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A CHARACTERIZATION OF THE SPHERICALLY COMPLETE NORMED SPACES WITH A DISTINGUISHED BASIS

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The theory of normed spaces over a trivially valued field (or *valued spaces*) was developed mainly by P. Robert in his series of papers [3]. He introduced the concept of distinguished basis, also called orthogonal bases in the literature, and in order to deal with spaces that posses distinguished bases, he restricted himself to V-spaces ([3], p. 16), that is, complete valued spaces E such that

$$\|E\| = \{\|x\| : x \in E\} \subset \{0\} \cup \{\rho^n : n \in \mathbf{Z}\}$$

for some real number $\rho > 1$. K.-W. Yang, [5], has given a different proof of the fact that V-spaces have a distinguished basis. All V-spaces are easily shown to be spherically complete.

In this note we give a characterization of all valued spaces which are spherically complete and have a distinguished basis. These spaces need not be V-spaces. Moreover, we answer a question of Robert ([3], p. 8), by giving examples of valued spaces without a distinguished basis.

For notations, we refer to [3] and [4].

THEOREM: *Let E be a complete valued space over a field K (i.e., a non-archimedean Banach space over a field with the trivial valuation). Then, the following are equivalent:*

(i) *E has a distinguished (or orthogonal) basis, and it is spherically complete.*

(ii) *Every strictly decreasing sequence in $\|E\|$ converges to zero.*

PROOF: Assume (ii). Let $X \subset E$ be a maximal orthogonal subset of E ([3], p. 9). It is very easy to prove that our hypothesis (ii) implies the closed linear span of X, $[X]$, is spherically complete. Then by Ingleton's Theorem ([4], Ex. 4.H; the proof also works when K is trivially valued), if $[X] \neq \cdot E$, there is a linear projection $P: E \to [X]$ of norm one, and for any $z \in E \setminus [X]$, $z - Pz$ is orthogonal to $[X]$ and different from zero, contradicting the maximality of X.

Conversely, assume E has a distinguished basis X and is spherically complete, and that there is a sequence in $\|E\|$ strictly decreasing and bounded away from zero. Since for every nonzero element of E there is some basic vector with the same norm, there must exist a sequence (x_n) in X with strictly decreasing norms but not convergent to zero.

Call F the closed vector subspace $[x_n : n \in \mathbf{N}]$. Then F is linearly isometric to the quotient of E by the subspace generated by the other members of X, hence it must be spherically complete (Cf. [4], Th. 4.2). But it is not: consider the sequence of closed balls

$$B(x_1 + \dots + x_n, \|x_n\|), \qquad n \in \mathbf{N}.$$

Remarks: (1) For non-archimedean Banach spaces over a *non-trivially* valued field, the same is true: a proof can be found in [4], Th. 5.16. That proof also works in our setting, but it is much more elaborated than the one given above; our proof is also valid when the valuation is not trivial, with a minor modification: in that case one cannot be sure that the set of norm values of a basis is the same as $\|E\| \setminus \{0\}$, and one has to change (x_n) into $(\lambda_n x_n)$ for suitable $\lambda_n \in K$.

(2) It is not difficult to prove that a valued space is spherically complete and has a distinguished basis if and only if it is linearly isometric with a space $c_0(I:s)$ defined as the set

$$\{x : I \to K \mid |x(i)|\, s(i) \to 0 \text{ for the Frechet filter on } I\}$$

(where I is any nonempty set) endowed with the norm

$$\|x\|_s = \max\{s(i) \mid x(i) \neq 0\}$$

where $s: I \to [0, +\infty)$ is a function whose range does not contain any strictly decreasing sequence with a positive limit.

Consequently, one can give examples of valued spaces with a distinguished basis, apart from V-spaces.

(3) Now we can produce several examples of valued spaces without a distinguished basis:

(a) Over the real field: the fields ${}^{\rho}\mathbf{R}$ introduced by A. Robinson, regarded as valued spaces over $\mathbf{R}$ (trivially valued), are spherically complete (see [1]), and have $\|{}^{\rho}\mathbf{R}\| = [0, +\infty)$.

(b) Over any field K: the field E of formal power series with coefficients in K and rational exponents, with the set of exponents relative to nonzero coefficients well-ordered is spherically complete ([2], p. 38), and has $\|E\|$ dense in $[0, +\infty)$.

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