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Compositio Mathematica, tome 49, nº 2 (1983), p. 283-289
<http://www.numdam.org/item?id=CM_1983__49_2_283_0>

Compositio Mathematica 49 (1983) 283–289

AN IMPROVED ESTIMATE ON THE DISTRIBUTION MOD 1 OF POWERS OF REAL MATRICES

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1. Introduction

This note is an addition to a former paper by the authors together with V. Losert [5] which was concerned with metrical results on the asymptotic distribution modulo 1 of powers of matrices. (For the basic concepts of the theory we refer to the classical monographs of Kuipers–Niederreiter [3] and Hlawka [2].) As the main result of [5] it was proved that for any strictly increasing sequence $(p(n))_{n=1}^{\infty}$ of positive integers and for almost all real $(s \times s)$-matrices A with at least one eigenvalue of modulus larger than 1 the sequence $(A^{p(n)})_{n=1}^{\infty}$ is uniformly distributed in $\mathbb{R}^{s^2}$ modulo 1 and that its discrepancy $D(N, A)$ can be estimated by

$$D(N, A) \leqq C(A, \varepsilon) N^{-1/2} (\log N)^{s^2 + (3/2) + \varepsilon} \quad (\varepsilon > 0) \tag{1}$$

For the case that the matrix A has a real eigenvalue of modulus larger than 1 we established the sharper bound

$$D(N, A) \leqq C(A) N^{-1/2} (\log N)^{s^2 + (1/2)} (\log\log N)^{1/2} \tag{2}$$

(again for almost all such matrices A).

The object of the present paper is to improve the inequality (1) by the following result.

THEOREM: *Let $(p(n))_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then for almost all (in the sense of the s^2-dimensional Lebesgue measure) real $(s \times s)$-matrices A with at least one eigenvalue of modulus* *larger than* 1 *there exists a constant* $C(A)$ *such that the discrepancy* $D(N, A)$ *of the sequence* $(A^{p(n)})_{n=1}^{N}$ *can be estimated by*

$$D(N, A) \leqq C(A) N^{-1/2} (\log N)^{s^2+1}. \tag{3}$$

The improvement is effected by using an Erdös–Koksma type argument [1] instead of the method of Leveque [4] based on the properties of trigonometric functions which we had employed to establish (1).

2. Proof of our theorem

It is obviously sufficient to consider the set M of all real $(s \times s)$-matrices with pairwise different eigenvalues and with at least one nonreal eigenvalue of modulus larger than 1. For each $\rho \equiv s\ (2)$ we denote by M_ρ the set of all matrices of M with exactly ρ real and $\tau = \frac{1}{2}(s - \rho)$ pairs of conjugate complex eigenvalues. Then for any $A \in M_\rho$ there exists an invertible transformation matrix X and an "almost diagonal" matrix Y of the form

$$Y = \begin{matrix} \lambda_1 & \dots & & & 0 \\ \vdots & \ddots & \lambda_\rho & \Lambda_1 & \vdots \\ 0 & \dots & & \ddots & \Lambda_\tau \end{matrix} \tag{4}$$

such that $A = X^{-1} Y X$. In (4) the Λ_j are (2×2)-matrices

$$\Lambda_j = \begin{pmatrix} r_j \cos \phi_j & r_j \sin \phi_j \\ -r_j \sin \phi_j & r_j \cos \phi_j \end{pmatrix} \quad (1 \leqq j \leqq \tau) \tag{5}$$

where the r_j are the moduli of the nonreal eigenvalues and ϕ_j are the corresponding arguments $(0 < \phi_j < \pi)$.

For any $(s \times s)$-matrix G with integer entries not all zero we shall have to consider the Weyl sum

$$S(N; G, X, Y) = \sum_{n=1}^{N} e\Bigg(\sum_{i=1}^{\rho} \alpha_i(G, X) \lambda_i^{p(n)} + \\ + \sum_{j=1}^{\tau} \gamma_j(G, X) r_j^{p(n)} \cos(p(n)\phi_j + y_i(G, X)) \Bigg) \tag{6}$$

(with $e(x) := e^{2\pi i x}$) occuring as formula (3.14) of [5], where $\alpha_i(G, X)$,

$\gamma_j(G,X)$ and $y_j(G,X)$ are defined in chapter 3 of [5]. There we also have proved the estimate

$$|\gamma_j(G,X)| \geqq C(X)\,\|G\|^{-5s^2} \tag{7}$$

for almost all (with respect to the s^2-dimensional Lebesgue measure) transformation matrices X; here $C(X) > 0$ and $\|G\|$ denotes the maximum of the moduli of the elements of the matrix G. Now by the same measure-theoretic argument as used in [5] to deduce Theorem I from Proposition 3.8 our theorem is established if we can prove the following result.

PROPOSITION: *Let X be a fixed transformation matrix such that* (7) *holds for all integer-valued, nonvanishing matrices G and fix all eigenvalues of A except one pair of conjugate complex ones, say λ, $\bar{\lambda}$. Put $r = |\lambda|$, $\phi = \arg\lambda$ (w.l.o.g. $0 < \phi < \pi$). Then for almost all $(r,\phi) \in [a,b] \times]0,\pi[$ (with arbitrary $a > 1$) the discrepancy of the sequence $(A^{p(n)})_{n=1}^{N}$ satisfies the estimate* (3).

To prove this proposition we first write the Weyl sum of (6) as a function of r and ϕ

$$S(N;G,X,Y) = \sum_{n=1}^{N} e(C_n + \gamma(G,X)r^{p(n)}\cos(p(n)\phi + y(G,X))) \tag{8}$$

where the constants C_n do not depend on r and ϕ and therefore are fixed because of the hypothesis of the proposition. (The index j has been dropped for short.) Next we need a simple auxiliary result.

LEMMA: *Let the matrix X be fixed again, $N \in \mathbb{N}$, $n \in \mathbb{N}$ with $1 \leqq n \leqq N$, G as above with $1 \leqq \|G\| \leqq \sqrt{N}$. Then for almost all $\phi \in]0,\pi[$ there exists a positive constant $K(\phi)$ not depending on N, n and G such that*

$$|\cos(p(n)\phi + y(G))| \geqq K(\phi)N^{-c(s)}\left(c(s) = \frac{s^2}{2} + 3\right). \tag{9}$$

PROOF: We put $\phi = \pi t$, $y(G) = \pi(\frac{1}{2} + z(G))$ and define for any arbitrary integer u and N, n, G as above

$$\mathfrak{U}_N(G,n,u) := \{t \in]0,1[: |p(n)t - z(G) - u| < N^{-c(s)}\}. \tag{10}$$

For the Lebesgue measure of each of these sets we obviously have

$$\mu(\mathfrak{U}_N(G,n,u)) \leqq 2p(n)^{-1}N^{-c(s)}. \tag{11}$$

If for an integer u there exists at least one nonempty set $\mathfrak{U}_N(G,n,u)$, we certainly have

$$|u| \leqq N^{-c(s)} + p(n)t + |z(G)| \leqq 4p(n) \tag{12}$$

(the bound for $z(G)$ easily follows from the definition of $y(G,X)$ given in [5]), hence for arbitrary $N \in \mathbb{N}$ and all corresponding n and G

$$\mu(\bigcup_{u\in\mathbb{Z}} \mathfrak{U}_N(G,n,u)) \leqq 32N^{-c(s)}. \tag{13}$$

Consequently we get for the Lebesgue measure of the union

$$\mathfrak{U}(N) := \bigcup_{n=1}^{N} \bigcup_{1\leqq\|G\|\leqq\sqrt{N}} \bigcup_{u\in\mathbb{Z}} U_N(G,n,u) \tag{14}$$

the estimate $\mu(\mathfrak{U}(N)) \leqq KN^{-2}$ (the constant K only depending on s), hence the series $\sum \mu(\mathfrak{U}(N))$ converges. By the Borel–Cantelli lemma for almost all $t \in]0,1[$ there exists an integer $N_0(t)$ such that for all $N > N_0(t)$, for all n with $1 \leqq n \leqq N$, for all G with $1 \leqq \|G\| \leqq \sqrt{N}$ and for all integers u the inequality

$$|p(n)t - z(G) - u| \geqq N^{-c(s)} \tag{15}$$

holds. Choosing u such that the left-hand side of (15) is $\leqq \frac{1}{2}$ we finally get

$$|\cos(p(n)\phi + y(G))| = |\sin \pi(p(n)t + z(G) - u| =$$
$$= \sin|\pi(p(n)t + z(G) - u)| \geqq 2\,|p(n)t - z(G) - u| \geqq K(\phi)N^{-c(s)} \tag{16}$$

for almost all $\phi \in]0,\pi[$, which proves the assertion of our lemma.

Therefore we now fix $\phi \in]0,\pi[$ such that (9) holds and are left with showing the estimate (3) for almost all $r \in [a,b]$ (with arbitrary $a > 1$ and b). We write (8) in the form

$$S(N;G,Y) = \sum_{n=1}^{N} e(f(n,r)), \tag{17}$$

$$f(n,r) := \gamma(G,X)r^{p(n)}\cos(p(n)\phi + y(G,X)) + C_n \tag{18}$$

and define (employing a method due to Erdös and Koksma [1])

$$h_\sigma(n,r) := f(\sigma + (n-1)q, r) \tag{19}$$

for $n = 1, 2, \ldots, N_\sigma := [(N-\sigma)q^{-1}] + 1$, $\sigma = 1, \ldots, q$, where the positive integer q is given (for sufficiently large N) by

$$q = q(N) := \left[\left(\frac{s^2}{2} + 5\right)\log N(\log a)^{-1}\right]. \tag{20}$$

We further put

$$w = w(N) = [\log N] \tag{21}$$

and get by a straightforward calculation

$$\left|\sum_{n=1}^{N_\sigma} e(h_\sigma(n,r))\right|^{2w} = \sum_{[n_1,\ldots,n_w]} P[n_1,\ldots,n_w]^2 +$$

$$+ 2 \sum_{[n_1,\ldots,n_w] > [m_1,\ldots,m_w]} \sum P[n_1,\ldots,n_w]P[m_1,\ldots,m_w]\cos(2\pi F(r)). \tag{22}$$

Here $[n_1,\ldots,n_w]$ denotes the equivalence class of all w-tuples which can be obtained from the special w-tuple $(n_1,\ldots,n_w)$ with $n_1 \leqq \ldots \leqq n_w \leqq N_\sigma$ by a permutation of the entries, $P[n_1,\ldots,n_w]$ being the cardinality of this equivalence class. $[n_1,\ldots,n_w] > [m_1,\ldots,m_w]$ means that for some $k \in \{1,\ldots,w\}$ we have $n_k > m_k$ and $n_j = m_j$ for $k < j \leqq w$ (obviously this is a total order). Finally for each pair $([m_1,\ldots,m_w], [m'_1,\ldots,m'_w])$ the function $F(r)$ is defined by

$$F(r) := \sum_{i=1}^{w} h_\sigma(m_i, r) - \sum_{i=1}^{w} h_\sigma(m'_i, r). \tag{23}$$

In order to establish a lower estimate for $|F'(r)|$ on $[a,b]$ we first infer from (19), (18) and our lemma

$$|h'_\sigma(m,r)h'_\sigma(m-1,r)^{-1}| \geqq a^{q(N)}KN^{-c(s)} > N > 2w \tag{24}$$

for sufficiently large N. Thus we have for $[m_1,\ldots,m_w] > [m'_1,\ldots,m'_w]$

$$|F'(r)| \geqq |h'_\sigma(m_k,r)| - (2w-1)|\,h'_\sigma(m_k-1,r)| \geqq$$

$$|f'(\sigma + (m_k-2)q, r)| \geqq |\gamma(G,X)|\cos(p(\sigma + (m_k-2)q)\phi + y(G,X)) \geqq$$

$$\geqq KN^{-6s^2-3} \tag{25}$$

in view of (7) and (9). By a similar argument as in [3], page 35, (4.4) we conclude for arbitrary fixed $[n_1, \ldots, n_w]$

$$\sum_{[m_1,\ldots,m_w]<[n_1,\ldots,n_w]} (\min_{[a,b]} |F'(r)|)^{-1} \leqq 2K^{-1}N^{6s^2+3} \sum_{j=1}^{N^w} \frac{1}{j}$$
$$\leqq K_1 N^{6s^2+3} w \log N, \tag{26}$$

so the second mean-value theorem yields from (22) (the monotony of $F'(r)$ on $[a,b]$ is easily verified by repeating the above argument for $|F''(r)|$)

$$\int_a^b \left| \sum_{n=1}^{N_\sigma} e(h_\sigma(n,r)) \right|^{2w} dr \leqq (b-a) w!\, N_\sigma^w +$$
$$+ K_2 N^{6s^2+3} N_\sigma^w w!\, w \log N \leqq K_3 w!\, w N^{6s^2+3} N_\sigma^w \log N \tag{27}$$

where we have used the obvious combinatorical relations

$$P(n_1, \ldots, n_w] \leqq w!, \quad \sum_{n_1 \leqq \ldots \leqq n_w \leqq N_\sigma} P[n_1, \ldots, n_w] = N_\sigma^w \tag{28}$$

We now consider subsets of $[a,b]$ defined by

$$\mathfrak{m}(N,G,\sigma) := \left\{ r \in [a,b] : \left| \sum_{n=1}^{N_\sigma} e(h_\sigma(n,r)) \right| \geqq N_\sigma^{1/2} \psi(N_\sigma) \right\} \tag{29}$$

with $\psi(x) := (\log x^2)^{1/2} e^{4s^2+3}$ and infer from (27) for their Lebesgue measure

$$\mu(\mathfrak{m}(N,G,\sigma)) N_\sigma^w \psi(N_\sigma)^{2w} \leqq K_3 w!\, w N_\sigma^w N^{6s^2+3} \log N$$
$$\Rightarrow \mu((N,G,\sigma)) \leqq K_3 w^w N^{6s^2+3} \log N \psi([Nq^{-1}])^{-2w}. \tag{30}$$

Thus we have for the Lebesgue measure of the union

$$\mathfrak{m}(N) := \bigcup_{\sigma=1}^{q} \bigcup_{1 \leqq \|G\| \leqq \sqrt{N}} \mathfrak{m}(N,G,\sigma)$$
$$\mu(\mathfrak{m}(N)) \leqq K_3 q N^{7s^2+3} \log N w^w \psi(N^{\frac{1}{2}})^{-2w}. \tag{31}$$

A short calculation shows $\mu(\mathfrak{m}(N)) < N^{-2}$ for sufficiently large N (by the definitions of q, w and ψ), hence the series $\sum \mu(\mathfrak{m}(N))$ converges and the

Borel–Cantelli lemma implies that for almost all $r \in [a, b]$ there exists a positive integer $N_0(r)$ such that for all integers $N > N_0(r)$ the inequality

$$\left| \sum_{n=1}^{N_\sigma} e(h_\sigma(n, r)) \right| < N_\sigma^{1/2} \psi(N_\sigma) \tag{32}$$

holds for all $\sigma = 1, \ldots, q(N)$ and for all matrices G with integer entries and with $1 \leqq \|G\| \leqq \sqrt{N}$. Finally we conclude from (17) and (19)

$$S(N; G, X, Y) = \left| \sum_{\sigma=1}^{q} \sum_{n=1}^{N_\sigma} e(h_\sigma(n, r)) \right| \leqq$$

$$q \max (N_\sigma^{\frac{1}{2}} \psi(N_\sigma)) \leqq 2q \left(\frac{N}{q} \right)^{1/2} \psi(N) \leqq C N^{1/2} \log N. \tag{33}$$

The inequality of Erdös–Turán–Koksma (cf. [3], page 116 and choose $m = [\sqrt{N}]$) now immediately yields the desired estimate (3).

REFERENCES

[1] P. Erdös and J.F. Koksma: On the uniform distribution mod 1 of lacunary sequences. *Indagationes Math. 11* (1949) 79–88.

[2] E. Hlawka: Theorie der Gleichverteilung. Mannheim–Wien–Zürich, Bibl. Inst. 1979.

[3] L. Kuipers and H. Niederreiter: Uniform distribution of sequences. New York–Sydney–Toronto, John Wiley and Sons 1974.

[4] W.J. Le Veque: The distribution mod 1 of trigonometric sequences. *Duke Math. Journal 20* (1953) 367–374.

[5] V. Losert, W.G. Nowak and R.F. Tichy: On the asymtotic distribution of the powers of ($s \times s$)-matrices. *Comp. Math.*, to appear.

(Oblatum 21-IV-1982)

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