; we do not know if any of these surfaces can be made algebraic (except for $C\times C$ and TC).

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2. Summary of general results

In this section we shall summarize some results in the general theory of equivalence of embedding (see also [9]), which we shall need in the sequel. In this discussion, Y is a submanifold of the complex manifold X, and $\mathcal{I}_Y$ is the sheaf of ideals of functions vanishing on Y.

DEFINITION 2.1: $Y^{\mathcal{O}(k)} = X^{\mathcal{O}}/\mathcal{I}_Y^{(k+1)}$ is the structure sheaf of the *kth (infinitesimal) neighborhood* of Y in X.

$$Y^{\hat{\mathcal{O}}} = \lim_{\overleftarrow{k\to\infty}} Y^{\mathcal{O}(k)}$$

is the structure sheaf of the *formal neighborhood* $\hat{Y}$ of Y in X.

DEFINITION 2.2: Let $i: Y\to X$, $i': Y\to X'$ be embeddings. A *kth order equivalence* {*formal, actual*} of these embeddings is a bimorphism of ringed spaces $\phi:(Y, {}_Y\mathcal{O}^{(k)} \cong (Y, {}_Y\mathcal{O}'^{(k)})$

$\{aY, {}_Y\hat{\mathcal{O}}) \cong (Y, {}_Y\hat{\mathcal{O}}'), (Y, {}_X\mathcal{O} \mid Y) \cong (Y, {}_X\mathcal{O} \mid Y)\}$ such that $\phi \circ i = i'$. (Here: ${}_X\mathcal{O} \mid Y$ is the topological restriction of ${}_X\mathcal{O}$ to Y.)

It is easily verified that an "actual" equivalence is induced by a biholomorphic map $\bar{\phi}$ of a neighborhood of Y in X' such that $\bar{\phi} \circ i = i'$. We shall refer to a kth (formal, actual) equivalence by the symbol *kEQ(FEQ, AEQ)*.

DEFINITION 2.3: $\mathbf{N}^* = \mathcal{I}_Y/\mathcal{I}_Y^2 \underset{X^{\mathcal{O}}}{\otimes} Y^{\mathcal{O}}$ is the *conormal sheaf* of the embedding $i : Y \to X$.

The injection di : $TY \to TX$ induces the exact sequence

$$0 \to TY \to TX \to N \to 0$$

where, by definition, N is the *normal bundle* to Y in X. Then $\mathbf{N}^*$ is the sheaf of sections of the dual N^* to N.

In [11] Spencer and Nirenberg introduced the basic machinery to study equivalence of embeddings; the following theorem summarizes their results which we need.

THEOREM 2.4: *Let $i : Y \to X$, $i : Y \to X'$ be two embeddings with the same normal bundle, N. Suppose α_k is a kEO of these embeddings. The obstruction $\delta(\alpha_k)$ to extending α_k to a $(k+1)EQ$ lies in*

$$\begin{aligned} &H^1(Y, TY \otimes N^*) && \text{if } k = 0 \\ &H^1(Y, TX\,|_Y \otimes S^{k+1}(N^*)) && \text{if } k > 0, \end{aligned}$$

where $S^k(N^*)$ is the kth symmetric product of N^* (the bundle of homogeneous forms on N of degree k).

If there is a k_0 such that these groups all vanish for $k \geq k_0$, then any k_0EQ can be extended to an *FEQ*. This will be the case when Y is a hypersurface in X and thus N is a line bundle, if $N < 0$, or if $N > 0$ and dim $Y \geq 2$. In these cases it has been proved [3, 4] that the *FEQ* can be taken to be an actual (convergent) equivalence. In the case where Y is a curve and $N > 0$ there is no such finiteness of equivalences theorem, but this seems to be because X is not necessarily algebraic. We shall elaborate on this situation in another article "Embeddings of curves in ruled surfaces".

In [9] we made some observations which are necessary to the present work. Let $i : Y \to X$ have the normal bundle sequence

$$0 \to TY \to TX \to N \to 0$$

and let $b \in H^1(\mathrm{Hom}(N, TY) = H^1(TY \otimes N^*)$ be the obstruction to splitting this sequence.

PROPOSITION 2.5: (i) $b = \delta(\alpha_0)$ *(as defined in* 2.4), *where* α_0 *is the OEQ of i and the embedding* z_N *of Y as the zero section of the normal bundle N.*

(ii) *Two embeddings* $i : Y \to X$, $i : Y \to X'$ *are* 1*EQ if and only if they have the same normal bundle sequence (i.e., the classes b, b′ are cohomologous).*

In this paper we are mainly concerned with comparing two particular embeddings: the *diagonal embedding* $\Delta : Y \to Y \times Y$, and the embedding $z_T : Y \to TY$ as the zero section of the tangent bundle. For a more general context to which our results will apply we make this definition.

DEFINITION 2.6: An embedding $i : Y \to X$ is *pluricanonical* if the normal bundle is $S^{(k)}TY$ for some $k > 0$ (if $k = 1$, we call the embedding *canonical*).

In [9] we obtained the following result.

THEOREM 2.7: *There is a canonical* 1*EQ* α_1 *of* z_T *and* Δ. *The obstruction* $\delta(\alpha_1)$ *lies in* $H^1(Y, TY \otimes S^2(T^*))$, *and is Atiyah's* [2] *obstruction to the existence of a holomorphic affine connection on Y.*

The following is an elaboration of this result:

COROLLARY 2.8: *Let Y be a compact complex manifold.*

(i) α_1 *extends to a* 2*EQ if and only if Y has a holomorphic affine connection.*

(ii) *if* α_1 *extends to a* 2*EQ, it extends to an AEQ.*

(iii) *if Y is a curve,* α_1 *extends if and only if Y has genus* 1.

(iv) *if* dim $Y = 2$ *and* α_1 *extends, Y has an affine structure (see* [7] *for a classification of such surfaces).*

(v) *if Y is Kahler and* α_1 *extends, then* $c_i(TY) = 0$ *for all* $i > 0$, *where* c_i *is the ith Chern class; if Y is not Kahler,* $c_i(TY) = 0$ *for* $i \geq (\dim Y)/2$.

REMARKS: 1. Except in case dim $Y = 1$ we do not know if any 2*EQ* of z_T with Δ implies that α_1 extends. In particular we do not know (if $\dim Y > 1$) if a $2EQ$ (or even an AEQ) of z_T with Δ implies the existence of an affine connection on Y.

2. If Y is an HNR in X (i.e., there is a holomorphic contraction of a neighborhood of Y onto X), and if X has a $2EQ$ extending α_1, then also Y has a $2EQ$ extending α_1. Thus if X contains a rational curve which is an HNR, α_1 cannot extend on X.

PROOF: (i) is a restatement of 2.7. So if α_1 extends, there is a holomorphic affine connection, with holomorphic Christoffel symbols Γ^i_{jk}. The "geodesics" of this connection are then solutions of the holomorphic differential equation

$$(*) \qquad \frac{d^2\gamma^i}{dt^2} = \sum_{j,k} \Gamma^i_{jk}(\gamma(t)) \frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt},$$

where t is a holomorphic parameter. Let $\gamma(y, \xi, t)$ be the solution of $(*)$ with $\gamma(y, \xi, 0) = y$, and $\frac{d\gamma}{dt}(y, \xi, 0) = \xi \in T_yY$. Let $\exp(\xi) = \gamma(y, \xi, 1)$, where ξ is in some small ball around $0 \in T_yY$. Now we define a map ϕ of a neighborhood of the zero section of TY to a neighborhood of $\Delta \subset Y \times Y$ by

$$\phi(\xi) = (y, \exp \xi), \text{ where } \xi \in T_yY.$$

The map ϕ is holomorphic because the solutions of $(*)$ depend holomorphically on the initial conditions. We wish to compute the derivative, $D\phi \,|_{(y,0)}$. It is clear that this matrix has the form

$$\begin{pmatrix} I & I \\ 0 & A \end{pmatrix}$$

To compute A we notice that

$$\exp(y, \xi, at) = \exp(y, a\xi, t) \text{ for } a \in \xi.$$

Using this it is easy to check that $A = I$. This proves that ϕ is a biholomorphic map on a neighborhood of the zero section of TY. This proves (ii) and the proofs of (iii), (iv), and (v) can be found in [7] and [9].

3. The diagonal embeddings.

In this section we shall show that for curves C the diagonal embedding is analytically equivalent to a specific embedding as a section of a ruled surface over C, although these embeddings are algebraically quite different. In order to do that we need to refer explicitly to Atiyah's [1] construction of ruled surfaces.

CONSTRUCTION 3.1: Let Y be a compact complex manifold, and $L \to Y$ a line bundle over Y. Choose a coordinate covering $\{U_i\}$ for L with the transition functions $\{g_{ij} \in H^0(U_i \cap U_j, \mathcal{O}^*)\}$. The isomorphism class of L is determined by the class $\{g_{ij}\} \in H^1(Y, \mathcal{O}^*)$. Let $b \in H^1(Y, L)$, and represent b on this covering by $b_{ij} \in H^0(U_i \cap U_j, \mathcal{O})$ (the isomorphism $L \mid U_i \cap U_j \cong \mathcal{O} \mid U_i \cap U_j$ being used here is that given by the second index). We construct the affine line bundle $A_L(b)$ as follows:

$$A_L(b) = \bigcup_i (U_i \times \mathbb{C})/E$$

where E is the equivalence relation: $(z_i, \xi_i)E(z_j, \xi_j)$ precisely when $z_i = z_j \in U_i \cap U_j$ and

$$\xi_i = g_{ij}(z_j)(\xi_j - b_{ij}(z_j)).$$

The cocycle conditions on $\{g_{ij}\}$, $\{b_{ij}\}$ guarantee that this defines an affine fiber space over Y (and conversely [1]). Now, we can consistently compactify each fiber by adding the point at infinity, obtaining a $\mathbb{P}^1$-bundle $X_L(b)$ over Y with a distinguished section σ_∞.

Another way to view this construction is as follows. Given the data $\{g_{ij}\}$, $\{b_{ij}\}$ we find the relations

$$(3.2) \qquad \begin{cases} \zeta_i = g_{ij}(\zeta_j - b_{ij}\eta_j) \\ \eta_i = \eta_j \end{cases}$$

are transition functions for a plane bundle V over Y ((ζ_i, η_i) are coordinates for V in U_i). Then $X_L(b)$ is precisely the projectivized bundle $\mathbb{P}(V)$, with $\xi_i = \zeta_i/\eta_i$ the local inhomogeneous coordinate. The transition functions (3.2) describe the exact sequence

$$0 \to L \to V \to \mathcal{O} \to 0$$

with $b \in H^1(\mathrm{Hom}(\mathcal{O}, L))$ the class defining this extension. The infinite section is given by the equation $\eta_i = 0$ and is just $\mathbb{P}(L) \to \mathbb{P}(V)$. An easy computation shows the normal bundle to this embedding to be L^{-1} (and since σ_∞ is an HNR, the normal bundle sequence splits).

Now, let $i : Y \to X$ be an HNR embedding of Y as a hypersurface with normal bundle N. Then the normal bundle sequence splits. $TX\,|_Y = TY \otimes N$, and i has a natural $1EQ$ α_1 with the zero section z_N of N. The obstruction to extending α_1 to a $2EQ$ is

$$\delta(\alpha_1) = H^1(Y, TX\,|_Y \otimes N^{-2}) = H^1(Y, TY \otimes N^{-2}) \otimes H^1(Y, N^{-1}).$$

Since i is an HNR, the first term vanishes (see [9]), and $\delta(\alpha_1) = b \in H^1(Y, N^{-1})$.

PROPOSITION 3.3: *In the above situation, $i : Y \to X$ is $2EQ$ with σ_∞ in $X_{N^{-1}}(b)$.*

PROOF: *Let* $\pi : U \to Y$ be the holomorphic contraction. Cover the image of Y in X with local coordinate neighborhood U_i, and let z_i be a coordinate for $Y \cap U_i$, and w_i a defining function for $Y \cap U_i : Y \cap U_i = \{w_i = 0\}$. We may assume that $U_i = \pi^{-1}(Y \cap U_i) \cap U$, and extends z_i to U_i by $z_i \circ \pi$. Then (z_i, w_i) are coordinates in U_i (shrinking U if necessary, and we have transition functions of the form

$$\begin{aligned} z_i &= f_{ij}(z_j) \\ w_i &= g_{ij}(z_j)(w_j + h_{ij}(z_j)w_j^2 + \cdots) \end{aligned} \tag{3.4}$$

The class b is represented by $\{h_{ij}\}$. $A_{N^{-1}}(b)$ has coordinates $\{z_i, \xi_i\}$ with transition functions

$$\begin{aligned} z_i &= f_{ij}(z_j) \\ \xi_i &= g_{ij}^{-1}(z_j)(\xi_j - h_{ij}(z_j)). \end{aligned}$$

At σ_∞ in $X_{N^{-1}}(b)$ we introduce the local coordinate $\eta_i = \xi_i^{-1}$. Transition functions along σ_∞ are

$$\begin{aligned} z_i &= f_{ij}(z_j) \\ \eta_i &= g_{ij}(z_j)\frac{\eta_j}{1 - h_{ij}(z_j)\eta_j} \end{aligned}$$

or

$$\eta_i = g_{ij}(z_j)(\eta_j + h_{ij}(z_j)\eta_j^2 + \cdots). \tag{3.5}$$

The correspondence $w_i = \eta_i$ is consistent (by (2.4) and (3.5)) up to second order under coordinate changes so defines a $2EQ$ between i and σ_∞ in $X_{N^{-1}}(b)$.

For the diagonal embedding Δ of a curve C the equivalence of Proposition 3.3 can be made an analytic equivalence. As we have seen, the obstruction $\delta(\alpha_1)$ to extending the natural $1EQ$ of Δ and z_T to a $2EQ$ is the Chern class $c \in H^1(C, K)$ (where K is the canonical bundle). We shall write X_K for $X_K(c)$.

THEOREM 3.6: *For any curve C the diagonal embedding $\Delta : C \to C \times C$ is analytically equivalent to the infinite section σ_∞ of X_K.*

PROOF: Every Riemann surface has a projective structure (in fact many, see [6]). By that we mean that there is a covering $\{U_i\}$ of C by coordinate neighborhoods so that the transition functions are given by projective transformation:

$$z_i = \frac{a_{ij}z_j + b_{ij}}{c_{ij}z_j + d_{ij}} = f_{ij}(z_j) \tag{3.7}$$

where a_{ij}, b_{ij}, c_{ij}, d_{ij} are constants and $a_{ij}d_{ij} - b_{ij}c_{ij} = 1$. This gives us a neighborhood of $\Delta \subset C \times C$ covered by the $\{U_i \times U_i\}$ with coordinates (z_i, w_i) and transition functions (3.7) and

$$w_i = \frac{a_{ij}w_j + b_{ij}}{c_{ij}w_j + d_{ij}} \tag{3.8}$$

Let $t_i = w_i - z_i$ so $\Delta \cap (U_i \times U_i) = \{t_i = 0\}$, and take (z_i, t_i) as local coordinates on $U_i \times U_i$. We compute the new transition functions to be (3.7) and

$$t_i = \frac{t_j}{(c_{ij}z_j + d_{ij}^2)} \cdot \left[\frac{1}{1 + \dfrac{c_{ij}t_j}{c_{ij}z_j + d_{ij}}}\right] = \frac{f'_{ij}(z_j)t_j}{1 - b_{ij}t_j}$$

where $b_{ij} = f''_{ij}(z_j)/2f'_{ij}(z_j)$ is a representative of the class $\pi i c(TC) = -\pi i c(K) \in H^1(K)$ (see [9]).

Now σ_∞ in X_K has a coordinate cover $\{W_i\}$ with coordinates (z_i, η_j) and transition functions (3.7) and

$$\eta_i = \frac{f'_{ij}(z_j)\eta_j}{1 - c_{ij}\eta_j} \tag{3.10}$$

where $\{c_{ij}\}$ represents $c(K)$. Comparing (3.9) and (3.10) we see that the functions $\phi_i : U_i \times U_i \to W_i$ defined by

$$\phi_i(z_i, t_i) = (z_i, (-\pi i)t_i)$$

agree on the overlaps and define a holomorphic map of a neighborhood of Δ with a neighborhood of σ_∞.

For $g > 0$ this equivalence is transcendental: it does not extend to a meromorphic map between $C \times C$ and X_K. In fact:

PROPOSITION 3.11: *There is no birational map between* $C \times C$ *and* X_K.

PROOF: If X_K and $C \times C$ were birationally equivalent, $C \times C$ would contain a smooth rational curve Γ. Let $p : C \times C \to C$ be the projection on the first factor. If C is not rational, then Γ cannot be contained in a fiber of p. But then $p \mid \Gamma$ makes Γ a branched cover of C, which is impossible since C is of positive genus.

REMARKS: 1. Of course, in case $g = 1$, $X_K = C \times \mathbb{P}^1$, and the diagonal map $C \to C \times C$ is analytically equivalent to a constant section of X_K.

2. In case $g = 0$, we have that X_K and $\mathbb{P}^1 \times \mathbb{P}^1$ coincide. The equivalence defined in Theorem 3.6 can be easily seen to extend to a global biholomorphic map.

An argument can be made for the thinking of X_K as a "better" tangent bundle for a curve C than TC. There is the following result of Atiyah [1], which, together with 3.6, justifies this. We give a new proof for completeness.

Let $f : C \to \mathbb{P}^n$ be an embedding. For $p \in C$, denote by $df(p)$ the line in $\mathbb{P}^n$ tangent to $f(C)$ at $f(p)$. Then df is a map of C into the Grassmanian $G(n+1, 2)$ of lines in $\mathbb{P}^n$. Let B represent the tautological projective bundle on $G(n+1, 2)$: for $x \in G(n+1, 2)$, B_x is the line x. Then $df^*(B)$ is a projective bundle over C.

PROPOSITION 3.12: *For any embedding* $f: C \to \mathbb{P}^n$, $df^*(B) \cong X_K$.

PROOF: The map f is given by certain sections $\sigma_0, \ldots, \sigma_n$ of an ample line bundle L. Cover C with coordinate neighborhoods $\{U_\alpha\}$, local coordinates z_α, and so that σ_i is realized by the function ξ_i^α in U_α. Let $\{g_{\alpha\beta}\}$ be the transition functions for L. Then

$$\xi_i^\alpha = g_{\alpha\beta}\xi_i^\beta, \qquad i = 0, \ldots, n.$$

Associate to each $p \in C$ the plane P_p in $\mathbb{C}^{n+1}$ spanned by

$$v_0^\alpha(p) = (\xi_0^\alpha(p), \ldots, \xi_n^\alpha(p)), \qquad v_1^\alpha(p) = \left(\frac{d\xi_0^\alpha}{dz_\alpha}(p), \ldots, \frac{d\xi_n^\alpha}{dz_\alpha}(p)\right)$$

so long as $p \in U_\alpha$. Since

$$v_0^\alpha = g_{\alpha\beta} v_0^\beta$$
$$v_1^\alpha = \frac{dz^\beta}{dz^\alpha}\left(\frac{dg_{\alpha\beta}}{dz^\beta} v_0^\beta + g_{\alpha\beta} v_1^\beta\right),$$

the plane P_p is well defined, and $P = UP_p$ defines a plane bundle on C with transition matrices

$$f_{\alpha\beta} = \begin{bmatrix} g_{\beta\alpha} & \dfrac{dg_{\beta\alpha}}{dz_\alpha}\dfrac{dz_\alpha}{dz_\beta} \\ 0 & g_{\beta\alpha}\dfrac{dz_\alpha}{dz_\beta} \end{bmatrix} \tag{3.13}$$

It is easily checked that $\mathbb{P}(P) = df^*(B)$. Now by (3.13) we obtain the exact sequence

$$0 \to L^{-1} \to P \to L^{-1} \otimes \Theta \to 0, \tag{3.14}$$

and

$$\left\{-\frac{d}{dz_\beta}\log g_{\alpha\beta}\right\} \tag{3.15}$$

defines the class in $H^1(C, \mathrm{Hom}(L^{-1} \otimes \Theta, L^{-1})$ obstructing the splitting of (3.14). Finally $\mathrm{Hom}(L^{-1} \otimes \Theta, L^{-1}) = K$, and (3.15) represents the Chern class of L^{-1}. Since L is ample, its Chern class is non-zero, and

3.14 does not split, nor does (3.14) tensored with L:

$$0 \to \mathcal{O} \to P \otimes L \to \Theta \to 0.$$

Thus $\mathbb{P}(P \otimes L) = X_K$. But $\mathbb{P}(P \otimes L) = \mathbb{P}(P) = df^*(B)$.

REMARK: The proposition is easily seen to fail in higher dimensions. If X is a projective variety with tangent bundle Θ, and $c = H^1(\Omega^1)$ is the Chern class of ample bundle L, then $X_{\Omega^1}(c)$ is $df^*(B)$ (defined analogously as above), where $f: X \to \mathbb{P}^n$ is the map given by the sections of L.

We conclude with a description of the diagonal in the symmetric product, which will be needed in the following section. Introduce an equivalence relation E on $C \times C: (x, y)E(y, x)$. $C^{(2)} = C \times C/E$ is the symmetric product of $\mathbb{C}$ with itself. Let Δ_s be the image of the diagonal. Since the map $C \times C \to C^{(2)}$ is $2:1$, branched along Δ, $C^{(2)}$ is an analytic variety which is nonsingular off Δ_s. But it is also nonsingular along Δ, and the diagonal map $C \to \Delta_s$ is an embedding, as we can see by introducing proper coordinates. Let U be a coordinate neighborhood in C, with coordinate z. Then $U \times U$ is a coordinate neighborhood in $C \times C$, with coordinates z_1, z_2. Then we can use as coordinates on $U_s = U \times U/E$:

$$\zeta = \tfrac{1}{2}(z_1 + z_2) \quad \eta = \xi^2, \text{ with } \xi = \tfrac{1}{2}(z_1 - z_2) \tag{3.12}$$

and Δ_s is given in U_s as $\eta = 0$.

THEOREM 3.13: *The normal bundle of Δ_s in $C^{(2)}$ is K^{-2}, where K is the canonical bundle of C. The obstruction to finding a 1EQ between Δ_s and the zero section of K^{-2} is the Chern class of C.*

In particular if $g \neq 1$, Δ_s and z are not first order equivalent, and if $g = 1$, they are in fact analytically equivalent embeddings.

We prove the theorem by exhibiting the normal bundle sequence.

PROOF: Let $\{U_\alpha\}$ be a coordinate cover of C, with coordinates z_α and coordinate transformations $z_\alpha = f_{\alpha\beta}(z_\beta)$. Then $U_{\alpha(s)} = U_\alpha \times U_\alpha/E$ form a coordinate cover of a neighborhood of Δ_s with coordinates ζ_α η_α defined as in (3.12). The coordinate transformation in $U_{\alpha(s)} \cap U_{\beta(s)}$

can be traced. We have

$$\zeta_\alpha = \tfrac{1}{2}[f_{\alpha\beta}(\zeta_\beta + \xi_\beta) + f_{\alpha\beta}(\zeta_\beta - \xi_\beta)] \tag{3.14}$$
$$\xi_\alpha = \tfrac{1}{2}[f_{\alpha\beta}(\zeta_\beta + \xi_\beta) - f_{\alpha\beta}(\zeta_\beta - \xi_\beta)]$$

from which we obtain

$$\zeta_\alpha = f_{\alpha\beta}(\zeta_\beta) + \frac{f''_{\alpha\beta}(\zeta_\beta)}{2}\xi_\beta^2 + \cdots$$
$$\xi_\alpha = f'_\alpha\beta(\zeta_\beta) + \frac{f'''_{\alpha\beta}(\zeta_\beta)}{6}\xi_\beta^3 + \cdots$$

and finally

$$\zeta_\alpha = f_{\alpha\beta}(\zeta_\beta) + \frac{f''_{\alpha\beta}}{2}(\zeta_\beta)\eta_\beta + \cdots$$
$$\eta_\alpha = (f'_{\alpha\beta})^2(\zeta_\beta)\eta_\beta + \cdots$$

The transition matrices for the tangent bundle of $C^{(2)}$ along Δ_s are

$$\begin{pmatrix} f'_{\alpha\beta} & \dfrac{f''_{\alpha\beta}(\zeta_\beta)}{2} \\ 0 & (f'_{\alpha\beta})^2(\zeta_\beta) \end{pmatrix} \tag{3.15}$$

These matrices tell us how the normal bundle sequence

$$0 \to \Theta_C \to \Theta_{C^{(2)}} \to N \to 0 \tag{3.16}$$

goes: the transition function for N is the lower right entry, which is the transition function for K^{-2}, and the extension (3.17) is defined by the class in $H^1(C, \mathrm{Hom}(K^{-2}, K^{-1})) = H^1(K)$ given by the quotients of the entries in the first row:

$$\left\{\frac{f''_{\alpha\beta}(\zeta_\beta)}{2f'_{\alpha\beta}(\zeta_\beta)}\right\}.$$

Comparing with [9], we see that this is the Chern class of C. Thus Δ_s and Z are first order equivalent if and only if (3.16) splits, which happens if and only if the Chern class is zero.

If $g = 1$, K is trivial, and the embedding $\Delta_s \to C^{(2)}$ is analytically trivial. Cover C with affine coordinates $\{z_\alpha$ in $U_\alpha\}$ with transition function $z_\alpha = z_\beta + c_{\alpha\beta}$. Then (using (3.14)) the transition function in $U_{\alpha(S)} \cap U_{\beta(s)}$ are

$$\zeta_\alpha = \alpha_\beta + c_{\alpha\beta}, \quad \eta_\alpha = \eta_\beta,$$
$$\{w \in \mathbb{C}; |w| < \epsilon\}$$

so the map $\phi : C \times \{w \in \mathbb{C}, |w| < \epsilon \to C^{(2)}$ given by $\zeta_\alpha = z_\alpha$, $\eta_\alpha = w$ is an isomorphism taking $C \times \{0\}$ to Δ_s.

REMARK: Except in the case $g = 1$, Δ_s is not an HNR, since the normal bundle sequence doesn't split.

4. Pluricanonical embeddings

In this section we shall describe all pluricanonical embeddings of a curve of genus greater than 1. In this case the normal bundle is negative, and Grauert's theorem applies.

THEOREM [3] 4.1: *Let $i : Y \to X$ be an embedding of a compact manifold Y in a manifold X, of codimension* 1, *with negative normal bundle N. Let $k_0 > 0$ be such that $H^k(Y, TX|_Y \otimes N^{-\nu}) = 0$ for all $k > k_0$. Then any k_0EQ of i with another embedding $i' : Y \to X'$ extends to an actual equivalence.*

Thus, for such embeddings, i is analytically determined by the k_0th neighborhood of Y in X. For Y a curve, the space of embeddings is parametrized by the set of inequivalent k_0th neighborhoods, since an embedding can be trivialized on a 2-set open covering and thus there is no cocycle condition to extending a k_0th neighborhood to an actual neighborhood (see [9], or below for details). Now, for curves of genus $g > 1$, we can estimate k_0 in terms of the degree of N.

COROLLARY 4.2: Let *C be a curve of genus $g > 1$, and $N \to C$ a line bundle of degree $-n$, $n > 0$. Let k_0 be the integer of the theorem*

(i) *if* $n < 2g - 2$, $k_0 \leq \left[\frac{4g-4}{n}\right]$.

(ii) *if* $n = 2g - 2$, $k_0 = 1$ *unless* $N^2 = TC^2$, *then* $k_0 = 2$

(iii) *if* $2g - 2 < n < 4g - 4$, $k_0 = 1$

(iv) *if* $n = 4g - 4$, $k_0 = 0$ *unless* $N = (TC)^2$, *then* $k_0 = 1$

(v) *if* $n > 4g - 4$, $k_0 = 0$; *i.e., the only embedding of C in a surface with normal bundle N is (up to analytic equivalence) the zero section of N.*

PROOF: Let $i: C \to S$ be an embedding with normal bundle N. From the exact sequence

$$0 \to TC \otimes N^{-k} \to TS \otimes N^{-k} \to N^{-k+1} \to 0$$

we obtain

$$\cdots \to H^1(C, TC \otimes N^{-k}) \to H^1(C, TS \otimes N^{-k}) \to H^1(C, N^{-k+1}) \to 0$$

so the middle term vanishes if the outside terms do. This is guaranteed if these bundles have degree greater than $2g-2$, so we have vanishing of all kth order obstructions if

$$2-2g+kn > 2g-2, \quad (k-1)n > 2g-2$$

or

$$kn > 4g-4, (k-1)n > 2g-2.$$

(i) if $n < 2g-2$ the first inequality dominates, so obstructions vanish for all k greater than

$$k_0 = \left[\frac{4g-4}{n}\right].$$

(iii) if $2g-2 < n < 4g-4$, both inequalities say that obstructions vanish for $k \geq 2$. Thus $k_0 = 1$.

(v) if $n > 4g-4$, both inequalities say that obstructions vanish for all $k > 0$, so $k_0 = 0$.

(ii) if $n = 2g-2$, both $TC \otimes N^{-k}$, and N^{-k+1} have degree $(k-1)$ $(2g-g)$ so obstructions vanish for $k2$. Thus we may take $k_0 = 2$. However, if $N^2 \neq (TC)^2$ then $TC \otimes N^{-2}$ and N^{-1} are bundles of degree $2g-2$ different from K, so again H^1 vanishes and we may take $k_0 = 1$.

(iv) Here the two bundles have degree $(2k-1)(2g-2), 2(k-1)(2g-2)$ respectively, so if $k > 1$, H^1 vanishes, and thus we may take $k_0 = 1$. However, to reduce this to $k_0 = 0$ we need only check the first bundle (recall theorem 2.4) and if $N \neq (TC)^2$ this bundle is of degree $2g-2$ but different from K, so H^1 vanishes and we may take $k_0 = 0$.

COROLLARY 4.3: *Let C be a curve of genus $g > 1$. The only pluricanonical embeddings (up to analytic equivalence) are the following:*

(i) *the zero section of* $(TC)^\nu (\nu \geq 2)$

(ii) *the diagonal of the symmetric product* $C^{(2)}$. *Here the normal bundle is* $(TC)^2$.

Finally, we give a prescription for parametrizing all canonical embeddings (up to analytic equivalence).

Let C be a curve of genus $g > 1$. Cover C with coordinate neighborhoods (u_i, z_i) so that $U_i \cap U_j \cap U_k = \phi$ if i, j, k are all distinct (we can find such covers so long as we allow the U_i to be multiply connected. Let

$$z_i = f_{ij}(z_j)$$

be the transition function of this covering.

Now suppose $i : C \to S$ be a canonical embedding with normal bundle sequence

$$0 \to TC \to TS\,|_C \to TC \to 0. \tag{4.4}$$

Let $b \in H^1(C, \mathrm{Hom}(TC, TC)) = H^1(C, \mathcal{O})$ describe the extension (4.4). Represent b in a covering by the cocycle $b_{ij} \in \mathcal{O}(U_i \cap U_j)$. We construct a surface S_b as follows: for Δ_i a small disc with coordinate w_i, S_b is covered by $U_i \times \Delta_i$ and has transition functions

$$\begin{aligned} z_i &= f_{ij}(z_j)(1 + b_{ij}(z_j) w_j) \\ w_i &= f'_{ij}(z_j) w_j. \end{aligned} \tag{4.5}$$

Since there are no triple intersections there is no consistency condition to check and S_b is a well-defined surface. Define $i_b : C \to S_b$ by $i_b \mid U_i(z_i) = (z_i, 0)$.

PROPOSITION 4.5: *$i : C \to S$ is $1EQ$ to i_b.*

The proof is as in [9].

Now i_b is $1EQ$ to $i_{b'}$ if and only if $b = tb'$ for some $t \neq 0$. Thus, up to first order equivalences embeddings of C are parametrized by $\mathbf{P}H^1(X, \mathcal{O}) \cup \{0\}$.

Let α_1 be the first order equivalence of proposition 4.5 and now compute the obstruction to extending it to a $2EQ$. From

$$0 \to K \to T_S \otimes K^2 \to K \to 0$$

we obtain the exact sequence

$$(4.7) \qquad \begin{array}{ccccccccc} H^0(K) & \to & H^1(K) & \to & H^1(T_S \otimes K^2) & \to & H^1(K) & \to & 0 \\ \| & & \| & \nearrow & & \searrow & \| & \nearrow & \\ \mathbb{C}^g & \longrightarrow & \mathbb{C} & & & & \mathbb{C} & & \end{array}$$

PROPOSITION 4.8: *If* $b = 0$, *i.e.*, (4.4) *splits, then* $\dim H^1(T_S \otimes K^2) = 2$. Otherwise $\dim H^1(T_S \otimes K^2) = 1$.

PROOF: Of course, if (4.4) splits, the first map in (4.7) is the zero map and thus $\dim H^1(T_S \otimes K^2) = 2$. What we have to show is this:

LEMMA 4.9: *Let*

$$(4.10) \qquad 0 \to K \to V \to K \to 0$$

be an exact sequence where V *is a plane bundle on a curve* C *and* K *the canonical bundle.* (4.10) *splits if and only if the coboundary map* $\delta : H^0(K) \to H^1(K)$ *is the zero map.*

PROOF: Let $b \in H^1(\mathrm{Hom}(K, K)) = H^1(\mathcal{O})$ be the class of the extension (4.10). We'll show that the coboundary map coincides with the natural map

$$(4.11) \qquad H^1(\mathcal{O}) \otimes H^0(K) \to H^1(K)$$

with first term b. But by Serre duality (4.11) is non-degenerate, so if the coboundary map is zero, $b = 0$.

To see this, let $\{b_{ij} \in \mathcal{O}(U_i \cap U_j)\}$ represent the class b relative to a suitable covering $\{U_i\}$. A C^∞ splitting of (4.10) is given by

$$b_{ij} = \phi_i - \phi_j \quad (\phi_i \in C^\infty(U_i))$$

and $\theta = \{-\bar{\partial}\phi_i\}$ defines the Dobleault representative of b. Let $w \in H_0(K)$ and $\tilde{w}$ its lift according to the C^∞ splitting. Locally, if w is given by w_i, then $\tilde{w}$ is given by

$$\begin{pmatrix} -\phi_i w_i \\ w_i \end{pmatrix}$$

and its coboundary is $\bar{\partial}\tilde{w}$, given locally by

$$\begin{pmatrix} -\bar{\partial}\phi_i \wedge w_i \\ 0 \end{pmatrix}$$

Thus $\delta(w) = \theta \wedge w$, where θ is the Dolbeault representative of b; but this is just the duality of Serre.

Now the space of second order extensions of i_b is $\mathbb{P}H^1(T_S \otimes K^2) \cup \{0\}$. For any $c \in H^1(T_S \otimes K^2)$ represent it on the cover (U_i, z_i) by the vector

$$\begin{pmatrix} c_{ij}(z_j) \\ d_{ij}(z_j) \end{pmatrix} \in \mathcal{O}^2(U_i \cap U_j)$$

and modify the transition functions (4.5) to read

$$\begin{aligned} z_i &= f_{ij}(z_j)(1 + b_{ij}(z_j)w_j + c_{ij}(z_j)w_j^2) \\ w_i &= f'_{ij}(z_j)(w_j + d_{ij}(z_j)w_j^2), \end{aligned} \tag{4.12}$$

giving us a surface S_c for $c \in H^1(T_S \otimes K^2)$. It is easy to check that two such extensions (given by C, C') of α_1 are equivalent if and only if $c = tc'$, $t \neq 0$, $i_c : c \to s_c$ is defined as before.

By the proposition, for $b \neq 0$ there are only two distinct embeddings correspondings to i_b and i_b, given by the nonzero class in $H^1(TS \otimes K^2)$. For $b = 0$ we obtain again the zero section embedding i_0, and am embedding i_p as p runs over $\mathbb{P}^1$. This $\mathbb{P}^1$ has a distinguished point corresponding to the diagonal embedding Δ. This is the only HNR embedding of C besides i_0. In summary

THEOREM 4.13: *Let $i: C \to S$ be a canonical embedding. If the normal bundle sequence splits the embedding is AEQ to one of the i_p, $p \in \mathbb{P}^1$ or to z_T. If $b \neq 0$ is the obstruction class to splitting, i is AEQ to one of i_b, $i_{b'}$.*

REMARKS: 1. Notice that Theorem 4.13 includes Theorem 3.6 as a special case; we felt that the proof in section 3 was worth exhibiting because of its directness.

2. For each embedding in the theorem we have constructed only locally the surface S. We have global constructions, so that S is algebraic in case of z_T or Δ. We do not know if any of the other embeddings can be realized algebraically.

3. In another paper [10], we shall find that the only algebraic pluricanonical embeddings of $\mathbb{P}^1$ are the zero sections of $\mathbb{P}(\mathcal{O}(2n)\otimes\mathcal{O})$ (in fact the only algebraic embeddings of $\mathbb{P}^1$ are as the zero section of $\mathbb{P}\mathcal{O}(n)\oplus\mathcal{O})$, or the linear or quadratic embeddings in $\mathbb{P}^2$). Canonical embeddings of elliptic curves remain elusive: since the canonical bundle is trivial there seems to be no way to get a grip.

5. Singularities obtained by blowing down.

In this section we continue the study of the particular embeddings Δ and σ_∞ in $C\times C$ and X_K respectively for C a curve of genus $g>1$. In both cases C is negatively embedded, so can be blown down. More precisely, let S_Δ, S_σ be the topological spaces obtained by identifying C to a point, and $\pi_\Delta: C\times C\to S_\Delta$, $\pi_\sigma: X_K\to S_\sigma$ the natural projections. Then, by Grauert's theorem on negative embeddings [3], if we endow $S_\Delta(S_\sigma)$ with the structure sheaf $R^0\pi(\mathcal{O}_{C\times C})(R^0\pi(\mathcal{O}_{X_K}))$, then $S_\Delta(S_\sigma)$ becomes a normal analytic space with one singular point $p_\Delta(p_\sigma)$. By Theorem 3.6 these singular points are analytically equivalent, but not algebraically equivalent for, as we shall see S_Δ is algebraic whereas S_σ is not. The proof of the following result was suggested to us by Ron Donagi.

THEOREM 5.1: *S_Δ is algebraic.*

PROOF: Suppose first that C is not hyperelliptic. Let J be the Jacobian of C; we may view J as the set of line bundles on C of degree 0. J is an abelian variety of dimension g; in particular J is algebraic. Consider the holomorphic map $\phi: C\times C\to J$ defined by

$$\phi(p,q)=[p-q]$$

(by $[D]$ we mean the line bundle of the divisor D). Suppose $\phi(p_1,q_1)=\phi(p_2,q_2)$. Then by Abel's theorem $p_1-q_1\sim p_2-q_2$, or $p_1+q_2\sim p_2+q_1$. If (p_1,q_2), (p_2,q_1) are different point sets, this gives a meromorphic function with two zeros and two poles and thus C is hyperelliptic, which is the case being excluded. Thus, either $p_1=p_2$, $q_2=q_1$ or $p_1=p_2$ and $q_1=q_2$. This says precisely that ϕ is a proper modification blowing Δ down to a point, and thus defines a holomorphic homeomorphism of S_Δ with the variety $\phi(C\times C)$ in J. Thus $\phi(C\times C)$ is algebraic and S_Δ is its normalization, so S_Δ is algebraic as well.

In the hyperelliptic case we define ϕ again as above. Letting $j: C \to C$ be the hyperelliptic involution, we find that $\phi(p_1, q_1) = \phi(p_2, q_2)$ if and only if one of these holds: (a) $p_1 = q_1$ and $p_2 = q_2$, (b) $p_1 = p_2$ and $q_1 = q_2$, (c) $q_2 = j(p_1)$, $p_2 = j(q_1)$.

Let Γ be the graph in $C \times C$ of j. For $(p_1, q_1) \not\in \Delta \cup \Gamma \phi(p_1, q_1) = \phi(p_2, q_2)$ if and only if $p_2 = j(q_1)$, $q_2 = j(p_1)$. Thus ϕ is $2:1$ off $\Delta \cup \Gamma$. On $\Gamma - \Delta$, ϕ is easily seen to be $1:1$, and $\phi(\Delta) = 0$. Thus ϕ descends to a holomorphic map of S_Δ to $\phi(C \times C)$ which is $2:1$ of $f\pi_\Delta(\Gamma)$ and branched along $\pi_\Delta(\Gamma)$. By [13], again S_Δ is seen to be algebraic.

The above argument generalizes easily to the case of the graph of a map.

COROLLARY 5.2: *Let $f: C \to C'$ be a holomorphic map with C' of genus at least 2. Then the graph Γ of f is exceptional and $C \times C'/\Gamma$ is algebraic.*

PROOF: Let d be the degree of f. Suppose first that C' is not hyperelliptic. Let J be the Jacobian of C' and consider the map $\phi: C \times C' \to J$, $\phi(\beta, q) = [f(p) - q]$. If $\phi(p, q) = \phi(p', q')$ we must have (as above) $\{f(p), q'\} = \{f(p'), q\}$ as point sets. If $f(p) = f(p')$, $q = q'$, then p, p' are in a fiber of f. If $f(p) = q$, $f(p') = q'$, then (p, q), (p', q') are in Γ. Thus ϕ blows down Γ, and the induced map $\hat{\phi}: C \times C'/\Gamma \to J$ is a d sheeted branched cover of $\phi(C \times C')$, which is algebraic, and thus $C \times C'/\Gamma$ is also algebraic. The case where C' is hyperelliptic is again as in theorem 5.1; this time $\hat{\phi}: C \times C'/\Gamma \to J$ is a $2d$ sheeted branched cover, so again $C \times C'/\Gamma$ is algebraic.

PROPOSITION 5.3: *If C is non-hyperelliptic, the variety $V = \phi(C \times C)$ of 5.1 is the Grauert blowdown, i.e., V is normal.*

PROOF: We work near $\phi(\Delta) \subset J$, and we can represent ϕ near Δ by $\phi(p, q) = (\int_p^q \omega_1, \ldots, \int_p^q \omega_g)$ where $\{\omega_1, \ldots, \omega_g\}$ gives a basis for $H^0(K)$. Let $f_j = \int_p^q \omega_j$, and suppose $\omega_j = h_{j\alpha}(z_\alpha)\, dz_\alpha$ in local coordinates. Changing product coordinates (ζ_α, w_α) on $U_\alpha \times U_\alpha \subset C \times C$ to $(\zeta_\zeta, \xi_\alpha) = (\zeta_\alpha, w_\alpha - \zeta_\alpha)$ we get $h_{j\alpha}(z_\alpha) = h_{j\alpha}(\zeta_\alpha) + h'_{j\alpha}(\zeta_\alpha)\xi_\alpha + \cdots$. Then

$$f_j(\zeta_\alpha, \xi_\alpha) = \int_{\zeta_\alpha}^{\zeta_\alpha + \xi_\alpha} h_{j\alpha}(\zeta_\alpha)\, dz_\alpha + \int_{\zeta_\alpha}^{\zeta_\alpha + \xi_\alpha} h'_{j\alpha}(\zeta_\alpha)(z_\alpha - \zeta_\alpha)\, dz_\alpha + \cdots$$

$$= \xi_\alpha h_{j\alpha}(\zeta_\alpha) + \frac{h'_{j\alpha}(\zeta_\alpha)}{2} \xi_\alpha^2 + \cdots$$

Now we use 5.7 of [9]. By the above expansion $\sigma(f_j) = \omega_j \in H^0(\Delta, N^{-1}) = H^0(\Delta, K)$. By Noether's theorem $\{\omega_1, \ldots, \omega_g\}$ generate $\Sigma_{l\geq 1} H^0(\Delta, K^l)$. Then by 5.7 of [9], $V = \phi(C \times C)$ is the Grauert blow-down, and hence V is normal.

Now, we verify that S_σ is not algebraic (this example is due to Grauert [3]). More generally,

THEOREM 5.4: *Let Y be a compact complex manifold, $L \to Y$ a positive line bundle, and $b \in H^1(Y, L)$, $b \neq 0$. Then the infinite section σ_∞ of $X_L(b)$ is negatively embedded, and the blowdown X_σ of σ_∞ is not an algebraic variety.*

PROOF: It is easy to verify that if σ is a section of $A_L(b)$ then the expression of σ in local coordination gives a cochain whose coboundary is b. Thus, since $b \neq 0$, $A_L(b)$ can have no sections.

On the other hand, if X_σ is algebraic, there is a hypersurface H which does not contain the blowdown point σ_∞. The pullback of H is a hypersurface in $A_L(b)$ and $\pi \mid H: H \to Y$ is a proper map with discrete level sets, so is a branched d-sheeted cover for some d. Now we define a section $\Delta: Y \to A_L(b)$ by

$$s(p) = \frac{1}{d} \sum_{q \in \pi^{-1}(p) \cap H} q.$$

Since the fibers of $A_L(b)$ have an affine structure, s is well defined over regular points of $\pi \mid H$. However, in local coordinates at other points s is a bounded $\mathbb{C}$-valued function, so extends analytically and defines a section of $A_L(b)$, contradicting that $b \neq 0$. Thus X_σ is not algebraic.

PROPOSITION 5.5: *The singular points of S_σ and S_Δ are Gorenstein.*

PROOF: On X_K, near σ_∞ we use the coordinates (z_j, n_j) described in (3.5). Then consider the meromorphic two form ω defined locally by

$$\omega = \frac{dn_j \wedge dz_j}{n_j^2}$$

Using 3.5 one easily checks that ω is globally defined on X_K, has no zeros, and has C_K as a polar locus of order 2. (Incidentally, this shows that the canonical bundle of X_K is $[-2C_K]$,) This proves that S_σ is

Gorenstein. By the analytic equivalence of the two singularities S_Δ is also Gorenstein. (However the form ω does not extend to a global meromorphic form on S.)

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