COMPOSITIO MATHEMATICA

STEVEN B. BANK

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Compositio Mathematica, tome 46, nº 1 (1982), p. 15-32
<http://www.numdam.org/item?id=CM_1982__46_1_15_0>

COMPOSITIO MATHEMATICA, Vol. 46, Fasc. 1, 1982, pag. 15–32

A NOTE ON A THEOREM OF C.L. SIEGEL CONCERNING BESSEL'S EQUATION

Steven B. Bank*

1. Introduction

In [7], C.L. Siegel proved that if the complex number α is not one-half of an odd integer, then no solution $y(z) \not\equiv 0$ of the Bessel differential equation of order α,

$$z^2y'' + zy' + (z^2 - \alpha^2)y = 0, \tag{1}$$

can satisfy an algebraic differential equation of the first order whose coefficients belong to the field of rational functions. (In the case where α is one-half of an odd integer, every solution of (1) satisfies a first-order algebraic differential equation whose coefficients are rational functions.) In this paper, we seek to determine more extensive fields L of meromorphic functions which have the property that if α is not one-half of an odd integer, then no solution $y(z) \not\equiv 0$ of Bessel's differential equation of order α can satisfy a first-order algebraic differential equation having coefficients in L (i.e. an equation of the form $\Omega(z, y, y') = 0$ where Ω is a polynomial in y and y', whose coefficients belong to L and some coefficient is not identically zero.) Since the Bessel function $J_\alpha(z)$ of order α of the first kind is a solution of (1), and has the form $z^\alpha E_\alpha(z)$, where $E_\alpha(z)$ is an entire function whose order of growth is 1, we see that a field which contains some entire functions of order 1 (and hence possibly $E'_\alpha(z)/E_\alpha(z)$) may not have the above property. However, we show in Theorem 1 (see §2 below) that the field L_1 consisting of all meromorphic functions on the plane whose order of growth is less than 1 does possess the desired property. (The field L_1 is simply the field of quotients of the ring consisting of all entire functions whose order of

* This research was supported in part by the National Science Foundation (MCS-8002269).

0010-437X/8204015-18$00.20/0

growth is less than 1. Clearly L_1 is a differential field (i.e. closed under differentiation), and is far more extensive than the field of rational functions.)

Theorem 1 is proved by dividing the possible values of α into four classes: (i) α is an integer (which we can clearly assume to be nonnegative); (ii) α is not a rational number; (iii) α is a rational number m/n in lowest terms, where n is odd and is at least 3; (iv) α is a rational number m/n in lowest terms, where n is even and is at least 4. In each of the first three cases, we actually obtain stronger results than are indicated in the statement of Theorem 1. For example, if α is a nonnegative integer, and if L is any differential field of meromorphic functions defined in a fixed neighborhood of $z = 0$ such that L contains the field of rational functions and satisfies the condition that J'_α/J_α is transcendental over L, then no solution $y(z) \not\equiv 0$ of Bessel's equation of order α can satisfy a first-order algebraic differential equation having coefficients in L. In the case when α is not a rational number, we obtain the same result if L satisfies the additional requirement that $J'_{-\alpha}/J_{-\alpha}$ is also transcendental over L. If α belongs to Case (iii), we show that if we consider the field L_2 which is generated by the field of meromorphic functions of order less than 1 together with all the entire functions of order 1 having minimal type, then no solution $y(z) \not\equiv 0$ of Bessel's equation of order α can satisfy a first-order algebraic differential equation having coefficients in L_2. These stronger results, together with the Sturm comparison theorem for linear differential equations, allow us to prove (see Theorem 2 below) that if α is a real number which belongs to any of the Cases (i), (ii), or (iii), then no solution $y(z) \not\equiv 0$ of Bessel's equation of order α can satisfy a first-order algebraic differential equation having coefficients in L_2.

We conclude with two remarks. First, in the course of proving our main theorem, we make use of some of Siegel's results, and also the Valiron-Wiman theory of entire functions (see §3 below). Second, the author wishes to acknowledge valuable contributions to this paper by his colleague, Robert P. Kaufman.

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We now state our main results. The proofs will be concluded in §§11, 12.

THEOREM 1: *Let α be a complex number which is not one-half of an odd integer. Let L_1 denote the field of all meromorphic functions on* *the plane whose order of growth is less than 1. Then, no solution $y(z) \not\equiv 0$ of Bessel's equation of order α can satisfy a first-order algebraic differential equation having coefficients in L_1.*

THEOREM 2: *Let α be a real number which is either irrational, or an integer, or a rational number m/n is lowest terms where n is an odd integer greater than or equal to 3. Then, no solution $y(z) \not\equiv 0$ of Bessel's equation of order α can satisfy a first-order algebraic differential equation whose coefficients belong to the field generated by the field of meromorphic functions on the plane of order less than 1, together with all the entire functions of order 1 having minimal type.*

3. Preliminaries and notation

(a) Consider a linear differential equation,

$$Q_p(z)y^{(p)} + Q_{p-1}(z)y^{(p-1)} + \cdots + Q_0(z)y + Q_{-1}(z) = 0, \tag{2}$$

where the $Q_k(z)$ are polynomials, and let $A_k z^{m_k}$ denote the term of highest degree in $Q_k(z)$. Suppose that (2) possesses a solution $y(z)$ of the form $z^\lambda E(z)$, where λ is a complex number and $E(z)$ is an entire function which is not a polynomial. Then, if we denote by σ the order of growth of $E(z)$, it follows from the Valiron-Wiman theory (see [11; pp. 93–109], or [12; pp. 193–220], or [13; pp. 65–67]) that the following are true: (1) σ is a rational number which is at least $1/p$; (2) The maximum modulus $M(r, E)$ of $E(z)$ satisfies the relation,

$$\log M(r, E) = c_1 r^\sigma (1 + o(1)) \quad \text{as} \quad r \to \infty, \tag{3}$$

for some positive constant c_1; (3) For some nonzero complex number c_2, the function $c_2 z^\sigma$ is the first term of one of the expansions around ∞ of the algebraic function $u = u(z)$ defined by the equation,

$$\sum_{k=0}^{p} A_k z^{m_k - k} u^k = 0. \tag{4}$$

(b) We will have occasion to use the concept of the Nevanlinna characteristic $T(r, f)$ of a meromorphic function $f(z)$ on the plane (see [5; pp. 6–12] or [2; pp. 3–4]). The order of growth of $f(z)$ is defined as,

$$\limsup_{r \to \infty} (\log T(r, f)/\log r), \tag{5}$$

which agrees with the definition of order for entire functions since the relation,

$$T(r,f) \leq \log M(r,f) \leq 3T(2r,f) \quad \text{for} \quad r > 0, \tag{6}$$

holds for any entire function (see [5; p. 24]). For the derivative of a meromorphic function, we have the relation $T(r,f') = 0(T(2r,f))$ as $r \to \infty$ (see [2; p. 55]). Finally, if a meromorphic function $f(z)$ satisfies an algebraic equation,

$$f_n(z)(f(z))^n + f_{n-1}(z)(f(z))^{n-1} + \cdots + f_0(z) \equiv 0, \tag{7}$$

where $f_0, f_1, \ldots, f_n$ are meromorphic functions and $f_n \not\equiv 0$, then by using the elementary properties of the Nevanlinna characteristic (e.g. see [3; p. 108]), we have

$$T(r,f) \leq \sum_{j=0}^{n} T(r,f_j) + 0(1), \quad \text{as} \quad r \to \infty. \tag{8}$$

(c) If L is a differential field consisting of meromorphic functions defined on a fixed region R, and if R_1 is a subregion of R, then as is customary, we will identify L with the differential field obtained by restricting each element of L to R_1.

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An important role in our proof will be played by the following result of Siegel [8; pp. 60–62]:

LEMMA: *Let L be a differential field of meromorphic functions defined in a fixed region R, and let $A(z)$ and $B(z)$ be elements of L. Suppose that the differential equation,*

$$w'' + A(z)w' + B(z)w = 0, \tag{9}$$

possesses a particular solution $w_0(z)$ in a subregion R_1 *of R, such that $w_0(z)$ is not algebraic over L, but $w_0(z)$ does satisfy a first-order algebraic differential equation having coefficients in L. Then, in some subregion R_2 of R_1, the differential equation (9) possesses a solution $w_1(z) \not\equiv 0$ such that w_1'/w_1 is algebraic over L.*

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LEMMA A: *Let α be a nonzero rational number m/n in lowest terms where $n \geq 3$. Then no solution $y(z) \not\equiv 0$ of Bessel's equation of order α can satisfy a first-order algebraic differential equation having coefficients in the differential field L_1 consisting of all meromorphic functions on the plane whose order of growth is less than* 1.

PROOF: We assume the contrary, and let $y_1(z) \not\equiv 0$ be a solution of (1) which satisfies a first-order algebraic differential equation,

$$\sum f_{ij}(z)y^i(y')^j = 0, \tag{10}$$

where the f_{ij} belong to L_1.

A fundamental set of solutions for Bessel's equation in this case is $\{J_\alpha(z),\ J_{-\alpha}(z)\}$ (see [1]), and we can write $J_\alpha(z) = z^{m/n}E_\alpha(z)$ and $J_{-\alpha}(z) = z^{-m/n}E_{-\alpha}(z)$, where E_α and $E_{-\alpha}$ are both entire functions. Hence, in some disk U not containing $z = 0$, we have,

$$y_1(z) = c_1 z^{m/n}E_\alpha(z) + c_2 z^{-m/n}E_{-\alpha}(z), \tag{11}$$

where c_1 and c_2 are constants, not both zero, and where $z^{m/n}$ and $z^{-m/n}$ denote branches of the power functions in U. Let V denote the image of U under a branch of $z^{1/n}$ and let $F(\zeta) = y_1(\zeta^n)$ for ζ in V. Hence, from (11), it easily follows that,

$$F(\zeta) \equiv c_3\zeta^m E_\alpha(\zeta^n) + c_4\zeta^{-m}E_{-\alpha}(\zeta^n) \quad \text{on } V, \tag{12}$$

for certain constants c_3 and c_4, not both zero. It now follows from (12) that we can write,

$$F(\zeta) = \zeta^{-m}h_1(\zeta), \tag{13}$$

where $h_1(\zeta)$ is an entire function. Since,

$$h_1(\zeta) = \zeta^m y_1(\zeta^n), \tag{14}$$

and since $y_1(z)$ satisfies equation (1) with $\alpha = m/n$, a routine calculation shows that $h_1(\zeta)$ satisfies the differential equation,

$$\zeta^2 h'' + (1-2m)\zeta h' + n^2\zeta^{2n}h = 0. \tag{15}$$

Now, since $y_1(z)$ is assumed to satisfy (10), it easily follows that $h_1(\zeta)$ satisfies a first-order algebraic differential equation whose coefficients are linear combinations (with rational functions of ζ for coefficients) of the functions $f_{ij}(\zeta^n)$. Since each $f_{ij}(z)$ belongs to L_1, it easily follows (e.g. see [9; p. 284 g]) that $f_{ij}(z)$ can be written as the quotient $\varphi_{ij}(z)/\psi_{ij}(z)$ of two entire functions of order less than 1. Hence, $f_{ij}(\zeta^n)$ is the quotient of two entire functions of order less than n, and it now follows from the elementary rules for the Nevanlinna characteristic, (e.g. [5; p. 15]) that each $f_{ij}(\zeta^n)$ belongs to the differential field L_3 consisting of all meromorphic functions on the plane whose order is less than n. Hence, $h_1(\zeta)$ is a solution of equation (15) which satisfies a first-order algebraic differential equation with coefficients in L_3. To apply Siegel's lemma (§4), we must show that $h_1(\zeta)$ is not algebraic over L_3. We note first that $h_1(\zeta)$ cannot be a polynomial, since it is easy to see that (15) has no polynomial solutions (except zero). Hence the Valiron-Wiman theory (§3(a)) can be applied to equation (15), and we see that $h_1(\zeta)$ is of order n, and, in fact, for some positive constant c_5,

$$\log M(r, h_1) = c_5 r^n(1 + o(1)), \quad \text{as } r \to \infty. \tag{16}$$

It now follows that $h_1(\zeta)$ is not algebraic over L_3, for in the contrary case, $h_1(\zeta)$ would satisfy an algebraic equation (7) over L_3, and it would follow from (8) that $h_1(\zeta)$ would be of order less than n.

From Siegel's lemma (§4), we see that in some disk V_1 of the ζ-plane, equation (15) possesses a solution $h_2(\zeta) \not\equiv 0$ such that,

$$h_2'(\zeta)/h_2(\zeta) \quad \text{is algebraic over } L_3. \tag{17}$$

We may assume that V_1 does not contain $\zeta = 0$, and is so small that the mapping $\zeta \to \zeta^n$ is one-to-one from V_1 onto a region U_1. Now, define $y_2(z) = \zeta^{-m} h_2(\zeta)$ for z in U_1, and $z = \zeta^n$. Then, a routine calculation shows that $y_2(z)$ satisfies equation (1) in U_1 with $\alpha = m/n$. Thus, $y_2(z)$ also has a representation of the form (11) in U_1, and so, as before, $F_2(\zeta) = y_2(\zeta^n)$ has a representation of the form (12) on V_1. Hence,

$$h_2(\zeta) = c_3 \zeta^{2m} E_\alpha(\zeta^n) + c_4 E_{-\alpha}(\zeta^n), \tag{18}$$

and therefore $h_2(\zeta)$ can be extended to be an entire function. As before, $h_2(\zeta)$ cannot be a polynomial, and the Valiron-Wiman theory then shows that $h_2(\zeta)$ is of order n, and

$$\log M(r, h_2) = c_6 r^n(1 + o(1)) \quad \text{as } r \to \infty, \tag{19}$$

for some positive constant c_6.

In view of (17), it follows from (8) that the order of growth of h_2'/h_2 is less than n. Hence, (see [5; p. 31]), the exponent of convergence of the sequence of poles of h_2'/h_2 is less than n. Since $h_2(\zeta)$ is not identically zero and solves equation (15), it follows from the uniqueness theorem for linear differential equations, that all zeros of $h_2(\zeta)$ are simple, with the possible exception of a zero at $\zeta = 0$. Hence, the zero-sequence of $h_2(z)$ has exponent of convergence less than n, and so if we let $G(\zeta)$ denote the canonical product which vanishes at the zero-sequence of $h_2(\zeta)$ (and which includes a factor ζ^q, where $q \geq 0$ is the multiplicity of the zero of $h_2(\zeta)$ at $\zeta = 0$), then (e.g. see [6; p. 330]),

$$G(\zeta) \quad \text{is of order less than } n. \tag{20}$$

From the Hadamard factorization theorem, we may write,

$$h_2(\zeta) = G(\zeta)e^{Q(\zeta)}, \tag{21}$$

where $Q(\zeta)$ is a polynomial of degree at most n (since $h_2(\zeta)$ has order n). However, in view of (19) and (20), we see that $Q(\zeta)$ must be of degree exactly n. Hence, $Q(\zeta) = a\zeta^n + \varphi(\zeta)$, where $a \neq 0$ and $\varphi(\zeta)$ is a polynomial of degree less than n, so that,

$$h_2(\zeta) = G_1(\zeta)e^{a\zeta^n}, \quad \text{where } G_1(\zeta) = G(\zeta)e^{\varphi(\zeta)}, \tag{22}$$

and thus,

$$G_1(\zeta) \quad \text{is of order less than } n. \tag{23}$$

Since $h_2(\zeta)$ satisfies equation (15), a routine calculation shows that $G_1(\zeta)$ satisfies the equation,

$$\zeta^2 G_1'' + (2na\zeta^{n+1} + (1 - 2m)\zeta)G_1' + H(\zeta)G_1 = 0, \tag{24}$$

where,

$$H(\zeta) = n^2(a^2 + 1)\zeta^{2n} + n(n - 2m)a\zeta^n. \tag{25}$$

We now show that $G_1(\zeta)$ cannot be a polynomial. If we assume the contrary, then from (22), it would follow that $h_2'(\zeta)/h_2(\zeta)$ is a rational function. Recall that earlier we showed that the function $y_2(z) = z^{-m/n}h_2(z^{1/n})$ defined on U_1 is a solution of Bessel's equation (1) with $\alpha = m/n$. Clearly, if $h_2'(\zeta)/h_2(\zeta)$ was a rational function, then $y_2'(z)/y_2(z)$ would be a rational function of $z^{1/n}$, and hence an algebraic function of z. However, Siegel proved [8; pp. 63–65] that if Bessel's equation of order α possesses a solution whose logarithmic derivative is an algebraic function of z, then α must be one-half of an odd integer. Since in our case, α is not one-half of an odd integer, it follows that $y_2'(z)/y_2(z)$ cannot be an algebraic function. Hence $h_2'(\zeta)/h_2(\zeta)$ cannot be a rational function, and so $G_1(\zeta)$ is an entire function which is not a polynomial.

We now apply the Valiron-Wiman theory (§3) to the solution $G_1(\zeta)$ of equation (24). Noting that $n - 2m \not\equiv 0$ (since m/n is not one-half of an odd integer), it follows that in both of the cases $a^2 + 1 \not\equiv 0$ and $a^2 + 1 = 0$, the Valiron-Wiman theory asserts that $G_1(\zeta)$ must be of order n, and

$$\log M(r, G_1) = c_7 r^n(1 + o(1)), \quad \text{as} \quad r \to \infty, \tag{26}$$

for some positive constant c_7. This is in direct contradiction to (23), and this contradiction establishes Lemma A.

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LEMMA B: *Let α be a nonzero rational number m/n in lowest terms where $n \geq 3$ and n is odd. Then no solution $y(z) \not\equiv 0$ of Bessel's equation of order α can satisfy a first-order algebraic differential equation having coefficients in the differential field L_2 consisting of all meromorphic functions $f(z)$ on the plane whose Nevanlinna characteristic $T(r, f)$ satisfies $T(r, f) = o(r)$ as $r \to \infty$.*

PROOF: The proof begins in a manner very similar to the proof of Lemma A. We assume the contrary, and let $y_1(z) \not\equiv 0$ be a solution of (1) which satisfies an equation (10) where the f_{ij} belong to L_2. Since $y_1(z)$ has the representation (11), it follows as in the proof of Lemma A, that the function $h_1(\zeta)$ defined by (14) is an entire solution of equation (15).

Since $y_1(z)$ satisfies equation (10), it easily follows that $h_1(\zeta)$ satisfies a first-order algebraic differential equation whose coefficients are linear combinations (with rational functions of ζ for coefficients) of the functions $f_{ij}(\zeta^n)$. Since each $f_{ij}(z)$ belongs to L_2, it follows from a result of Miles [4; pp. 372–373] that $f_{ij}(z)$ can be written as the quotient of two entire functions $\varphi_{ij}(z)/\psi_{ij}(z)$ which belong to L_2. From (6) it then follows that $\log M(r, \varphi_{ij})$ and $\log M(r, \psi_{ij})$ are each $o(r)$ as $r \to \infty$. Hence $f_{ij}(\zeta^n)$ is the quotient of two entire functions which belong to the differential field L_4 consisting of all meromorphic functions $\varphi(\zeta)$ for which $T(r, \varphi) = o(r^n)$ as $r \to \infty$. It follows that $f_{ij}(\zeta^n)$ belongs to L_4, and hence $h_1(\zeta)$ is a solution of equation (15) which also satisfies a first-order algebraic differential equation with coefficients in L_4. As in the proof of Lemma A, we see that $h_1(\zeta)$ cannot be a polynomial, and hence the Valiron-Wiman theory (applied to (15)) shows that h_1 satisfies the relation (16) for some $c_5 > 0$. It now follows that $h_1(\zeta)$ is not algebraic over L_4 for in the contrary case, we would have $T(r, h_1) = o(r^n)$ as $r \to \infty$ by (8), and this would contradict (16) in view of (6). Hence, we may apply Siegel's lemma (§4), and it follows that equation (15) possesses a solution $h_2(\zeta) \not\equiv 0$ in some disk, with the property that $h_2'(\zeta)/h_2(\zeta)$ is algebraic over L_4. From (8), it then follows that,

$$T(r, h_2'/h_2) = o(r^n) \quad \text{as } r \to \infty. \tag{27}$$

As in the proof of Lemma A, $h_2(\zeta)$ has the representation (18) for some constants c_3 and c_4, and thus can be extended to be an entire function of order n satisfying the growth relation (19) for some $c_6 > 0$.

Now, it is well-known that the entire functions $E_\alpha(z)$ and $E_{-\alpha}(z)$ appearing in the representations for $J_\alpha(z)$ and $J_{-\alpha}(z)$, are even functions, and thus $h_2(\zeta)$ is an even function of ζ by (18). Hence, $h_2(\zeta^{1/2})$ is an entire function of order $n/2$, and since $n/2$ is not an integer, it follows (e.g. see [6; p. 339]) that $h_2(\zeta^{1/2})$ and $h_2(\zeta)$ have infinitely many zeros. Let $\zeta = 0$ be a zero of $h_2(\zeta)$ of multiplicity $k \geq 0$. The zeros of $h_2(\zeta)$ in $0 < |\zeta| < \infty$ are all simple (since $h_2(\zeta)$ satisfies equation (15)), and since $h_2(\zeta)$ is even, these zeros can be arranged in a sequence $B = (\zeta_1, -\zeta_1, \zeta_2, -\zeta_2, \ldots)$ where $|\zeta| \leq |\zeta_2| \leq \cdots$. Let $q(r)$ denote the counting function of this sequence (i.e. for $r > 0$, $q(r)$ is the number of elements of this sequence lying in $|\zeta| \leq r$). Since $(\zeta_1, -\zeta_1, \zeta_2, -\zeta_2, \ldots)$ is also the sequence of poles of $h_2'(\zeta)/h_2(\zeta)$ in $0 < |\zeta| < \infty$, it follows from (27) (e.g. see [2; p. 25] and the definition of $T(r, h_2)$) that $q(r) = o(r^n)$ as $r \to \infty$. Since $q(|\zeta_j|) \geq 2j$ for all j, we have,

$$|\zeta_j|^n/j \to \infty \quad \text{as } j \to \infty. \tag{28}$$

Now, for $\lambda = 1, 2, \ldots$, set

$$D_\lambda(\zeta) = (1 - \zeta)\exp(\zeta + (\zeta^2/2) + \cdots + (\zeta^\lambda/\lambda)). \tag{29}$$

Since n is an odd integer which is at least 3, we have $n = 2p + 1$ where $p \geq 1$. Clearly,

$$D_n(\zeta/\zeta_j)D_n(-\zeta/\zeta_j) = D_p((\zeta/\zeta_j)^2) \quad \text{for each } j. \tag{30}$$

Now, it is well-known (e.g. see [6; p. 297]) that,

$$|D_p(\zeta) - 1| \leq 3|\zeta|^{p+1} \quad \text{for } |\zeta| \leq 1/2. \tag{31}$$

Now set,

$$u_j(\zeta) = D_n(\zeta/\zeta_j)D_n(-\zeta/\zeta_j) - 1 \quad \text{for } j = 1, 2, \ldots. \tag{32}$$

Then, if $R > 0$ is given, let J be an index such that $|\zeta_j| \geq 2^{1/2}R$ for $j \geq J$. If $|\zeta| \leq R$ and $j \geq J$, it follows from (30) and (31) that,

$$|u_j(\zeta)| \leq 3R^{n+1}/|\zeta_j|^{n+1}. \tag{33}$$

In view of (28), it now follows that the series $\Sigma_{j=0}^{\infty} |u_j(\zeta)|$ converges uniformly on $|\zeta| \leq R$. Since R is arbitrary, we see that the infinite product,

$$G(\zeta) = \prod_{j=1}^{\infty} D_n(\zeta/\zeta_j)D_n(-\zeta/\zeta_j), \tag{34}$$

converges to an entire function.

We now assert that,

$$\log M(r, G) = o(r^n) \quad \text{as } r \to \infty. \tag{35}$$

To prove (35), let $r > 0$ and let ζ be a point on $|\zeta| = r$. From (30) and (34), we can write,

$$\log|G(\zeta)| = I_1(\zeta) + I_2(\zeta), \tag{36}$$

where,

$$I_1(\zeta) = \sum_{|\zeta_j| \leq 2r} \log|D_p((\zeta/\zeta_j)^2)|, \tag{37}$$

and

$$I_2(\zeta) = \sum_{|\zeta_j| > 2r} \log|D_p((\zeta/\zeta_j)^2)|. \tag{38}$$

To estimate I_1 and I_2, we will need the following fact (see [6; pp. 330–331]): There exist positive constants A_1 and A_2 such that for all ζ,

$$|D_p(\zeta)| \leq \exp(A_1|\zeta|^p) \quad \text{and} \quad |D_p(\zeta)| \leq \exp(A_2|\zeta|^{p+1}). \tag{39}$$

From (37) and (39), we see that

$$I_1(\zeta) \leq A_1 r^{n-1} \sum_{|\zeta_j| \leq 2r} |\zeta_j|^{-(n-1)}. \tag{40}$$

Since the counting function for the sequence $(\zeta_1, \zeta_2, \ldots)$ is $q(t)/2$, the sum in (40) can be written as a Stieltjes integral $(1/2) \int_{|\zeta_1|/2}^{2r} t^{-(n-1)}\, dq(t)$. Integrating by parts, we thus obtain,

$$I_1(\zeta) \leq A_1 2^{-n} q(2r) + (A_1/2) r^{n-1} \int_{|\zeta_1|/2}^{2r} (n-1) t^{-n} q(t) dt. \tag{41}$$

From the second inequality in (39), we see that

$$I_2(\zeta) \leq A_2 r^{n+1} \sum_{|\zeta_j| > 2r} |\zeta_j|^{-(n+1)}. \tag{42}$$

The sum in (42) can be written, $(1/2) \int_{2r}^{\infty} t^{-(n+1)}\, dq(t)$. Since $q(t) = o(t^n)$ as $t \to \infty$, integration by parts now shows that,

$$I_2(\zeta) \leq (A_2/2) r^{n+1} \int_{2r}^{\infty} (n+1) t^{-n-2} q(t)\, dt. \tag{43}$$

If $\delta > 0$ is given, we can choose $r_0 > 1$ such that $q(t) < \delta t^n$ for $t \geq r_0$. Partitioning the interval of integration in (41) into the two intervals $[|\zeta_1|/2, r_0]$ and $[r_0, 2r]$, it now follows from (36), (41), and (43) that there are two constants K_1 and K_2 (which depend only on A_1, A_2, and n) such that for $r \geq r_0$,

$$(\log M(r, g))/r^n \leq K_1 \delta + r^{-1}(K_2 \delta r_0^{n+1}/|\zeta_1|^n). \tag{44}$$

It is now clear that (35) holds.

Since $G(\zeta)$ vanishes precisely at the zeros of $h_2(\zeta)$ in $0 < |\zeta| < \infty$, and since $h_2(\zeta)$ is of order n, we have a representation,

$$h_2(\zeta) = \zeta^k G(\zeta) e^{Q(\zeta)}, \tag{45}$$

where $Q(\zeta)$ is a polynomial of degree at most n. The degree of $Q(\zeta)$ must be exactly n, for in the contrary case, it would follow from (35) and (45) that $\log M(r, h_2) = o(r^n)$ as $r \to \infty$, which contradicts the relation (19) that was established earlier. Hence, $Q(\zeta) = a\zeta^n + \varphi(\zeta)$, where $a \neq 0$ and $\varphi(\zeta)$ is a polynomial of degree less than n, so that,

(46) $$h_2(\zeta) = G_1(\zeta)e^{a\zeta^n}, \quad \text{where} \quad G_1(\zeta) = \zeta^k G(\zeta)e^{\varphi(\zeta)}.$$

As in the proof of Lemma A, the entire function $G_1(\zeta)$ satisfies the differential equation (24), where $H(\zeta)$ is given by (25). Since $h_2(\zeta)$ was shown earlier to have infinitely many zeros, $G_1(\zeta)$ is not a polynomial. Hence, the Valiron-Wiman theory can be applied to equation (24), and as in the proof of Lemma A, we see that in both of the cases $a^2 + 1 \neq 0$ and $a^2 + 1 = 0$, the function $G_1(\zeta)$ is of order n, and

(47) $$\log M(r, G_1) = c_7 r^n(1 + o(1)) \quad \text{as} \quad r \to \infty,$$

for some positive constant c_7. However, (47) is clearly impossible in view of the representation for $G_1(\zeta)$ in (46), since $\varphi(\zeta)$ is of degree less than n, and the function $G(\zeta)$ satisfies $\log M(r, G) = o(r^n)$ as $r \to \infty$ by (35). This contradiction establishes Lemma B.

7

LEMMA C: *Let α be a complex number which is not a rational number. Let L be a field of meromorphic functions each defined in a fixed neighborhood U of $z = 0$, and assume that both of the functions $J'_\alpha(z)/J_\alpha(z)$ and $J'_{-\alpha}(z)/J_{-\alpha}(z)$ are transcendental over L. Then, if $y(z) \not\equiv 0$ is any solution of Bessel's equation of order α in a subregion of U, the function $y'(z)/y(z)$ is transcendental over L.*

PROOF: We assume the contrary and let $y(z)$ be a solution of (1) which is nowhere zero in a disk D of the form $|z - z_0| < \epsilon$, where $z_0 \neq 0$, $\epsilon < |z_0|$, and D is contained in U, and where $y(z)$ has the property that $u_0(z) = y'(z)/y(z)$ is algebraic over L. Hence, there exist elements $F_0(z), F_1(z), \ldots, F_q(z)$ in L, with $F_q \not\equiv 0$, such that $u = u_0(z)$ satisfies the equation,

(48) $$F_q(z)u^q + F_{q-1}(z)u^{q-1} + \cdots + F_0(z) = 0.$$

Now a fundamental set of solutions for (1) in this case is $\{J_\alpha(z), J_{-\alpha}(z)\}$, and we may write, $J_\alpha(z) = z^\alpha E_\alpha(z)$ and $J_{-\alpha}(z) = z^{-\alpha}E_{-\alpha}(z)$, where $E_\alpha(z)$ and $E_{-\alpha}(z)$ are entire functions. Let $f(z)$ be a fixed branch of z^α in D, and let $h(z) = 1/f(z)$. Then fE_α and $hE_{-\alpha}$ form a fundamental set of solutions of (1) in D, so there exist constants c_1 and c_2, not both zero, such that,

$$y(z) = c_1 f(z)E_\alpha(z) + c_2 h(z)E_{-\alpha}(z) \quad \text{on } D. \tag{49}$$

Since J_α'/J_α and $J_{-\alpha}'/J_{-\alpha}$ are assumed transcendental over L, while y'/y is assumed algebraic over L, clearly both c_1 and c_2 are nonzero.

For each $n = 1, 2, \ldots$, let $u_n(z)$ denote the analytic function on D obtained by analytically continuing the function $u_0(z)$ counterclockwise around the circle $|z| = |z_0|$ precisely n times. We note that under this analytic continuation, $E_\alpha(z)$ and $E_{-\alpha}(z)$ both return to their original values, while $f(z)$ becomes $e^{2\pi i n\alpha} f(z)$, and $h(z)$ becomes $e^{-2\pi i n\alpha} h(z)$. In addition

$$f'(z) \equiv \alpha z^{-1} f(z), \quad \text{and} \quad h'(z) \equiv -\alpha z^{-1} h(z) \quad \text{on } D. \tag{50}$$

Hence,

$$u_n = (c_1 d_n f Q + c_2 d_{-n} h R)/(c_1 d_n f E_\alpha + c_2 d_{-n} h E_{-\alpha}), \tag{51}$$

where $d_n = e^{2\pi i n\alpha}$, $d_{-n} = e^{-2\pi i n\alpha}$, and where

$$Q = E_\alpha' + \alpha z^{-1} E_\alpha, \quad \text{and} \quad R = E_{-\alpha}' - \alpha z^{-1} E_{-\alpha}. \tag{52}$$

Now, since the functions $F_j(z)$ are meromorphic (and hence single-valued) on U, each of these functions returns to its original value under the analytic continuation around $|z| = |z_0|$. It follows that each of the functions $u_0(z), u_1(z), \ldots$, solves the algebraic equation (48) in D, and thus these functions cannot all be distinct. Hence, there exist distinct nonnegative integers k and n such that,

$$u_n(z) \equiv u_k(z) \quad \text{on } D. \tag{53}$$

Using the representation (50) for u_n, and the corresponding representation for u_k, and substituting into equation (51), (and using the fact that c_1 and c_2 are both nonzero), we obtain,

$$\lambda E_{-\alpha}(z)Q(z) \equiv \lambda E_\alpha(z)R(z) \quad \text{on } D, \tag{54}$$

where $\lambda = e^{2\pi i m\alpha} - e^{-2\pi i m\alpha}$, and $m = n - k$. Since α is not rational, clearly $\lambda \neq 0$ so,

$$E_{-\alpha}(z)Q(z) \equiv E_\alpha(z)R(z) \quad \text{on } D. \tag{55}$$

Multiplying the relation (55) by $f(z)h(z)$, and using (50) and the representations for Q and R given in (52), it follows from (55) that

$$fE_\alpha(hE_{-\alpha})' \equiv hE_{-\alpha}(fE_\alpha)' \quad \text{on } D. \tag{56}$$

Of course, (56) contradicts the fact that fE_α and $hE_{-\alpha}$ are linearly independent solutions of (1) on D, and this contradiction proves Lemma C.

8

COROLLARY 1: *Let α be a complex number which is not a rational number. Let L be a differential field of meromorphic functions each defined in a fixed neighborhood U of $z=0$, and assume that L contains the field of rational functions. Suppose that both of the functions $J'_\alpha(z)/J_\alpha(z)$ and $J'_{-\alpha}(z)/J_{-\alpha}(z)$ are transcendental over L. Then, no solution $y(z)\not\equiv 0$ of Bessel's equation of order α in a subregion of U can satisfy a first-order algebraic differential equation having coefficients in L.*

PROOF: We assume the contrary, and let $y_1(z)\not\equiv 0$ be a solution of (1) in a subregion of U which satisfies a first-order algebraic differential equation with coefficients in L. Then by Lemma C, the function $y_1'(z)/y_1(z)$ is transcendental over L. We assert that $y_1(z)$ must also be transcendental over L. To see this, we assume the contrary. Then by differentiating the minimal polynomial of y_1 over L, we see that y_1' belongs to the field $L(y_1)$ generated by L and y_1. Thus, the field $L(y_1)$ is an algebraic extension of L which contains y_1'/y_1, and this contradicts the fact that y_1'/y_1 is transcendental over L by Lemma C. This proves the assertion, and hence we can apply Siegel's lemma (§4) which shows that equation (1) possesses a solution $y_2(z)\not\equiv 0$ in a subregion of U, with the property that $y_2'(z)/y_2(z)$ is algebraic over L. This contradicts the conclusion of Lemma C and thus establishes the result.

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LEMMA D: *Let α be an integer, which we may assume to be nonnegative. Let L be a field of meromorphic functions each defined in a fixed neighborhood U of $z=0$, and assume that the function*

$J'_\alpha(z)/J_\alpha(z)$ is transcendental over L. Then, if $y(z) \not\equiv 0$ is any solution of Bessel's equation of order α in a subregion of U, the function $y'(z)/y(z)$ is transcendental over L.

PROOF: The proof closely parallels the proof of Lemma C. We assume the contrary, and let $y(z)$ be a solution of (1) which is nowhere zero on a disk D of the form $|z - z_0| < \epsilon$, where $z_0 \not\equiv 0$, $\epsilon < |z_0|$, and D is contained in U, and where $y(z)$ has the property that $u_0(z) = y'(z)/y(z)$ is algebraic over L. Hence, $u = u_0(z)$ satisfies an equation of the form (48), where $F_0, \ldots, F_q$ are elements of L, with $F_q \not\equiv 0$. Now, a fundamental set of solutions of (1) in this case (e.g. see [1]) is $\{J_\alpha(z), K_\alpha(z)\}$, where

$$K_\alpha(z) = G_\alpha(z) + (\log z)J_\alpha(z), \tag{57}$$

and where $G_\alpha(z)$ is meromorphic in the plane. Of course, $J_\alpha(z)$ is an entire function in this case. Let $\varphi(z)$ be a fixed branch of $\log z$ in D, so that $\{J_\alpha(z), G_\alpha(z) + \varphi(z)J_\alpha(z)\}$ is a fundamental set of solutions of (1) in D. Hence, there exist constants c_1 and c_2, not both zero, such that,

$$y(z) = c_1 J_\alpha(z) + c_2(G_\alpha(z) + \varphi(z)J_\alpha(z)) \quad \text{on } D. \tag{58}$$

Since J'_α/J_α is assumed transcendental over L, while y'/y is assumed algebraic over L, clearly $c_2 \not\equiv 0$.

As in Lemma C, let $u_n(z)$ denote the analytic function on D obtained from $u_0(z)$ by analytic continuation around $|z| = |z_0|$ in the positive direction n times. Of course, G_α and J_α return to their original values, while $\varphi(z)$ becomes $\varphi(z) + 2\pi i n$. Hence,

$$u_n = (Q + (\varphi + 2\pi i n)c_2 J'_\alpha)/(R + (\varphi + 2\pi i n)c_2 J_\alpha) \tag{59}$$

where,

$$Q = c_1 J'_\alpha + c_2 G'_\alpha + c_2 z^{-1} J_\alpha \quad \text{and} \quad R = c_1 J_\alpha + c_2 G_\alpha. \tag{60}$$

Since each $F_j(z)$ in (48) returns to its original value under the analytic continuation, it follows as in the proof of Lemma C, that each $u_n(z)$ solves the algebraic equation (48), and hence, there exist distinct nonnegative integers k and n such that, $u_n(z) \equiv u_k(z)$ on D. Using the representations (59) and (60), a simple calculation gives,

$$c_2^2 G'_\alpha J_\alpha + c_2^2 z^{-1} J_\alpha^2 \equiv c_2^2 G_\alpha J'_\alpha \quad \text{on } D. \tag{61}$$

Since $c_2 \not\equiv 0$, and since $G_\alpha = K_\alpha - \varphi J_\alpha$, it now follows from (61) that $J_\alpha K'_\alpha \equiv K_\alpha J'_\alpha$ on D. Of course, this is impossible since $\{J_\alpha, K_\alpha\}$ is a linearly independent set of solutions of (1), and this contradiction establishes Lemma D.

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COROLLARY 2: *Let α be an integer which we may assume to be nonnegative. Let L be a differential field of meromorphic functions each defined in a fixed neighborhood U of $z = 0$, and assume that L contains the field of rational functions. Suppose that the function J'_α/J_α is transcendental over L. Then, no solution $y(z) \not\equiv 0$ of Bessel's equation of order α in a subregion of U can satisfy a first-order algebraic differential equation having coefficients in L.*

PROOF: This corollary follows from Lemma D exactly the way Corollary 1 followed from Lemma C.

11

PROOF OF THEOREM 1: We are given that α is not one-half of an odd integer, and L_1 is the field of meromorphic functions of order less than 1.

Suppose first that α is a nonzero rational number m/n in lowest terms with $n \geq 3$. Then the conclusion of Theorem 1 follows in this case from Lemma A.

In the remaining cases, α is either an integer or is not a rational number. In view of Corollaries 1 and 2, it obviously suffices to show that J'_α/J_α is transcendental over L_1 for any complex α. We know that $J_\alpha(z) = z^\alpha E_\alpha(z)$, where $E_\alpha(z)$ is an entire even function of order 1, and thus $E_\alpha(z^{1/2})$ is an entire function of order 1/2. Thus, (e.g. see [6; p. 339]) the exponent of convergence of the zero-sequence of $E_\alpha(z^{1/2})$ is 1/2, and so the exponent of convergence of the zero-sequence of $E_\alpha(z)$ is 1. Of course, the zeros of $E_\alpha(z)$ in $0 < |z| < \infty$ are simple since $E_\alpha(z)$ is easily seen to satisfy the differential equation,

$$z^2 E''_\alpha + (2\alpha + 1) z E'_\alpha + z^2 E_\alpha = 0. \tag{62}$$

Hence, the sequence of poles of $E'_\alpha(z)/E_\alpha(z)$ has exponent of convergence equal to 1, and thus, the same is true for the sequence of poles of $J'_\alpha(z)/J_\alpha(z)$. Therefore, the order of $J'_\alpha(z)/J_\alpha(z)$ is at least 1 by [5; p. 31]. (In fact, the order of J'_α/J_α is precisely 1.) It follows that J'_α/J_α is transcendental over L_1, for in the contrary case, it would follow from (8) that the order of J'_α/J_α is less than 1. Thus, the conclusion of Theorem 1 holds in all cases.

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PROOF OF THEOREM 2: We first observe that the field described in the statement of Theorem 2 is precisely the field L_2 of all meromorphic functions $f(z)$ for which $T(r, f) = o(r)$ as $r \to \infty$. The fact that the field in Theorem 2 is contained in L_2 follows from the elementary properties of $T(r, f)$ together with relation (6). On the other hand, the two fields must then coincide since by [4; pp. 372–373] and relation (6), every element f in L_2 can be written as a quotient φ_1/φ_2 of two entire functions such that $\log M(r, \varphi_j) = o(r)$ as $r \to \infty$ for $j = 1, 2$.

Now, let α be a real number. If α is a nonzero rational number m/n in lowest terms where n is an odd integer greater than or equal to 3, the conclusion follows from Lemma B.

In the remaining cases, the real number α is either an integer or irrational. In view of Corollaries 1 and 2, it obviously suffices to show that for any real number α, the function J'_α/J_α is transcendental over L_2. If we assume the contrary, then by (8),

$$T(r, J'_\alpha/J_\alpha) = o(r) \quad \text{as } r \to \infty. \tag{63}$$

However, it is easy to see that if $g_\alpha(x) = x^{1/2}J_\alpha(x)$ for $x > 0$, then $g_\alpha(x)$ satisfies the equation,

$$y'' + \psi(x)y = 0, \tag{64}$$

where $\psi(x) = 1 + x^{-2}((1/4) - \alpha^2)$. Since α is real, it follows that both $\psi(x)$ and $g_\alpha(x)$ are real-valued for $x > 0$, and clearly there exists $x_0 > 0$ such that $\psi(x) > 1/4$ for all $x \geq x_0$. Comparing the equation (64) with the equation $y'' + (1/4)y = 0$ which has $\sin(x/2)$ for a solution, it follows from the Sturm comparison theorem [10; p. 29] that between any two consecutive zeros of $\sin(x/2)$ on (x_0, ∞), there is a zero of $g_\alpha(x)$, and hence a zero of $J_\alpha(x)$. Writing $J_\alpha(z) = z^\alpha E_\alpha(z)$, where $E_\alpha(z)$ is entire, and letting $q(r)$ denote the number of zeros of $E_\alpha(z)$ in $0 < |z| \leq r$, it easily follows that $q(r) \geq r/7$ for all sufficiently large r. Since the zeros of $E_\alpha(z)$ in $0 < |z| < \infty$ are simple (see (62)), it therefore follows that the number $n(r, E'_\alpha/E_\alpha)$ of poles of E'_α/E_α in $|z| \leq r$ satisfies,

$$n(r, E'_\alpha/E_\alpha) \geq r/7 \quad \text{for all sufficiently large } r. \tag{65}$$

However, from (63) and the elementary properties of the Nevanlinna characteristic, it follows that $T(r, E'_\alpha/E_\alpha) = o(r)$ as $r \to \infty$, and hence $n(r, E'_\alpha/E_\alpha) = o(r)$ as $r \to \infty$ (see [2; p. 25]). Of course, this last estimate is in direct contradiction to (65), and this establishes Theorem 2.

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(Oblatum 25-IX-1980)

Department of Mathematics
University of Illinois
1409 W. Green St.
Urbana, Illinois 61801
U.S.A.