COMPOSITIO MATHEMATICA

JOHN LITTLE

# Translation manifolds and the converse of Abel's theorem

*Compositio Mathematica*, tome 49, nº 2 (1983), p. 147-171

<http://www.numdam.org/item?id=CM_1983__49_2_147_0>

Compositio Mathematica 49 (1983) 147–171


# TRANSLATION MANIFOLDS AND THE CONVERSE OF ABEL'S THEOREM

John Little

## §0. Introduction

(a) *Historical Survey*

The connection between translation manifolds and Abel's theorem was first realized during the last quarter of the nineteenth century. As an outgrowth of his differential geometric research on minimal surfaces, Sophus Lie wanted to find all surfaces $S \subset \mathbb{C}^3$ (or $\mathbb{R}^3$) which could be swept out in more than one way by translating one curve rigidly along another.

In terms of coordinates, this condition means that $S$ can be described parametrically in two ways, as the locus

$$\begin{aligned} x_1 &= \alpha_{11}(t_1) + \alpha_{21}(t_2) = \beta_{11}(u_1) + \beta_{21}(u_2) \\ x_2 &= \alpha_{12}(t_1) + \alpha_{22}(t_2) = \beta_{12}(u_1) + \beta_{22}(u_2) \\ x_3 &= \alpha_{13}(t_1) + \alpha_{23}(t_2) = \beta_{13}(u_1) + \beta_{23}(u_2) \end{aligned} \tag{0.1}$$

where

$$\alpha_i(t_i) = (\alpha_{i1}(t_i), \alpha_{i2}(t_i), \alpha_{i3}(t_i))$$

and

$$\beta_i(u_i) = (\beta_{i1}(u_i), \beta_{i2}(u_i), \beta_{i3}(u_i))$$

are the two sets of curves which sweep out $S$. To ensure that the parametizations were really distinct, Lie worked with the hypothesis that the

$t_i$ could be expressed as functions of the $u_j$ in such a way that $\dfrac{\partial t_i}{\partial u_j} \neq 0$ for all $i$ and $j$.

Lie studied these surfaces $S$ by considering the tangent lines to the curves $\alpha_i$ and $\beta_i$, and made the fundamental discovery that these tangents cut the plane at infinity in the points of an algebraic quartic curve $C$ (possibly singular or reducible, but always reduced). Explicitly, the tangents,

$$\begin{aligned} \dot{\alpha}_i(t_i) &= (\alpha'_{i1}(t_i), \alpha'_{i2}(t_i), \alpha'_{i3}(t_i)) \\ \dot{\beta}_i(u_i) &= (\beta'_{i1}(u_i), \beta'_{i2}(u_i), \beta'_{i3}(u_i)), \end{aligned} \tag{0.2}$$

viewed as the homogeneous coordinates of points in $\mathbb{P}^2$, all satisfy a polynomial relation $F(\xi, \eta, \zeta) = 0$ of degree four, which is the equation of the curve $C$.

Lie then applied Abel's theorem to rewrite (0.1) in terms of the three Abelian integrals on $C$. Namely, $\alpha_i$ and $\beta_i$ can be parametrized as follows:

$$\begin{aligned} \alpha_1 &= \left(\int_{p_1}^{Q_1} \omega_1, \int_{p_1}^{Q_1} \omega_2, \int_{p_1}^{Q_1} \omega_3\right) \\ \alpha_2 &= \left(\int_{p_2}^{Q_2} \omega_1, \int_{p_2}^{Q_2} \omega_2, \int_{p_2}^{Q_2} \omega_3\right) \\ \beta_1 &= \left(\int_{p_3}^{Q_3} \omega_1, \int_{p_3}^{Q_3} \omega_2, \int_{p_3}^{Q_3} \omega_3\right) \\ \beta_2 &= \left(\int_{p_4}^{Q_4} \omega_1, \int_{p_4}^{Q_4} \omega_2, \int_{p_4}^{Q_4} \omega_3\right) \end{aligned} \tag{0.3}$$

where

$$\omega_1 = \frac{\xi\, d\xi}{\partial F/\partial \eta}, \omega_2 = \frac{\eta\, d\xi}{\partial F/\partial \eta}, \omega_3 = \frac{\zeta\, d\xi}{\partial F/\partial \eta},$$

the $p_i$ are a fixed collinear 4-tuple of points $C \cap l_0$ for some line $l_0 \subset \mathbb{P}^2$, and the $Q_i$ are the variable collinear 4-tuples $C \cap l$ for lines $l$ near $l_0$.

The double parametrization (0.1) is the reflection of the addition formula for the Abelian integrals, and $S$ is the locus in $\mathbb{C}^3$ obtained by substituting the expressions (0.3) into (0.1).

Thus Lie was led to a recipe for constructing all surfaces $S$ as in (0.1), since this construction can be applied to any reduced quartic $C$ in $\mathbb{P}^2$. In a sense Lie had reduced the problem to the simpler, and completely algebraic, one of finding all the reduced quartics.

Lie proved these results in a rather *ad hoc* way, by showing that the coefficients of the Taylor expansions of the $\alpha_i$ and $\beta_i$ satisfied certain relations, which then implied that the $\alpha_i$ and $\beta_i$ had to lie on an algebraic curve. The article [16] is an excellent survey of Lie's work, and includes a bibliography of his papers on the subject. This survey also contains some interesting examples of the surfaces $S$ obtained from singular and reducible $C$, one of which was Lie's original motivation for studying these surfaces. See also §4 below.

Lie then proceeded to consider higher dimensional translation manifolds swept out by curves in two different ways. He conjectured that any such manifold would be parametrized by Abelian integrals on an algebraic curve as well, but he did not complete a proof of this, even in the case of three-folds.

The proof in the case of translation surfaces was greatly simplified by Darboux [4]. He showed that a sort of *converse* of Abel's theorem could be used to deduce that the $\alpha_i$ and $\beta_i$ were pieces of an algebraic curve in $\mathbb{P}^2$.

The first complete proof of Lie's conjecture was given by Wirtinger in 1938, [18]. (Although Poincaré had published an argument to this effect in [11], his proof is difficult to understand.) Wirtinger's proof is based on yet another algebraicity criteria for analytic curves such as the $\alpha_i, \beta_i$. Then, as in the other proofs, he applied Abel's theorem to obtain the parametrization of any doubly translation type hypersurface $S \subset \mathbb{C}^n$ in terms of Abelian integrals.

In the paper cited above and in a later one [12], Poincaré pointed out the possibility of using these results to study the moduli of curves and their Jacobian varieties. Specifically, if we apply the above construction to a smooth, canonically embedded (hence non-hyperelliptic) curve $C$ of genus $n$ in $\mathbb{P}^{n-1}$, then the associated translation manifold $S$ is nothing other than the lifting to $\mathbb{C}^n$ of the theta-divisor $W_{n-1}$ in the Jacobian of $C$. Poincaré went on to try to deduce a period relation for non-hyperelliptic Jacobians of dimension 4 (the Poincaré asymptotic period relation) from the fact that the theta-divisor of a Jacobian must be a translation manifold in two ways. This gives some idea of the fascinating history (and rich promise for the future) of this circle of ideas. As Wirtinger says in the paper cited above, the amazing thing about Lie's discovery is the way the simply-stated differential-geometric criterion (0.1) leads to a class of objects with as many deep applications in algebraic geometry as the theta divisors in Jacobians.

(b) The purpose of this paper is two-fold. First, we give the first complete account of a new, simpler, and (perhaps) more conceptual proof of the Lie-Wirtinger results. The idea behind this proof is that Darboux's method (the converse of Abel's theorem, Corollary 1.3) can be extended to the case of hypersurfaces $S \subset \mathbb{C}^n$, for any $n$. This realization is due to Bernard Saint-Donat ([15]); see also the sketch in Mumford [10].) By systematically developing the theory for singular and reducible curves, in Theorem 3.9 below we are able to give a precise, intrinsic description of the doubly translation type hypersurfaces in $\mathbb{C}^n$ satisfying a slight additional condition (see Proposition 3.7). It is primarily the present, deeper understanding of the local structure of singular curves which makes this unified treatment possible.

Second, we show that these results are sufficient to prove that the Jacobians of non-hyperelliptic curves are exactly the principally polarized abelian varieties whose theta-divisors have a local double parametrization analogous to (0.1) at some point. This is the principal result of this paper and the precise statement is given in Theorem 5.1.

Even though most of what follows can be phrased purely algebraically, I have tried so stay as close in spirit as possible to the concrete, geometric style of the earlier papers cited above. In particular, we will work exclusively with analytic and algebraic varieties defined over $\mathbb{C}$.

§1 is a compendium of facts about generalized Jacobians of singular curves, Abel's theorem and its converse which will be used later. §2 and §3 are devoted to the new proof of the Lie-Wirtinger results. In §4 we present a few examples. Finally, §5 is devoted to the proof of the characterization of Jacobian varieties mentioned above.

This is an expanded version of a part of the author's Ph.D. thesis, done at Yale under the direction of Bernard Saint-Donat. I would like to thank him for introducing me to this fascinating subject.

## §1. Abel's theorem and its converse

(a) In this section we briefly review the salient facts about generalized Jacobians, Abel's theorem and its converse which will be used later. We will work over $C$.

Let $C \subset \mathbb{P}^n$ be a reduced curve (possibly singular or reducible). Let $\pi : \tilde{C} \to C$ be the normalization of $C$. $C$ has a canonical dualizing sheaf $\omega_c$, whose sections may be identified with meromorphic differentials, $\omega$, on $\tilde{C}$ satisfying

$$\sum_{P \to Q} \mathrm{Res}_p(f\omega) = 0 \tag{1.1}$$

for all $f \in \mathcal{O}_{C,Q}$ and all $Q \in C$. The notation is that of Serre [17], an excellent reference for this material, together with Rosenlicht's papers [13] and [14] which treat the case of reducible curves as well.

Let $C_{ns} = C - \operatorname{Sing} C$. As usual, we define the generalized Jacobian of $C$, denoted $J(C)$, to be the group

$$J(C) = \left\{\begin{matrix}\text{Cartier divisors on } C \text{ supported on} \\ C_{ns} \text{ of degree zero on each component}\end{matrix}\right\} \Big/ \left\{\begin{matrix}\text{divisors of meromorphic} \\ \text{functions on } C\end{matrix}\right\}$$

As in the smooth case, $J(C)$ may be realized analytically as a smooth commutative complex Lie group – an extension of an abelian variety by a linear group $(\mathbb{C}^*)^a \times \mathbb{C}^b$. This is a consequence of

THEOREM 1.1 (Abel's theorem [14]): *A divisor $D$ is equivalent to the zero divisor if and only if there is an integral 1-chain $\gamma$ on $\tilde{C} - \pi^{-1}(\operatorname{Sing} C)$ with $\partial\gamma = D$ such that $\int_\gamma \omega = 0$ for all $\omega \varepsilon H^0(\omega_C)$.*

Integrating forms $\omega$ over 1-chains induces a map

$$H_1(\tilde{C} - \pi^{-1}(\operatorname{Sing} C)) \to H^0(\omega_C)^*,$$

and the image, which we will call $\Lambda_C$, is a discrete subgroup of rank at most $2\dim H^0(\omega_C)$, the "lattice of periods" of $C$.

Choosing a non-singular base point $p_i$ on each irreducible component $C_i$ of $C$, and a basic $\omega_j$ of $H^0(\omega_C)$, we define the *Abel map*.

$$\begin{aligned} u: C_{ns} &\longrightarrow H^0(\omega_C)^*/\Lambda_C \\ p &\mapsto \left[\omega \mapsto \int_{p_i}^{p} \omega\right] \bmod \Lambda_C, \end{aligned} \tag{1.2}$$

if $p \in C_i$. $u$ extends by linearity to a map on divisors of $C$, and (via Theorem 1.1) sets up an isomorphism $J(C) \simeq H^0(\omega_C)/\Lambda_C$.

If $C$ is connected, $\dim H^0(\omega_C) = p_a(C)$, the arithmetic genus of $C$, which can be computed in terms of the genera $g_i$ of the components of $\tilde{C}$ and the local invariants $\delta$ of the singular points.

$$p_a(C) = g_1 + \ldots + g_r - r + 1 + \sum_{p \in C} \delta_p \tag{1.3}$$

From now on in this section we will also assume $C$ is a *Gorenstein* curve. That is, $\omega_C$ is locally free of rank one, so the divisors of differentials determine a divisor class on $C$, called the *canonical class*. This property may be expressed locally at the singular points: $C$ is Gorenstein if and only if $d_p = 2\delta_p$ for all $P \in C$, where $d_p = \text{length}(\tilde{\mathcal{O}}_p/C_p)$ and $C_p$ is the conductor – see Serre [17]. As in the smooth case, most of the geometry of $C$ is reflected in the properties of the "subvarieties of special divisors" of $J(C)$. Let $C = C_1 \cup \ldots \cup C_r$ be the decomposition of $C$ into irreducible components. We will say a divisor $D$ has *multidegree* $d = (d_1, \ldots, d_r)$ if $\deg D \,|\, C_i = d_i$. The *degree* of $D$ is $\sum d_i$, and we will say $D$ is *effective* if $D = \sum n_i Q_i$ with all $n_i \geq 0$.

The set of effective divisors of multidegree $d$ may be identified with $(C_1)_{ns}^{(d_1)} \times \ldots \times (C_r)_{ns}^{(d_r)}$, where $(C_i)_{ns} = C_i \cap C_{ns}$ and $x^{(n)}$ is the $n^{\text{th}}$ *symmetric* power of $x$. Fixing our nonsingular base points $p_i \in C_i$, the image under the Abel map (1.2) of this set of divisors is a subset of $J(C)$, which we will denote $W_d$.

Now we refer to §§5, 6of [9]. Because the Abel map is only defined on $C_{ns}$, $W_d$ is Zariski-closed in $J(C)$ only in certain cases. However, the noncompact group variety $J(C)$ may be compactified (embedded as a Zariski-open subset of a projective algebraic variety $\overline{J(C)}$ in such a way that the Abel map $u \circ \pi : \tilde{C} - \pi^{-1}(\text{Sing}\, C) \to J(C)$ extends to a holomorphic map $\tilde{u} : \tilde{C} \to \overline{J(C)}$. Hence the proper mapping theorem implies that the Zariski-closure of $W_d$ in $J(C)$ is an analytic subvariety of $J(C)$, and $W_d$ is a constructible, Zariski-dense subset. In fact for each $d$, $\overline{W_d}$ is an irreducible analytic subvariety of $J(C)$. For a non-functorial but elementary way to do all this, when the singularities of $C$ are planar, see [9]. The restrictive hypothesis that $C$ lie on a smooth surface is not necessary to obtain Jambois' results. All that is needed is the weaker hypothesis that $C$ be a *Gorenstein* curve. The argument of [15] extends.

(b) Now let $C \subset \mathbb{P}^n$ be any reduced curve. Let $H_0$ be a hyperplane meeting $C$ transversely in $d = \deg C$ distinct points $P_i(H_0)$. If $H$ is any hyperplane sufficiently close to $H_0$, $H$ will also meet $C$ in $d$ distinct points $P_i(H)$ and Abel's theorem implies that

$$\sum_{i=1}^{d} \int_{P_i(H_0)}^{P_i(H)} \omega = 0 \tag{1.4}$$

for all $\omega \in H^0(\omega_C)$. (We agree to perform the integrations over paths lying completely within some fixed (small) neighborhood of $H_0$, so that no periods enter.)

All the results of this paper are based on the following sort of *converse* *of Abel's theorem* ([8], [4], [3]). The idea is that addition formulas like (1.4) come only from Abelian integrals on algebraic curves.

THEOREM 1.2 ([8], p. 367): *Let $H_0$ be a fixed hyperplane in $\mathbb{P}^n$ and let $U \supset H_0$ be an open neighborhood of $H_0$ (in the classical topology). Let $\Gamma_1, \ldots, \Gamma_d$ be $d$ analytic curves in $U$, which meet $H_0$ transversely in $d$ distinct points. Suppose there are holomorphic 1-forms $\omega_i$ on $\Gamma_i$ such that for all $H$ near $H_0$:*

$$\mathrm{Tr}(\omega_i)(H) \underset{\mathrm{def}}{=} \sum_i \omega_i(H \cdot \Gamma_j) = 0, \tag{1.5}$$

*then there is a algebraic curve $\Gamma \subset \mathbb{P}^n$ of degree $d$ (possibly singular or reducible) and a differential $\omega \in H^0(\omega_\Gamma)$ such that $\Gamma_i \subset \Gamma$ and $\omega | \Gamma_i = \omega_i$.*

For our purposes it will be more convenient to use an "integrated" form of this statement, namely

COROLLARY 1.3: *Let $H_0$, $U$, $\Gamma_i$ be as in the theorem. Assume there exist "parameters" $T_i: \Gamma_i \to C$ (that is, invertible functions $T_i$ such that $T_i^{-1}$ parametrizes $\Gamma_i$ for each $i$) such that for all hyperplanes $H$ near $H_0$,*

$$\sum_{i=1}^{d} T_i(H \cdot \Gamma_i) = 0 \tag{1.6}$$

*then $\Gamma_i \subset \Gamma$ for some algebraic curve $\Gamma$ of degree $d$ and $dT_i$ are the local expansions of some $\omega \in H^0(\omega_\Gamma)$.*

PROOF: Use affine coordinates on the dual projective space $(\mathbb{P}^n)^*$ near $H_0$, take the total differential of (1.6) with respect to these coordinates, and apply the theorem. Q.E.D.

REMARKS: (1) The first proof of Corollary 1.3 (for curves in $\mathbb{P}^2$) appeared in Darboux, [4], and this was reproduced in Blaschke–Bol [3]. Griffiths [8] reinterpreted this proof as an application of Poincaré residues and generalized the theorem to the case of intersections of varieties of higher dimension with linear subspaces of $\mathbb{P}^n$.

It would be very interesting to know if, after ruling out some degenerate cases, corollary 1.3 (suitably reformulated) remains true in positive characteristic.

(2) The author has obtained such a result in arbitrary characteristic in the case $n = 2$, $d = 3$ by a method different from that of Griffiths. Details will appear elsewhere.

## §2. Translation manifolds

Let $(S, 0)$ be a non-singular germ of analytic hypersurface in $\mathbb{C}^n$, contained in no linear subspace of $\mathbb{C}^n$.

DEFINITION 2.1: $S$ is a *translation manifold* if there exist $n-1$ holomorphic curves $\alpha_i : \Delta \to \mathbb{C}^n$ ($\Delta$ is the open unit disc in $\mathbb{C}$) with $\alpha_i(0) = 0 \in \mathbb{C}^n$ for all $i$, such that every point $x \in S$ can be written uniquely in the form

$$x = \sum_{i=1}^{n-1} \alpha_i(t_i) \text{ (vector addition in } \mathbb{C}^n) \tag{2.1}$$

where $(t_1, \ldots, t_{n-1}) = (t_1(x), \ldots, t_{n-1}(x)) \in \Delta^{n-1}$. The $\alpha_i$ are called the generating curves of $S$.

Geometrically, this means that the hypersurface $S$ is swept out as the curves $\alpha_i$ are translated rigidly along each other, hence the name "translation manifold". If the component functions of the $\alpha_i$ are

$$\alpha_i(t_i) = (\alpha_{i1}(t_i), \ldots, \alpha_{in}(t_i)),$$

then (2.1) is equivalent to the following parametrization. Let $X_1, \ldots, X_n$ be coordinates in $\mathbb{C}^n$, then $S$ is the locus

$$X_j = \sum_{i=1}^{n-1} \alpha_{ij}(t_i) \qquad (j = 1, \ldots, n). \tag{2.2}$$

REMARKS: More general translation manifolds are obtained if the hypothesis that $\alpha_{ij}$ be holomorphic functions of $t_i$ is dropped, but they will not be studied here.

We will be interested primarily in hypersurfaces $S$ which admit two parametrizations such as (2.2) which are distinct (in a sense to be made precise). If $S$ is given as a translation manifold in two ways:

$$X_j = \sum_{i=1}^{n-1} \alpha_{ij}(t_i) = \sum_{i=1}^{n-1} \beta_{ij}(u_i)$$

(where the $\alpha_i$ and the $\beta_i$ are the two sets of generating curves) we will say $S$ is doubly of translation type.

DEFINITION 2.2: (1) the two parametrizations in (2.3) are *distinct* if the vectors $\dot{\alpha}_i(0) = (\alpha'_{i1}(0), \ldots, \alpha'_{in}(0))$ and $\dot{\beta}_i(0) = (\beta'_{i1}(0), \ldots, \beta'_{in}(0))$ are pairwise linearly independent. (That is, the tangent lines to the $\alpha_i$ and $\beta_i$ at 0 are distinct.)

(2) $S$ will be called a *nondegenerate* doubly translation type hypersurface if none of the vectors $\ddot{\alpha}_i(0) = (\alpha''_{i1}(0), \ldots, \alpha''_{in}(0))$, $\ddot{\beta}_i(0) = (\beta''_{i1}(0), \ldots, \beta''_{in}(0))$ is tangent to $S$ at 0.

## §3. A new proof of Lie and Wirtinger's results

Let $(S, 0) \subset (\mathbb{C}^n, 0)$ be a non-degenerate doubly translation type hypersurface as in (2.3). The fundamental construction we will use is Lie's original one. Namely we will pass from the generating curves $\alpha_i$ and $\beta_i$ themselves to the sets of tangent lines to these curves, which we will call $\dot{\alpha}_i$ and $\dot{\beta}_i$.

Since $T_Y(\mathbb{C}^n)$ is canonically identified with $\mathbb{C}^n$ for all $Y \in \mathbb{C}^n$, we can view the tangent lines to the $\alpha_i$ and $\beta_i$ as lying in a single ambient space $\mathbb{C}^n$. Via the standard identification

$$\mathbb{P}^{n-1} = \{\text{lines through 0 in } \mathbb{C}^n\},$$

the tangents to $\alpha_i$ and $\beta_i$ sweep out arcs in $\mathbb{P}^{n-1}$, denoted by $\dot{\alpha}_i$ and $\dot{\beta}_i$ as above. In parametric form, the $\dot{\alpha}_i$ and $\dot{\beta}_i$ are given (in homogeneous coordinates) by

$$\dot{\alpha}_i(t_i) = (\alpha'_{i1}(t_i), \ldots, \alpha'_{in}(t_i))$$

$$\dot{\beta}_i(u_i) = (\beta'_{i1}(u_i), \ldots, \beta'_{in}(u_i)).$$

In terms of the tangents, the two conditions of Definition 2.2 become:

(1′) Since the tangent lines to the $\alpha_i$ and $\beta_i$ at 0 all lie in the tangent hyperplane to $S$ at 0, which corresponds to a hyperplane $H_0 \subset \mathbb{P}^{n-1}$, this condition says $\dot{\alpha}_i(0)$ and $\dot{\beta}_i(0)$ are $2n-2$ distinct points of $H_0$.

(2′) Condition (2) implies that $\dot{\alpha}_i$ and $\dot{\beta}_i$ cross $H_0$ transversely.

LEMMA 3.1: *By restricting the domains of $t_i$ and $u_i$ if necessary, we may assume that the Gauss maps $\alpha_i(t_i) \to \dot{\alpha}_i(t_i)$ and $\beta_i(u_i) \to \dot{\beta}_i(u_i)$ are injective for all $i$.*

PROOF: Condition (2) in Definition 2.2 implies that all the $\ddot{\alpha}_i(0)$ and $\ddot{\beta}_i(0)$ are non-zero. The lemma then follows, using the inverse function theorem. Q.E.D.

REMARKS: Lemma 3.1 says in particular that none of the generating curves of a nondegenerate doubly translation type manifold can be a line. This rules out "cylindrical" translation manifolds.

As usual, we will call the map $g: S \to (\mathbb{P}^{n-1})^*$ which associates to each $p \in S$ the tangent hyperplane $T_p(s)$ the projectivized *Gauss map* on $S$.

LEMMA 3.2: *By restricting $t_i$ and $u_i$ again, if necessary, we may assume $g$ is injective.*

PROOF: For this, all we need is that for one of the sets of generating curves, say the $\alpha_i$, none of the vectors $\ddot{\alpha}_i(0)$ is tangent to $S$ at zero.

Note that, without loss of generality, we may assume that $T_0(S)$ is the hyperplane $X_n = 0$, and that for all $p \in S$, $T_p(S)$ has the form

$$X_n = \sum_{j=1}^{n-1} C_j(t_1, \ldots, t_{n-1}) X_j \tag{3.1}$$

where $C_j$ are functions satisfying $C_j(0, \ldots, 0) = 0$. The $C_j(t_1, \ldots, t_{n-1})$ are the affine coordinates of the point $g(P) \in (\mathbb{P}^{n-1})^*$ where

$$P = \sum_{i=1}^{n-1} \alpha_i(t_i).$$

To prove the lemma, it suffices to show that the differential of $g$ at 0 has maximal rank

$$\left[\text{or in terms of matrices, } \det\left(\left.\frac{\partial C_i}{\partial t_j}\right|_{t_1 = \ldots = t_{n-1} = 0}\right) \neq 0\right].$$

The lemma will then follow by the inverse function theorem.
Now, if $P = \sum \alpha_i(t_i)$, then $\dot{\alpha}_i(t_i) \in T_p(S)$ so from (3.1)

$$\alpha'_{in}(t_i) = \sum_{j=1}^{n-1} C_j(t_1, \ldots, t_{n-1}) \alpha'_{ij}(t_i). \tag{3.2}$$

Differentiating (3.2) with respect to $t_k$, we have

$$\left.\begin{aligned} 0 &= \sum_{j=1}^{n-1} \frac{\partial C_j}{\partial t_k} \alpha'_{ij}(t_i), \text{ if } k \neq i \\ \alpha''_{in}(t_i) &= \sum_{j=1}^{n-1} \left[\frac{\partial C_j}{\partial t_i} \alpha'_{ij}(t_i) + C_j \alpha''_{ij}(t_i)\right], \text{ if } k = i. \end{aligned}\right\} \tag{3.3}$$

These equations can be rewritten in matrix form:

$$\begin{pmatrix} \dfrac{\partial C_1}{\partial t_1} & \cdots & \dfrac{\partial C_{n-1}}{\partial t_1} \\ \vdots & & \vdots \\ \dfrac{\partial C_1}{\partial t_{n-1}} & \cdots & \dfrac{\partial C_{n-1}}{\partial t_{n-1}} \end{pmatrix} \begin{pmatrix} \alpha'_{11} \dots \alpha'_{n-1,1} \\ \vdots \quad \vdots \\ \alpha'_{1,n-1} \dots \alpha'_{n-1,n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha''_{in} - \sum_j C_j \alpha''_{ij} \, 0 \dots 0 & & \\ 0 & \ddots & \vdots \\ 0 \dots \alpha''_{n-1,n} - \sum_j C_j \alpha''_{n-1,j} & & \end{pmatrix} \tag{3.4}$$

Evaluating at $t_l = 0$, all $l$, we see that $\det(\partial C_i/\partial t_j) = 0$ if and only if $\alpha''_{kn}(0) - \sum_j C_j(0,\dots,0)\alpha''_{kj} = 0$ for some $k$, that is $\ddot{\alpha}_k(0)$ is tangent to $S$ at $0 \in C^n$. This would contradict the hypothesis that $S$ is nondegenerate. Q.E.D.

The basis of all that follows is this proposition. It is here that this proof differs from the one given by Wirtinger. This is also proved in Mumford, [10].

PROPOSITION 3.3: *Let $S$, $\alpha_i$, $\beta_i$, $H_0$ be as above. Then the $\dot{\alpha}_i$ and $\dot{\beta}_i$ lie on an algebraic curve of degree $2n-2$ in $\mathbb{P}^{n-1}$, possibly singular or reducible.*

PROOF: The starting point is the parametrization (2.3) for $S$. Let $z \in S$ be any point. For this $z$ we have (unique) $t_i(z)$ and $u_i(z)$ in the unit disc such that

$$z = \sum_{i=1}^{n-1} \alpha_i(t_i(z)) = \sum_{i=1}^{n-1} \beta_i(u_i(z)). \tag{3.5}$$

The tangent lines to $\alpha_i$ and $\beta_i$ at the points $\alpha_i(t_i(z))$ and $\beta_i(u_i(z))$ all lie in $T_z(S)$, hence in $\mathbb{P}^{n-1}$, the corresponding points on $\dot{\alpha}_i$ and $\dot{\beta}_i$ lie on the image of $T_z(S)$, a hyperplane we will call $H_z$. In this way, each $z \in S$ defines a hyperplane $H_z \subset \mathbb{P}^{n-1}$ meeting the $\dot{\alpha}_i$, $\dot{\beta}_i$ in $2n-2$ (distinct) points.

Conversely, any hyperplane $H \subset \mathbb{P}^{n-1}$ which is sufficiently close to $H_0$ (in the obvious sense) will meet the holomorphic curves $\dot{\alpha}_i$ and $\dot{\beta}_i$ in

$2n-2$ points which correspond via Lemma 3.1 to unique points on $\alpha_i$ and $\beta_i$. By definition, the points on the $\alpha_i$ sum to a point $z_1 \in S$, while those on the $\beta_i$ sum to a point $z_2 \in S$. However, the points $H \cap \alpha_i$ and $H \cap \beta_i$ span only a hyperplane in $\mathbb{P}^{n-1}$, so we must have $T_{z_1}(S) = T_{z_2}(S)$. By Lemma 3.2 then, $z_1 = z_2$. In sum, after restricting the $t_i$ and $u_i$ as necessary, there is a one-to-one correspondence

$$\{\text{points of } S\} \leftrightarrow \left\{\begin{array}{l}\text{sets of } 2n-2 \text{ points} \\ \text{one from each } \mathring{\alpha}_i, \mathring{\beta}_i \\ \text{spanning a hyperplane}\end{array}\right\},$$

defined via $z \mapsto \{H_z \cap \mathring{\alpha}_i, H_z \cap \mathring{\beta}_i\}$.

Now, consider linear maps $\Phi: \mathbb{C}^n \to \mathbb{C}$ with the property that $\Phi$, restricted to each of the tangent lines $\mathring{\alpha}_i(0)$ and $\mathring{\beta}_i(0)$ is not identically zero. Since the set of such $\Phi$ is the complement of a finite set of hyperplanes in $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$, it is non-empty (in fact, it contains a linear basis of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ – this remark will be used later).

Using Lemma 3.1, it is easy to see that for any such $\Phi$, $\Phi \circ \alpha_i$ and $\Phi \circ \beta_i$ define *parameters* (in the sense of Corollary 1.3) on the $\mathring{\alpha}_i$, $\mathring{\beta}_i$ respectively.

For a more symmetrical statement, we let $\mathring{\alpha} = \Gamma_i$ and $\mathring{\beta}_i = \Gamma_{n-1+i}$. Further, let $\Phi \circ \alpha_i = T_i$ and $-\Phi \circ \beta_i = T_{n-1+i}$. Equation (3.5) and the above remarks imply that for any $H$ near $H_0$ in $\mathbb{P}^{n-1}$, $H = H_z$ for some $z \in S$, and we have

$$\begin{aligned}\sum_{j=1}^{2n-2} T_j(\Gamma_j \cap H_z) &= \sum_{i=1}^{n-1} (\Phi \circ \alpha_i)(t_i(z)) - \sum_{i=1}^{n-1} (\Phi \circ \beta_i)(u_i(z)) \\ &= \Phi\left(\sum_{i=1}^{n-1} \alpha_i(t_i(z))\right) - \Phi\left(\sum_{i=1}^{n-1} \beta_i(u_i(z))\right) \\ &= \Phi(z) - \Phi(z) \\ &= 0.\end{aligned}$$

By Corollary 1.3 (applied to the curves $\Gamma_j$ with parameters $T_j$), the $\Gamma_j$ lie on an algebraic curve $C$ of degree $2n-2$ in $\mathbb{P}^{n-1}$, and furthermore, for each $\Phi$, $d(\Phi \circ \alpha_i)$ and $-d(\Phi \circ \beta_i)$ are the local expansions of some differential $\omega \in \mathrm{H}^0(\omega_c)$. Q.E.D.

The curves which arise this way are very special. First, note that any such $C$ spans $\mathbb{P}^{n-1}$ (otherwise we would have a contradiction to the conclusion of Lemma 3.2). Second, the set of $\Phi$ for which we can apply Corollary 1.3 is open in $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$. These two remarks combined yield.

Corollary 3.4: *Let $S$ be any doubly translation type hypersurface as in the proposition, and let $C \subset \mathbb{P}^{n-1}$ be the corresponding algebraic curve of degree $2n-2$. Then*

(a) $\dim H^0(\omega_c) \geq n$, *and*

(b) *the hyperplanes in $\mathbb{P}^{n-1}$ cut special divisors on $C$.*

Proof: (a) Without loss of generality, we can choose coordinates $x_1, \ldots, x_n$ in $\mathbb{C}^n$ such that none of the $\Phi_j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ defined by $\Phi_j(x_1, \ldots, x_n) = x_j$ vanish on any of the $\dot{\alpha}_i(0)$ or $\dot{\beta}_i(0)$. Then, as in the proof of the proposition, for each $j$, $d(\Phi_j \circ \alpha_i)$ and $-d(\Phi_j \circ \beta_i)$ are the local expansions of some $\omega_j \in H^0(\omega_c)$ on $\alpha_i$ and $\beta_i$ respectively.

Suppose the $\omega_j$ are linearly dependent in $H^0(\omega_c)$:

$$\sum_{j=1}^{n} a_j \omega_j \equiv 0$$

for some $a_j \in C$. Then by the definition of $\omega_j$ (since $\alpha_i(0) = 0$ and $\beta_i(0) = 0$) all the $\alpha_i$ and $\beta_i$ lie in the hyperplane $\sum a_j x_j = 0$ in $\mathbb{C}^n$. This is a contradiction by the remark following the proposition. Hence $\dim H^0(\omega_c) \geq n$.

(b) By part (a) the hyperplane $\sum_{j=1}^{n} a_j x_j = 0$ in $\mathbb{P}^{n-1}$ cuts $C$ in the divisor of zeros of $\sum_{j=1}^{n} a_j \omega_j \in H^0(\omega_c)$ (or a part of that divisor.) Q.E.D.

Our second corollary of Proposition 3.3 is the following explicit form of the parametrization of $S$ in (3.5).

Corollary 3.5: (*the theorem of Lie and Wirtinger*). *Let $\omega_j$ be the differentials chosen in Corollary* 3.4. *Then $S$ is the locus*

$$x_j = \sum_{i=1}^{n-1} \int_{P_i(H_0)}^{P_i(H)} \omega_j = -\sum_{i=n}^{2n-2} \int_{P_i(H_0)}^{P_i(H)} \omega_j, \tag{3.7}$$

*where $P_i(H) = \Gamma_i \cap H$ and $H$ ranges over all hyperplanes near $H_0$ in $\mathbb{P}^{n-1}$.*

Proof: This is immediate from the proposition and corollary 3.4. Q.E.D.

This is the exact analog of the parametrization of doubly translation type surfaces described in §0. The generating curves are parametrized by the Abelian integrals on $C$, and any double parametrization as in (3.5) is a direct consequence of Abel's theorem on some algebraic curve $C$ (in the form (1.4)).

By reversing the above construction, Corollary 3.5 gives, in effect, a recipe for generating all possible hypersurfaces doubly of translation type (this was Lie's original goal). Hence every nondegenerate doubly translation type hypersurface $S \subset \mathbb{C}^n$ may be constructed as follows:

1. Let $C$ be a reduced algebraic curve of degree $2n-2$ spanning $\mathbb{P}^{n-1}$ with $\dim H^0(\omega_C) \geq n$, whose hyperplane sections are special divisors (N.B. this last condition is not automatic for reducible curves C!).
2. Let $H_0$ be any hyperplane meeting $C$ transversely.
3. Fix some partition of the branches of $C$ centered along $H_0$ into two complementary sets of $n-1$ each $\{\Gamma_1, \ldots, \Gamma_{n-1}\}$ and $\{\Gamma_n, \ldots, \Gamma_{2n-2}\}$ in such a way that both sets of points $\{\Gamma_1 \cap H_0, \ldots, \Gamma_{n-1} \cap H_0\}$ and $\{\Gamma_n \cap H_0, \ldots, \Gamma_{2n-2} \cap H_0\}$ span $H_0$.
4. Let $\omega_1, \ldots, \omega_n$ be differentials in $H^0(\omega_C)$ corresponding to the coordinate hyperplanes in $\mathbb{P}^{n-1}$.
5. Construct the locus $S$ as in (3.7).

In principle, this reduces the problem of finding all doubly translation type manifolds $S$ to the algebraic problem of classifying all curves $C$ satisfying all the conditions above. A complete listing of the possibilities (even when $n = 3$) is difficult, though, because of the large number of different types of singular and reducible curves which can occur, not to mention the fact that one (reducible) $C$ can give rise to several different manifolds $S$ (see example 4.4).

We will be primarily interested in the manifolds $S$ which are parametrized by Abelian integrals on curves $C$ with $\dim H^0(\omega_C) = n$. There are several good reasons to restrict our attention in this way. First, the maximum arithmetic genus of an irreducible curve of degree $2n-2$ in $\mathbb{P}^{n-1}$ is $n$ and since irreducible curves are connected, $\dim H^0(\omega_C) = p_a(C) = n$ ([R1], corollary to theorem 16.). Thus, if $\dim H^0(\omega_C) > n$, $C$ is reducible (and furthermore, $C$ cannot appear as a singular fiber in a family of smooth canonical curves of genus $n$).

Second, in case $\dim H^0(\omega_C) = n + k (k > 0)$ we can complete $\omega_1, \ldots, \omega_n$ (as above) to a basis $\omega_1, \ldots, \omega_{n+k}$ of $H^0(\omega_C)$. The addition formula (1.4) holds for *all* the $\omega_i$, so we can construct a new doubly translation type manifold $S^1 \subset \mathbb{C}^{n+k}$ by integrating all the $\omega_i$ and summing as in (3.7). Our original $S$ thus appears as the projection of $S^1$ into the $\mathbb{C}^n$ defined by $X_{n+1} = \ldots = X_{n+k} = 0$.

EXAMPLE 3.6: The simplest example of this type may be obtained as follows (see [18], p. 429–431). In $\mathbb{P}^{2m} (m \geq 2)$, let $L_i \simeq \mathbb{P}^m$ be two linear subspaces in general position. Let $C_i \subset L_i$ be smooth canonical curves of genus $m+1$, meeting transversely at the point $L_1 \cap L_2$. Then $C$

$= C_1 \cup C_2$ is a (stable) curve of genus $2m + 2$ and degree $4m$ in $\mathbb{P}^{2m}$. It is clear that $\dim H^0(\omega_C) = 2m + 2$ as well.

Let $H_0$ be any hyperplane meeting $C$ transversely, and let $\Gamma_1, \ldots, \Gamma_m$ be some $m$ branches of $C_1$ centered along $H_0$. Similarly let $\Gamma_{m+1}, \ldots, \Gamma_{2m}$ be some $m$ branches of $C_2$ centered along $H_0$. Let $\Gamma_{2m+1}, \ldots, \Gamma_{3m}$ be the remaining branches of $C_1$ and $\Gamma_{3m+1}, \ldots, \Gamma_{4m}$ the remaining branches of $C_2$.

For a general $H_0$, $\{\Gamma_1, \ldots, \Gamma_{2m}\}$ and $\{\Gamma_{2m+1}, \ldots, \Gamma_{4m}\}$ will be complementary sets of $2m$ branches, each with the property that the points $\{\Gamma_i \cap H_0\} = \{P_i(H_0)\}$ ($i = 1, \ldots, 2m$ and $i = 2m + 1, \ldots, 4m$) span $H_0$.

Choose any basis $\{\omega_1, \ldots, \omega_{2m+2}\}$ for $H^0(\omega_C)$ and let $S' \subset \mathbb{C}^{2m+2}$ be the locus

$$X_j = \sum_{i=1}^{2m} \int_{P_i(H_0)}^{P_i(H)} \omega_j = - \sum_{i=2m+1}^{4m} \int_{P_i(H_0)}^{P_i(H)} \omega_j. \tag{3.8}$$

Projecting $S'$ into $\mathbb{C}^{2m+1}$ yields a doubly translation type hypersurface $S$. It is easy to generalize this example by allowing the $C_i$ to have different genera, allowing more than two components for $C$, allowing the $C_i$ themselves to degenerate and so on.

Remarks: In intrinsic terms, the $S'$ constructed in this example is nothing other than a piece of *product* of the theta-divisors of the $C_i$: $\Theta_1 \times \Theta_2 \subset J(C_1) \times J(C_2)$.

One important feature of this example is that (in general) if a hyperplane $H$ passes through $m$ specified points on $\Gamma_1, \ldots, \Gamma_m$, the points $H \cap \Gamma_{2m+1}, \ldots, H \cap \Gamma_{3m}$ on $C_1$ are uniquely determined, while the points $H \cap \Gamma_i$ for $i = m + 1, \ldots, 2m$ and $i = 3m + 1, \ldots, 4m$ (the points on $C_2$) are *completely arbitrary*. This says that in the parametrization of $S'$ (and $S$) some of the $u_i$ in the second parametrization vary independently of some of the $t_i$ in the first.

Somewhat surprisingly, if we exclude this kind of behavior, then $\dim H^0(\omega_C) = n$. The following proposition thus gives a criterion for determining whether a given $S$ is parametrized by the Abelian integrals on a curve $C$ with $\dim H^0(\omega_C) = n$.

Proposition 3.7 ([18], p. 395–397): *Let $S$, $\alpha_i(t_i)$, $\beta_i(u_i)$ and $C$ be defined as above. Assume further that*

(*) *For at least one $t_i$, say $t_1$, if we fix $t_j$ ($j = 2, \ldots, n - 1$) and vary $t_1$ to obtain $z(t_1) \in S$, then*

$$z(t_1) = \sum_{i=1}^{n-1} \beta_i(u_i(t_1))$$

*where* $\dfrac{d}{dt_1}u_i(t_1) \not\equiv 0$ *for all* $i$ *(that is, none of the* $u_i$ *is constant along* $z(t_1) \subset S$*). Then,* $\dim H^0(\omega_C) = n$ *and* $S$ *is not the projection of a doubly translation type manifold from a space of higher dimension.*

REMARKS: (1) Wirtinger studies the slightly more general problem of finding all doubly translation type manifolds $S$ of dimension $n-1$ in $\mathbb{C}^p$, $p \geq n$ and shows that under assumption (*), any such $S$ lies automatically in a $\mathbb{C}^n \subset \mathbb{C}^p$ that is, there is linear dependence among the component functions of the generating curves if $p > n$. $\dim H^0(\omega_C) = n$ follows immediately.

The proof is entirely elementary, but computational, so we do not reproduce it here.

(2) Note that (*) is always satisfied when $n = 3$.

When (*) is satisfied on $S$, the corresponding $C$ have almost all of the beautiful properties of smooth canonical curves.

COROLLARY 3.8: *Let* $S$ *satisfy the condition* (*) *of the proposition, then* $C$ *is a connected, canonically embedded ("non-hyperelliptic") curve of arithmetic genus* $n$ *in* $\mathbb{P}^{n-1}$ *(that is* $\mathcal{O}_C(1) \simeq \omega_C$*).*

PROOF: This follows by the Riemann-Roch formula for singular curves ([1] ch. VII). Let $L = \mathcal{O}_C(1)$. We have

$$\chi(L) = \deg L + 1 - p_a(C)$$

$\deg L = 2n - 2$ and $p_a(C) = \dim H^0(\omega_C) - b + 1$ where $b$ is the number of connected components of $C$. Hence,

$$\chi(L) = n + b - 2.$$

By duality $h^1(L) = \dim \mathrm{Hom}_{\mathcal{O}_C}(L, \omega_C) \geq 1$ since the hyperplane sections of $C$ are special (Corollary 3.4). Since $h^0(L) = n$, we deduce that $b + h^1(L) = 2$. Hence, $b = h^1(L) = 1$ and the corollary follows. Q.E.D.

In particular these curves $C$ are Gorenstein curves, although they can easily fail to be local complete intersections. For instance, see Example 4.3 below.

We conclude this section with a more intrinsic characterization of the $S$ satisfying condition (*). We use the notation and terminology of §1.

THEOREM 3.9: *Let* $(S, 0)$ *be a non-degenerate doubly translation type hypersurface in* $\mathbb{C}^n$ *satisfying the condition* (*) *of Proposition* 3.7, *and let* $C$

*be the associated canonically embedded curve in* $\mathbb{P}^{n-1}$. *Then there is a multidegree d and an effective divisor D of degree* $n-1$ *and multidegree d, with* $\dim H^0(\omega_C \otimes \mathcal{O}(-D)) = 1$, *such that* $(S_1, 0)$ *is a translate of the lifting to* $\mathbb{C}^n$ *of the germ* $(W_d, u(D))$ *from* $J(C)$.

PROOF: Recall that any double parametrization as in (3.5) determines not only the curve $C \subset \mathbb{P}^{n-1}$, but also a partition of the branches of $C$ along $H_0$ into two complementary subsets of $n-1$ each the $\{\dot{\alpha}_i\}$ and the $\{\dot{\beta}_i\}$. Comparing the first parametrization in (3.7) with the definition of the $W_d$ via the Abel map in §1, we see that $S$ is a translate of the lifting to $\mathbb{C}^n$ of $(W_d, u(D))$ where

$$D = \sum_{i=1}^{n-1} P_i(H_0)$$

(the points $\dot{\alpha}_i \cap H_0$) and $d$ is determined by the distribution of the points of $D$ among the components of $C$. $\dim H^0(\omega_C \otimes \mathcal{O}(-D)) = 1$ follows since $S$ is nondegenerate.

In intrinsic terms, the second parametrization of $S$ comes from the "reciprocity law" $W_d = k - W_{d_1}$ in $J(C)$, where $k$ is the "canonical point," $u((\omega))$ for $\omega \in H^0(\omega_C)$, and $d_1$ is the multidegree $\deg(\omega) - d$. Q.E.D.

## §4. Some examples

In this section, we give some examples to illustrate the results of §3. Our major tool will be theorem 3.9.

EXAMPLE 4.1: When $C \subset \mathbb{P}^{n-1}$ is a smooth canonical curve of genus $n$ and $H_0$ is a hyperplane meeting $C$ transversely, we can divide the $2n-2$ points $H_0 \cap C$ into two complementary sets $\{P_1, \ldots, P_{n-1}\}$ and $\{P_n, \ldots, P_{2n-2}\}$. Applying theorem 3.9, we see that the translation manifold corresponding to this subdivision of the points $H_0 \cap C$ is a translate of the germ of $W_{n-1}(C)$ at the point $u(D)$, where $D = \sum_{i=1}^{n-1} P_i$. In this case, since $C$ is irreducible, different subdivisions of $H_0 \cap C$ simply yield the germ of $W_{n-1}(C)$ at different points.

Extending the Abel map $u$ over all paths in $C$ we obtain the full divisor $\rho^{-1}(W_{n-1})$, where $\rho: \mathbb{C}^n \to J(C) \simeq \mathbb{C}^n/\Lambda_C$ is the natural projection. By Riemann's theorem, $\rho^{-1}(W_{n-1})$ is a translate of the divisor of zeroes of the Riemann theta function of $C$.

REMARKS: In this example $C$ is (necessarily) nonhyperelliptic, but it is easy to describe what happens for hyperelliptic curves as well. If $\Gamma$ is hyperelliptic $W_{n-1}(\Gamma)$ on $J(\Gamma)$ is constructed via the Abel map as before, and by definition the germ of $W_{n-1}(\Gamma)$ at any smooth point is a translation manifold. $W_{n-1}(\Gamma)$ is still symmetric about the canonical point $u(k_\Gamma)$ (that is $W_{n-1}(\Gamma) = u(k_\Gamma) - W_{n-1}(\Gamma)$), but in this case $k_\Gamma = (n-1)g_2^1$ and $W_1(\Gamma)$ is symmetric about the hyperelliptic point $u(g_2^1)$. The two parametrizations of $W_{n-1}(\Gamma)$ actually *coincide* (neither condition of Definition 2.2 is satisfied). If we apply the construction of proposition 3.3, we obtain the non-reduced canonical image in $\mathbb{P}^{n-1}$.

EXAMPLE 4.2: If $C$ is an irreducible curve with at most ordinary double points (nodes) as singularities, then, as is well known, the picture is very similar to that for smooth curves. As in §1, let $\pi: C \to C$ be the normalization. Let $g = g(C)$ and let $\delta$ be the number of nodes on $C$, the ith one obtained by identifying a pair $(a_i, b_i)$ of distinct points on $\tilde{C}$. We have $p_a(C) = n = g + \delta$.

$H^0(\omega_C)$ is spanned by the holomorphic differentials in $H^0(\omega_{\tilde{C}})$ and additional normalized differentials of the third kind $\omega_{b_i - a_i}$ which have simple poles at $b_i$ and $a_i$ with residues $+1$ and $-1$ respectively.

In this case $J(C) \simeq \mathbb{C}^n/\Lambda_C$, where $\Lambda_C \subset \mathbb{C}^n$ has rank $2g + \delta$; $J(C)$ is an extension of $J(\tilde{C})$ by a $\delta$-dimensional torus $(\mathbb{C}^*)^\delta$. By theorem 3.9, the doubly translation type manifold $S$ is the germ of $W_{n-1}(C)$ at some point $u(D)$, lifted to $\mathbb{C}^n$ and translated. In this case $\overline{W_{n-1}(C)}$ is still an irreducible divisor in $\mathbb{C}^n$, but it is periodic *only* with respect to translations by $x \in \Lambda_C$ (this will be proved in §5).

Once again there is an entire holomorphic function on $\mathbb{C}^n$ whose divisor of zeroes is a translate of $\overline{W_{n-1}(C)}$. For example, if $\delta = 2$, writing $z \in \mathbb{C}^{g+2}$ as $z = (\tilde{z}, z_{g+1}, z_{g+2})$ where $\tilde{z} \in \mathbb{C}^g$, we have that $W_{n-1}(C)$ is a translate of the divisor of zeroes of

$$\begin{aligned}\theta(\tilde{z}, z_{g+1}, z_{g+2}) = {} & \tilde{\theta}(\tilde{z} - \tfrac{1}{2}\tilde{u}(b_1 - a_1) - \tfrac{1}{2}\tilde{u}(b_2 - a_2)) \\ & + \tilde{\theta}(\tilde{z} + \tfrac{1}{2}\tilde{u}(b_1 - a_1) - \tfrac{1}{2}\tilde{u}(b_2 - a_2))e^{2\pi i(z_{g+1} - \int_{a_2}^{b_2} \omega b_1 - a_1)} \\ & + \tilde{\theta}(\tilde{z} - \tfrac{1}{2}\tilde{u}(b_1 - a_1) + \tfrac{1}{2}\tilde{u}(b_2 - a_2))e^{2\pi i(z_{g+2} - \int_{a_1}^{b_1} \omega b_2 - a_2)} \\ & + \tilde{\theta}(\tilde{z} + \tfrac{1}{2}\tilde{u}(b_1 - a_1) + \tfrac{1}{2}\tilde{u}(b_2 - a_2))e^{2\pi i(z_{g+2} + z_{g+1})}.\end{aligned}$$

Here $\tilde{u}: \tilde{C} \to \mathbb{C}^g$ is the Abel map on $\tilde{C}$, $\tilde{\theta}$ is the $g$ variable Riemann theta function of $\tilde{C}$, and $\omega_{b_i - a_i}$ are the normalized differentials of the third kind as above.

This $\theta$ may be viewed as a degenerate "limit" of ordinary theta functions, and the expression above may be derived in this way. See for instance [7], p. 55.

EXAMPLE 4.3: Very roughly speaking, the more singular $C$ is, the less transcendental $S$ is. At the far end of the scale from the smooth canonical curves in $\mathbb{P}^{n-1}$, consider the monomial curve $C$ parametrized by $\phi: \mathbb{P}^1 \to \mathbb{P}^{n-1}$, $\phi(s,t) = (s^{2n-2}, s^{n-2}t^n, s^{n-3}t^{n+1}, \ldots, t^{2n-2})$. It is easily checked that $C$ is a Gorenstein canonically-embedded curve of arithmetic genus $p_a(C) = n$ with a single singular point $p = (1, 0, \ldots, 0)$ for which $\delta_p = $ n. In fact $H^0(\omega_C)$ is spanned by the differentials

$$\frac{dt}{t^2}, \frac{dt}{t^3}, \ldots, \frac{dt}{t^n}, \frac{dt}{t^{2n}}$$

on $\tilde{C} \simeq \mathbb{P}^1$. $W_{n-1}(C)$ is parametrized as follows. We omit the limits of integration and write $t_i$ $i = 1, \ldots, n-1$ for the parameters on the chosen $n-1$ branches of $C$ through $H_0$:

$$x_1 = \int \frac{dt_1}{t_1^2} + \ldots + \int \frac{dt_{n-1}}{t_{n-1}^2} = \frac{-1}{t_1} + \ldots + \frac{-1}{t_{n-1}}$$

$$x_2 = \int \frac{dt_1}{t_1^3} + \ldots + \int \frac{dt_{n-1}}{t_{n-1}^3} = \frac{-1}{2t_1^2} + \ldots + \frac{-1}{2t_{n-1}^2}$$

$$x_n = \int \frac{dt_1}{t_1^{2n}} + \ldots + \int \frac{dt_{n-1}}{t_{n-1}^{2n}} = \frac{-1}{(2n-1)t_1^{2n-1}} + \ldots + \frac{-1}{(2n-1)t_{n-1}^{2n-1}}.$$

The $t_i$ can be eliminated between these expressions to yield a *polynomial* equation for $\overline{W_{n-1}(C)}$. For example, when $n = 3$, setting $x = -x_1$, $y = -2x_2$, $z = -3x_3$ we have

$$t_1^5 t_2^5 x = t_1^5 + t_2^5$$

$$t_1^2 t_2^2 y = t_1^2 + t_2^2$$

$$t_1 t_2 z = t_1 + t_2,$$

and a calculation shows that $4x + z^5 - 5zy^2 = 0$. Since this algebraic surface in $\mathbb{C}^3 \simeq J(C)$ is clearly irreducible, $W_2(C)$ is cut out by this equation. In the same vein the papers [5] and [6] contain many examples of other doubly translation type surfaces in $\mathbb{C}^3$ which are most instructive.

EXAMPLE 4.4: Finally, let $C \subset \mathbb{P}^2$ be a reducible quartic $C = C_1 \cup C_2$, where the $C_i$ are smooth conics meeting transversely. Let $f(x, y) = p_1(x, y)p_2(x, y) = 0$ be the affine equation of $C$. Then $H^0(\omega_C)$ is spanned by the differentials

$$\frac{x\,dx}{\partial f/\partial y}, \quad \frac{y\,dx}{\partial f/\partial y}, \quad \frac{dx}{\partial f/\partial y}.$$

Let $H_0$ be any line meeting $C$ transversely. Following the recipe of §3, there are two possible ways to split the branches of $C$ along $H_0$ into two complementary sets of two branches each, namely we can take

(a) two branches belonging to the same $C_i$, or

(b) one branch from each of the $C_i$.

These two choices yield *different* translation manifolds, namely $W_{(2,0)}$ and $W_{(1,1)}$ in $J(C)$.

Lie studied this case and showed that if the $C_i$ are defined over $\mathbb{R}$ and $C_1$ is a circle, then in case (a) the real points of the corresponding $S$ form a minimal surface (in the differential-geometric sense). Recall that any real minimal surface is a part of a complex tranlation surface whose generating curves are "minimal curves." See [2], pages 330–332. Noticing that $S$ actually had two different parametrizations as a translation surface, Lie was led to study all such surfaces.

## §5. Application to moduli of curves and Jacobian varieties

In this section, we will show that theorem 3.9 can be used to give a characterization of non-hyperelliptic Jacobians among all principally polarized abelian varieties. The idea of the proof is to consider the theta divsor of the abelian variety $A$, lifted to $\mathbb{C}^n$ (or equivalently, the divisor of zeros of the corresponding Riemann theta-function). We assume that near some point, the theta divisor, $\Theta$, has a double parametrization as in (2.3), and show (using Theorem 3.9) that $\Theta$ coinsides with the lifting of $W_d(C)$ to $\mathbb{C}^n$ for some $C$. Then by analyzing the global periodicity properties of these divisors we conclude that $C$ is a smooth curve of genus $n$ and $(A, \Theta)$ is the canonically polarized Jacobian of $C$.

THEOREM 5.1: *Let $(A, \Theta)$ be an $n$-dimensional principally polarized abelian variety. Suppose there is a point $p \in \Theta$ such that the germ of $\Theta$ at $p$ (lifted to $\mathbb{C}^n$ and translated to 0) is a nondegenerate doubly translation type manifold satisfying the condition (*) in Proposition 3.7. Then $(A, \Theta)$ is the canonically polarized Jacobian of a smooth nonhyperelliptic curve of genus $n$.*

Proof: First note that under our hypotheses, $\Theta$ must be an irreducible divisor in $A$. This is a consequence of the following easy lemma.

Lemma 5.2: *Let $A$ be an $n$-dimensional abelian variety and let $D$ be an effective divisor which defines a principal polarization on $A$. If $D = r_1D_1 + \ldots + r_kD_k$ where the $r_i > 0$ and $k \geq 2$, then the $D_i$ are all degenerate divisors (in the sense that corresponding Riemann forms are not positive definite).*

Proof of the Lemma: Suppose $D = \sum r_iD_i$ as above with $D_1$ nondegenerate (and $r_1 > 0$). Since $D_1$ is effective, it is an ample divisor by Lefschetz's theorem. Hence, by the Riemann-Roch theorem

$$\dim H^0(\mathcal{O}_A(D_1)) = \frac{D_1^n}{n!} \geq 1$$

On the other hand $D$ defines a principal polarization so

$$\dim H^0(\mathcal{O}_A(D)) = \frac{D^n}{n!} = 1.$$

Hence

$$\begin{aligned} 1 &= \frac{1}{n!}\left(\sum r_iD_i\right)^n \\ &= \frac{1}{n!}\sum_{e_1+\ldots+e_k=n} (e_1^n \ldots e_k)D_1^{e_1}\ldots D_n^{e_n}r_1^{e_1}\ldots r_n^{e_n} \\ &= \frac{D_1^n}{n!}r_1^n + \frac{1}{n!}\left(\sum_{i=2}^{k} D_1^{n-1}D_ir_i\right)r_1^{n-1} + \text{other terms.} \end{aligned}$$

Since $D_1$ is ample, $D_1^{n-1}D_i > 0$ all $i$, and all the other terms are at least zero since the $D_i$ are effective divisors on $A$. It follows easily that

$$\frac{D_1^n}{n!} = 1, r_1 = 1, \text{and } r_i = 0 \text{ for all } i \geq 2.$$

Hence $D = D_1$. Q.E.D.

On the other hand, the germ of a degenerate divisor (in the sense of the lemma) cannot be a nondegenerate translation manifold (the image of the Gauss map is not $(n-1)$-dimensional). Hence $\Theta$ must be an irreducible divisor.

Now, we fix an isomorphism $A \simeq C^n/\Lambda_A$ where $\Lambda_A$ is a lattice in $\mathbb{C}^n$. We consider $(\Theta, p)$ lifted to $\mathbb{C}^n$ and translated so that $p$ lies at the origin. By Theorem 3.9, we know there is a connected, reduced algebraic curve $C$ of degree $2n-2$ and arithmetic genus $p_a(C) = n$ in $\mathbb{P}^{n-1}$, and a non-singular divisor $D$ of multidegree $d = (d_1, \ldots, d_r)$ where $\sum d_i = n-1$, with $\dim H^0(\omega_C \otimes \mathcal{O}(-D)) = 1$, such that after translating

$$(\Theta, p) = (W_d(C), u(D)). \tag{5.1}$$

Furthermore, as in Corollary 3.4, the standard basis in $\mathbb{C}^n$ determines a basis of $H^0(\omega_C)$ and the "lattice of periods," $\Lambda_C$, of $C$.

We have two maps $\rho$ and $\sigma$

$$\begin{array}{ccc} & \mathbb{C}^n & \\ {}^{\rho}\swarrow & & \searrow^{\sigma} \\ J(C) \simeq \mathbb{C}^n/\Lambda_C & & A \simeq \mathbb{C}^n/\Lambda_A \end{array}$$

Equation (5.1) implies that (suitable translates of) $\sigma^{-1}(\Theta)$ and $\rho^{-1}(W_d)$ intersect in a non-empty open set (in the classical topology). It follows that $\rho^{-1}(W_d) \subseteq \sigma^{-1}(\Theta)$, since $\sigma^{-1}(\Theta)$ is an irreducible analytic subvariety of $C^n$, and by results quoted in §1, $\rho^{-1}(W_d)$ is a constructible subset of an irreducible analytic subvariety in $\mathbb{C}^n$. If we let $W$ be the Zariski closure of $W_d$ in $J(C)$, then we have $\rho^{-1}(W) = \sigma^{-1}(\Theta)$.

Now $\sigma^{-1}(\Theta)$ is periodic with respect to the full lattice $\Lambda_A \subset C^n$. That is, for all $\lambda \in \Lambda_A$, $\sigma^{-1}(\Theta) + \lambda = \sigma^{-1}(\Theta)$. Hence $\rho^{-1}(W)$ is also periodic with respect to $\lambda \in \Lambda_A$, or equivalently $W + \rho(\lambda) = W$ in $J(C)$ for all $\lambda \in \Lambda_A$. I claim this implies $\rho(\lambda) = 0$ in $J(C)$, or what is the same $\lambda \in \Lambda_C$. This follows from a general lemma about the subvarieties $W_d$ in $J(C)$. Let $d$ and $d'$ be two multidegrees on $C$; we will say $d \geq d'$ if $d_i \geq d_i'$ for all $i$. $C \subset \mathbb{P}^{n-1}$ is any canonically-embedded Gorenstein curve with $p_a(C) = n$.

LEMMA 5.3: *Let $d \geq d'$ be two multidegrees on $C$ with $d = (d_1, \ldots, d_r)$, $d' = (d_1', \ldots, d_r')$, where $d = \sum d_i \leq n-1$, and all $d_i$, $d_i' \geq 0$. Assume there is an effective divisor $D'$ of multidegree $d'$ with*

$$\dim H^0(\omega_C \otimes \mathcal{O}(-D')) = n - d'$$

*Then, if $a \in J(C)$ and*

$$W_{d'} + a \subseteq W_d, \tag{5.2}$$

*we have*

$$a \in W_{d-d'}.$$

PROOF: This is well-known when $C$ is a smooth curve, and we will prove it using the same method here. The point $a \in J(C)$ represents an isomorphism class of line bundles of multidegree zero, from which we choose a representative, $L$. Letting $P_1, \ldots, P_r$ be the fixed (smooth) base points on the components of $C$, the condition (5.2) on a translates into the following statement about $L$:

$$\dim H^0(L \otimes \mathcal{O}(D') \otimes (\sum (d_i - d_i')P_i)) \geq 1$$

for *all* $D'$ of multidegree $d'$. However, by hypothesis there exists a divisor of multidegree $d'$ imposing $d'$ independent conditions on sections of $\omega_C$, so by the Riemann-Roch theorem

$$\dim H^0(L \otimes \mathcal{O}(\sum (d_i - d_i')P_i)) \geq 1$$

as well. Hence the corresponding point $a \in J(C)$ lies in $W_{d-d'}$. Q.E.D.

Returning to the proof of theorem 5.1, and letting $d = d'$ in the lemma, we see that if there exists a $D$ of multidegree $d$ as in theorem 3.9 with $\dim H^0(\omega_C \otimes \mathcal{O}(-D)) = 1$ then any $a \in J(C)$ such that $W_d + a = W_d$ must be zero. This conclusion also clearly holds if $(W_d + a) \cap W_d$ is Zariski-dense in $W_d$.

In fact, this is the case for all the $a = \rho(\lambda)$ for $\lambda \in \Lambda_A : W + \rho(\lambda) = W$, and $W_d$ is a Zariski-dense constructible subset of $W$ such that $\dim(W - W_d) < n - 1$ at every point. Hence $(W_d + \rho(\lambda)) \cap W_d$ is Zariski-dense in $W_d$. It follows that $\rho(\lambda) = 0$ in $J(C)$, and hence $\Lambda_A \subseteq \Lambda_C$ in $\mathbb{C}^n$.

Now it follows immediately that $\Lambda_A = \Lambda_C$. Let $\mu \in \Lambda_C$ and note that $\rho^{-1}(W) + \mu = \rho^{-1}(W)$. Hence $\Theta + \sigma(\mu) = \Theta$ in $A$. This implies $\sigma(\mu) = 0$ in $A$, since $\Theta$ defines a principal polarization. So $\Lambda_C \subseteq \Lambda_A$, and the assertion above follows from the pair of inclusions.

Now if $C$ is our curve, rank $\Lambda_C \leq 2g + \delta$, where $g = g(\tilde{C})$ and $\delta = \sum \delta_p$ ($\delta_p$ are the local invariants of the singular points). But $p_a(C) = g + \delta = n$, so if rank $\Lambda_C = 2n$, than $C$ must be a smooth canonical curve of genus $n$ in $\mathbb{P}^{n-1}$.

Composing the Abel map $u$ on $C$ with $\sigma$ (which is the same as $\rho$) we obtain a (holomorphic) map $\phi: C \to A$, and it is clear that $A$ is birationally equivalent to $C^{(n)}$ (symmetric product) and $\Theta$ is birationally equivalent to $C^{(n-1)}$. (Since $C$ is irreducible the multidegree $d$ is just $n - 1$). It follows immediately that $(A, \Theta)$ is the canonically polarized Jacobian of $C$. Q.E.D.

REMARKS: (1) It may be true that the conclusion of theorem 5.1 is valid even without the hypothesis that (*) holds for the local parametrization of $\Theta$, near $p$.

One way to prove this might be to make a careful study of the $C$ of degree $2n-2$ in $\mathbb{P}^{n-1}$ with $\dim H^0(\omega_C) > n$. If Example 3.6 (including the generalizations mentioned in the remarks following that example) turned out to be typical of this case, that is, the corresponding translation manifolds were *always* projections of *products* of $W_d$'s in the Jacobians of the components of $C$, then (*) could be omitted from the hypotheses of theorem 5.1.

(2) It is hoped that this characterization of nonhyperelliptic Jacobians in terms of the geometry of the theta-divisor may eventually shed some light on moduli questions.

## REFERENCES


[1] A. ALTMAN, and S. KLEIMAN: Introduction to Grotendieck Duality Theory. *Lecture Notes in Mathematics, 146*. Berlin, Springer-Verlag, 1970.

[2] W. BLASCHKE and K. LEICHTWEISS: *Elementare Differentialgeometrie*, 5 th ed. Berlin, Springer-Verlag, 1973.

[3] W. BLASCHKE and G. BOL: *Geometrie der Gewebe*. Berlin, Springer-Verlag, 1938.

[4] G. DARBOUX: *Leçons sur la Théorie Générale des Surfaces*, 2nd ed. Paris, Gauthier–Villars, 1914.

[5] J. EIESLAND: On a Certain Class of Algebraic Translation Surfaces. *Amer. J. Math. 29* (1908) 363–386.

[6] J. EIESLAND: On Translation Surfaces Connected with a Unicursal Quartic. *Amer. J. Math. 30* (1909) 170–208.

[7] J. FAY: Theta Functions on Riemann Surfaces. *Lecture Notes in Mathematics 352*. Berlin, Springer-Verlag, 1973.

[8] P. GRIFFITHS: Variations on a Theorem of Abel. *Inv. Math. 35* (1976) 321–390.

[9] T. JAMBOIS: The Theorem of Torelli for Singular Curves. *Transactions of the A.M.S. 239* (1978) 123–146.

[10] D. MUMFORD: *Curves and their Jacobians*. Ann Arbor, University of Michigan Press, 1975.

[11] H. POINCARÉ: Sur les Surfaces de Translation et les Fonctions Abéliennes. *Bull. Soc. Math. de France 29* (1901) 61–86.

[12] H. POINCARÉ: Remarques Diverses sur les Fonctions Abéliennes. *J. de Math. (Liouville), 5th Series, 1* (1895) 219–314.

[13] M. ROSENLICHT: Equivalence Relations on Algebraic Curves. *Annals of Math. 56* (1952) 169–191.

[14] M. ROSENLICHT: Generalized Jacobian Varieties. *Annals of Math. 59* (1954) 505–530.

[15] B. SAINT-DONAT: Variétés de Translation et Théoreme de Torelli. *Comptes Rendus Series A 280* (1975) 1611–1612.

[16] G. SCHEFFERS: Das Abel'sche Theorem und das Lie'sche Theorem über Translationsflachen. *Acta Math. 28* (1904) 65–91.

[17] J.-P. SERRE: *Groupes Alge{briques et Crops de Classes*. Paris, Hermann, 1959.

[18] W. WIRTINGER: Lies Translationsmannigfaltigkeiten und Abel'sche Integrale. *Monatshefte für Math. und Phys. 46* (1938) 384–431.

(Oblatum 4-VI-1981)

Department of Mathematics
College of the Holy Cross
Worcester, MA 01610
U.S.A.