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HARMONIC ERGODICITY OF ACTIONS OF CERTAIN DISCRETE LINEAR GROUPS

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Let $A_1, A_2, \ldots, A_k, \ldots$ be a sequence of non-singular $n \times n$ matrices and let $a_1, a_2, \ldots, a_k, \ldots$ be a sequence of positive real numbers such that $\sum_1^\infty a_i = 1$. Consider the functional equation on $\mathbb{R}^n$ defined by

$$f(v) = \sum_{i=1}^{\infty} a_i f(vA_i) \quad \text{for all } v \in \mathbb{R}^n \tag{1.1}$$

where $\mathbb{R}^n$ is viewed as the space of row vectors. Equation (1.1) is a "mean value property" with $a_1, a_2, \ldots$ as the weights. When do there exist "non-trivial" measurable functions satisfying (1.1)? Evidently, there exist a multitude of functions which satisfy (1.1), but are constant a.e. (almost everywhere, with respect to the Lebesgue measure). Therefore "non-triviality" of a solution f should mean that f is not constant a.e..

Let Γ be the subgroup generated by $\{A_i \mid i \in \mathbb{N}\}$. If the action of Γ on $\mathbb{R}^n$ is not ergodic then there exist measurable functions f such that $f(vA_i) = f(v)$ for all $v \in \mathbb{R}^n$ and $i \in \mathbb{N}$, which are not constant a.e.. These functions evidently satisfy (1.1) for any choice of a_i, $i \in \mathbb{N}$, as above. We remark that if Γ is a solvable subgroup then its action on $\mathbb{R}^n$ is necessarily non-ergodic.

The action of Γ on $\mathbb{R}^n$, together with the probability measure μ defined by $\mu(\{A_i\}) = a_i$, $i \in \mathbb{N}$, defines a random walk on $\mathbb{R}^n$, with respect to the Γ-action. A measurable function satisfying (1.1) is nothing but a harmonic function with respect to the random walk; i.e. μ-harmonic function. If there exists no non-trivial bounded μ-harmonic function we shall say that the action is *μ-harmonically ergodic.* The definition can be extended (cf. §1 for details) to any action of Γ equipped with a distinguished quasi-invariant measure, the action on $\mathbb{R}^n$ being only an example. While μ-harmonic ergodicity implies ergodicity, the converse is not always true.

In this article we investigate μ-harmonic ergodicity of a certain interesting class of ergodic actions. We prove the following.

A. THEOREM (*cf. Theorem* 4.2): *Let $G = SL(n, \mathbb{R})$ and let N be the subgroup consisting of all upper triangular unipotent matrices. Let Γ be a* *discrete subgroup of* G *such that* G/Γ *is compact. Then there exists a probability measure* μ *supported on* Γ *such that the following assertions hold.*

(*i*) *If* H *is a closed subgroup of* G *containing* N *then the* Γ*-action on* $H\backslash G$ (*on the right*) *is* μ*-harmonically ergodic if and only if* H/N *is infinite*; *here* $H\backslash G$ *is considered to be equipped with a measure which is quasi-invariant under the* G*-action.*

(*ii*) *The* Γ*-action on* $\mathbb{R}^n$ *is* μ*-harmonically ergodic if and only if* $n \geqslant 3$.

Now let H be a closed subgroup containing N (notation as in the theorem) such that H/N is infinite. Then the different behaviour of the Γ-actions on $N\backslash G$ and $H\backslash G$, as asserted by the theorem, implies that they must be non-isomorphic – we note however that this can also be deduced directly from Lemma 3.2. We can conclude the following:

B. COROLLARY (*cf. Corollary* 4.4): *Let* H *and* N *be as above. Then the* Γ*-action on* $N\backslash G$ *is not a factor of the* Γ*-action on* $H\backslash G$. *In particular the actions are not isomorphic.*

It has been pointed out to the author that if the subgroup H as above is *non-amenable* then corollary can also be deduced from the work of R. Zimmer (cf. [6]). We emphasize that the corollary is applicable even when H is amenable – e.g. if H consists only of upper triangular (not necessarily unipotent) matrices.

Before concluding the introduction it would be worthwhile to mention, and acknowledge, that in proving Theorem A we make crucial use of a recent result of Y. Guivarch (cf. Theorem 4.1) on integral representations of μ-harmonic functions on Γ, the latter being as in Theorem A.

§1. Preliminaries

Let G be a locally compact second countable group. Let X be a standard Borel space. Let Φ: $X \times G \to X$ be a jointly measurable right action of G on X; we denote $\Phi(x, g)$ by xg. Let μ be a probability measure on G. A (Borel) measurable function f on X is said to be *μ-harmonic* if

$$f(x) = \int_G f(xg)\,d\mu(g) \quad \text{for all } x \in X$$

In the special case of G acting on itself on the right, the above reduces to the usual notion of μ-harmonic functions on G (cf. [1]).

Let X and G be as above. Suppose that there exists a σ-finite measure σ on X which is quasi-invariant under the G-action; that is, for any $g \in G$, the equality $\sigma(Eg) = 0$ holds for a Borel set E if and only if $\sigma(E) = 0$. Such an action of G on the measure space (X, σ) is also called a *non-singular* G-action. As before let μ be a probability measure on G. The non-singular G-action on (X, σ) is said to be *μ-harmonically ergodic* if every bounded μ-harmonic function on X is constant σ-almost everywhere.

1.1. REMARK: If the G-action on (X, σ) as above, is μ-harmonically ergodic, then it is ergodic; that is, if E is a Borel subset of X such that $\sigma(Eg\Delta E)=0$ for all $g \in G$, then either $\sigma(E)=0$ or $\sigma(X-E)=0$.

PROOF: It is well-known that if there exists a Borel subset E such that $\sigma(Eg\Delta E)=0$ for all $g \in G$, then there exists a Borel subset E_1 such that $E_1 g = E_1$ for all $g \in G$ and $\sigma(E\Delta E_1)=0$. Evidently, the characteristic function χ of E_1 is a μ-harmonic function (indeed for any μ). If the action is μ-harmonically ergodic, we obtain that χ is constant σ a.e.. Hence $\sigma(E)\sigma(X-E)=\sigma(E_1)\sigma(X-E_1)=0$, which proves the remark.

1.2. REMARK: It may be noted that when the action is non-transitive, there exist non-constant μ-harmonic functions (e.g. characteristic functions of orbits). The condition for μ-harmonic ergodicity demands that these be constant σ a.e..

Examples of μ-harmonically ergodic actions

(i) Let G be a group of type T, in the sense of [1]. As an example we note that any nilpotent Lie group is of type T. Let $X = G$ and consider the action by translation on the right. Let μ be a spread out probability measure on G; that is, for some $n \geqslant 1$, $\mu^n = \mu * \mu * \ldots * \mu$ (n copies), is not singular with respect to the Haar measure on G. Then the in view of Theorem I.3, [1], the above G-action is μ-harmonically ergodic.
(ii) Let $G = SL(d, \mathbb{R})$ be the special linear group. Let N be the subgroup of G consisting of all upper triangular unipotent matrices and let $X = N \backslash G$. Let μ be a probability measure on G such that for any $g_1, g_2 \in G$ there exists $n \geqslant 1$ such that $\mu^n * \delta_{g_1}$ and $\mu^n * \delta_{g_2}$ are not mutually singular; δ_{g_1} and δ_{g_2} denote point measures concentrated on g_1 and g_2 respectively. Then the G-action on $N \backslash G$ on the right is μ-harmonically ergodic (cf. [3], pp. 226).

1.3. REMARK: Let G be a group of type T and consider a non-singular action of G on (X, σ), as before. Let μ be a spread out probability measure on G. Then the G-action is μ-harmonically ergodic if and only if it is ergodic.

PROOF: We only need to show that, in this case ergodicity implies μ-harmonic ergodicity. Let f be a μ-harmonic function on X. For any $x \in X$ define a function f_x on G by $f_x(g) = f(xg)$. It is easy to verify that f_x is a μ-harmonic function on G (with respect to the action by translations on the right). Thus in view of the contention in Example (1) above, each f_x is a constant function on G; that is, $f(xg) = f(x)$ for all $x \in X$ and $g \in G$. By ergodicity this implies that f is constant σ a.e.. Hence the action is μ-harmonically ergodic.

The notion of μ-harmonic ergodicity can also be used to infer non-equivalence of certain non-singular actions. Let us recall some of the definitions. Let (X_1, σ_1) and (X_2, σ_2) be two non-singular G-spaces. Let E_1 be a G-invariant Borel subset of X_1. A (Borel) measurable map $\varphi: E_1 \to X_2$ is said to be *G-equivariant* if $\varphi(xg) = \varphi(x)g$ for all $x \in E_1$ and $g \in G$. The G-space (X_2, σ_2) is said to be a *factor* of (X_1, σ_1) if there exists a G-invariant Borel subset E_1 of full measure (i.e. $\sigma_1(X_1 - E_1) = 0$) and a G-equivariant map $\varphi: E_1 \to X_2$ such that the image measure $\sigma_2' = \varphi\sigma_1$ on X_2 (defined by $\sigma_2(B) = \sigma_1(\varphi^{-1}(B))$ for all Borel subsets B) is equivalent to σ_2. The non-singular G-spaces (X_1, σ_1) and (X_2, σ_2) are said to be *equivalent* (or *isomorphic*) if there exist G-invariant Borel subsets E_1 and E_2 of full measure in X_1 and X_2 respectively and a G-equivariant Borel isomorphism $\varphi: E_1 \to E_2$ such that $\varphi\sigma_1$ is equivalent to σ_2. It is straightforward to verify the following

1.4. PROPOSITION: *Let μ be a probability measure on G. If (X_1, σ_1) and (X_2, σ_2) are two equivalent non-singular G-spaces and if one of them is μ-harmonically ergodic then so is the other. If (X, σ) is a non-singular G-space which is μ-harmonically ergodic then every factor G-space of (X, σ) is also μ-harmonically ergodic.*

§2. Integral representation

Let G be a locally compact second countable group and let Γ be a lattice in G; that is, Γ is a discrete subgroup such that G/Γ admits a G-invariant probability measure. Let m be a right Haar measure on G. Let μ be a probability measure on Γ. We shall also view it as a measure on G such that $\mu(G - \Gamma) = 0$.

Two μ-harmonic functions on G are said to be equivalent if they agree almost everywhere with respect to the Haar measure on G. We shall denote by $\mathcal{H}_\mu(G, m)$ the space of equivalence classes of bounded μ-harmonic functions on G, equipped with the essential supremum norm with respect to the Haar measure. For any μ-harmonic function f on G and $g \in G$ the function $L_g f$, defined by $L_g f(h) = f(g^{-1}h)$, is a μ-harmonic function on G. Further, if f and f' are equivalent μ-harmonic functions then for any $g \in G$ the same is true of $L_g f$ and $L_g f'$. This induces a natural left action $g \mapsto L_g$ of G on $\mathcal{H}_\mu(G, m)$.

Let $\mathcal{H}_\mu(\Gamma)$ denote the space of bounded μ-harmonic functions on Γ, equipped with the supremum norm. There is also a natural Γ-action, on the left, on $\mathcal{H}_\mu(\Gamma)$, defined by associating to $f \in \mathcal{H}_\mu(\Gamma)$ and $\gamma \in \Gamma$ the function $L_\gamma f$. Let Y be a standard Borel space with a Γ-action on it, on the left. Let ρ be a σ-finite, Γ-quasi-invariant measure on Y and let λ be a probability measure on Y absolutely continuous with respect to ρ. We say that (Y, λ, ρ) is a *Poisson-Furstenberg representation space* for $\mathcal{H}_\mu(\Gamma)$ if the following conditions are satisfied.

(i) λ is μ-stationary; that is $\mu_*\lambda = \lambda$.

(ii) the map $j_\lambda : L^\infty(Y, \rho) \to \mathcal{H}_\mu(\Gamma)$ defined by $j_\lambda(\varphi)(\gamma) = \int_Y \varphi(\gamma y) d\lambda(y)$ for all $\gamma \in \Gamma$ is an isometric isomorphism of the Banach spaces.

2.1. Remark: Existence of a Γ-space satisfying (i) and a certain version of (ii) not involving ρ is well-known (cf. [2]). We do not know whether such a space always satisfies above stronger conditions. In the particular case that interests us here (cf. §4) the existence of a space satisfying the conditions is guaranteed by [3].

To each Γ-action on a standard Borel space Y is associated a G-action defined as follows. Consider the Γ-action on $G \times Y$ defined by $\gamma(g, y) = (g\gamma^{-1}, \gamma y)$ for all $\gamma \in \Gamma$, $g \in G$ and $y \in Y$. Let Y^G be the space of Γ-orbits in $G \times Y$. Then Y^G is a standard Borel space. Let η denote the quotient map of $G \times Y$ onto Y^G. The G-action on $G \times Y$ defined by $x(g, y) = (xg, y)$, for all x, $g \in G$ and $y \in Y$, quotients under η to a G-action on Y^G.

Let ρ be a Γ-quasi-invariant probability measure on Y and let σ be a probability measure on G equivalent to the Haar measure. Then the image of $\sigma \times \rho$ under η is a G-quasi-invariant measure on Y^G; we shall denote this by ρ^G.

2.2. Theorem: *Let the notations be as above. Let (Y, λ, ρ) be a Poisson-Furstenberg representation space for $\mathcal{H}_\mu(\Gamma)$. Then there exists an isometric isomorphism $j : L^\infty(Y^G, \rho^G) \to \mathcal{H}_\mu(G, m)$ such that*

$$j(\varphi)(g) = \int_Y \varphi \circ \eta((g, y)) d\lambda(y)$$

for all $\varphi \in L^\infty(Y^G, \rho^G)$, and $g \in G$.

Proof: Firstly, since ρ is quasi-invariant under the Γ-action and $\lambda < \rho$ it follows that the equation as above yields a well-defined map j of $L^\infty(Y^G, \rho^G)$ into $L^\infty(G)$. Further since λ is μ-stationary the image of j is contained in $\mathcal{H}_\mu(G, m)$. It is also obvious that j is a linear map and that its norm does not exceed 1. We shall now construct a map p which will turn out to be the inverse of j.

Let f be a μ-harmonic function on G. For each $g \in G$ let f_g be the function on Γ defined by $f_g(\gamma) = f(g\gamma)$. Then f_g is a μ-harmonic function on Γ and therefore is of the form $j_\lambda(\varphi_g)$ where φ_g is a uniquely defined element of $L^\infty(Y, \rho)$. Naively speaking, the quotient of the function $(g, y) \mapsto \varphi_g(y)$ on Y^G is the candidate for $p(f)$. However there is a hitch; it is not clear that this function is measurable. To avoid this difficulty we shall follow a slightly circuitous route. The technique being standard we shall avoid being too detailed.

First suppose that f is non-negative. For any Borel subset A of G we define a function f_A on Γ by

$$f_A(\gamma) = \int_A f_g(\gamma)\,d\sigma(g)$$

Then f_A is also a bounded non-negative μ-harmonic function on Γ. Therefore there exists a unique element $\varphi_A \in L^\infty(Y, \rho)$ such that $f_A = j_\lambda(\varphi_A)$. Evidently φ_A is also non-negative. We now define a measure β on $G \times Y$ as follows. If A and B are Borel subsets of G and Y respectively, then set

$$\beta(A \times B) = \int_B \varphi_A(y)\,d\rho(y)$$

It is easy to verify that if $\{A_i\}_{i=1}^k$ and $\{B_j\}_{j=1}^l$ are families of mutually disjoint subsets of G and Y respectively and $A = \cup_1^k A_i$ and $B = \cup_1^l B_j$ then

$$\beta(A \times B) = \sum_{j=1}^{l} \sum_{i=1}^{k} \beta(A_i \times B_j).$$

One then deduces that β extends to a unique Borel measure on $G \times Y$, which also we shall denote by β. It is easy to check that β is absolutely continuous with respect to $m \times \rho$ and that the Radon-Nikodym derivative is bounded by the essential supremum of f. Further evidently β is invariant under the Γ-action on $G \times Y$ defined earlier. Hence there exists a unique element of $L^\infty(Y^G, \rho^G)$, which we shall choose for $p(f)$, such that $p(f) \circ \eta((g, y)) = [d\beta/d(\sigma \times \rho)](g, y)$ for $\sigma \times \rho$ almost all (g, y).

If f is any bounded μ-harmonic function on G and $c > 0$ is a constant such that $f + c$ is non-negative then we define $p(f) = p(f + c) - c$, which is independent of the choice of c.

An easy verification shows that if f and f' be two bounded μ-harmonic functions which are equal m a.e. then $p(f) = p(f')$. Thus p defines a map of $\mathcal{H}_\mu(G, m)$ into $L^\infty(Y^G, \rho^G)$. It is also evident from the construction that p is linear and that its norm does not exceed 1. It is also straightforward to prove that p is the inverse of j. Since both j and p are of norm not exceeding 1 we conclude that j is an isometric isomorphism.

2.3. Corollary: *Let H be a closed subgroup of G and let σ_H be a finite G-quasi-invariant measure on $H \backslash G$. The Γ-action on $(H \backslash G, \sigma_H)$ is μ-harmonically ergodic if and only if the H-action on Y^G (obtained by restricting the G-action) is ergodic with respect to ρ^G.*

Proof: Since j as in Theorem 2.2 is evidently G-equivariant it sets up a 1-1 correspondence between the subspace of H-invariant elements in

$L^\infty(Y^G, \rho^G)$ and the subspace of H-invariant elements in $\mathcal{H}_\mu(G, m)$. The latter subspace is obviously in 1-1 correspondence with the space of equivalence classes with respect to σ_H of μ-harmonic functions on $H\backslash G$. Therefore there exists a non-trivial (i.e. not constant σ_H a.e.) μ-harmonic function on $H\backslash G$ if and only if there exists an H-invariant function which is not constant ρ^G a.e. This proves the corollary.

2.4. Proposition: *The H-action on (Y^G, ρ^G) as in Corollary* 2.3 *is ergodic if and only if the product Γ-action on $G/H \times Y$ (defined by $\gamma(x, y) = (\gamma x, \gamma y)$ for all $\gamma \in \Gamma$, $x \in G/H$ and $y \in Y$) is ergodic with respect to $m_H \times \rho$ where m_H is any G-quasi-invariant measure on G/H.*

Proof: If f is an H-invariant function on Y^G then $f \circ \eta$ is an H-invariant function on $G \times Y$ which is also invariant under the Γ-action on $G \times Y$. This enables us to define a Γ-invariant function φ on $G/H \times Y$ by $\varphi(gH, y) = f \circ \eta((g^{-1}, y))$. Conversely, every Γ-invariant function on $G/H \times Y$ may be seen to arise from an H-invariant function via the above procedure. This correspondence implies the proposition.

§3. Ergodic actions of lattices in SL(n, $\mathbb{R}$)

In order to prove the μ-harmonic ergodicity as in Theorem A using Corollary 2.4, we need to know the conditions for ergodicity of certain linear actions, which we now present.

Let $G = SL(n, \mathbb{R})$, the special linear group of $n \times n$ matrices, where $n \geqslant 2$. Let N denote the subgroup of G consisting of upper triangular unipotent matrices. The subgroups consisting of all upper, and respectively lower, triangular matrices will be denoted by P and P^-. Let H be any closed subgroup of G containing N. In the sequel we shall consider G/H, G/P etc. to be equipped with a G-quasi-invariant probability measure. $G/H \times G/P$ is equipped with the product measure. When $G/H \times G/P$ is identified canonically with $G \times G/H \times P$, where $H \times P = \{(x, y) \mid x \in H, y \in P\}$, the measure is $G \times G$ quasi-invariant. Finally, let Γ be any lattice in G.

3.1. Lemma: *Consider the product action of G on $G/H \times G/P$. There exists a unique open G-orbit Ω in $G/H \times G/P$, such that the complement of Ω has zero measure. The isotropy subgroup of any element in Ω is conjugate to $H \cap P^-$.*

Proof: Let $\sigma \in G$ be such that $\sigma^{-1}P\sigma = P^-$. It is well-known that P^-N is an open dense subset of G whose complement has zero Haar measure (cf. [5], Proposition 1.2.3.5 for a more general result). Hence the subset Ω_1 of $G \times G$ defined by

$$\Omega_1 = \{(g\sigma n, gp) \mid g \in G, n \in N \text{ and } p \in P\}$$
$$= \{(g'p\sigma n, g') \mid g' \in G, n \in N \text{ and } p \in P\}$$

is an open subset of $G \times G$ whose complement has zero Haar measure in $G \times G$. Let Ω be the image of Ω_1 in $G \times G/H \times P$. Evidently, since $N \subset H$, Ω is a single G-orbit under the action in question (identifying $G/H \times G/P$ with $G \times G/H \times P$). Further, in view of the above, Ω is open and the $G \times G$-quasi-invariant measure of its complement is zero. The isotropy subgroup of the $H \times P$-coset $(\sigma, I)^-$ containing (σ, I), where I is the identity matrix, is $\sigma H \sigma^{-1} \cap P = \sigma(H \cap \sigma^{-1} P \sigma)\sigma^{-1} = \sigma(H \cap P^-)\sigma^{-1}$. Since $(\sigma, I)^- \in \Omega$ it now follows that the isotropy subgroup every element of Ω is conjugate to $H \cap P^-$.

3.2. Lemma: *Let G, Γ, P, N and H be as above; recall that H contains N. Then the product Γ-action on $G/H \times G/P$ is ergodic if and only if H/N is infinite.*

Proof: Let Ω be the G-orbit as in Lemma 3.1. Since the complement of Ω has zero measure it is enough to show that the Γ-action on Ω is ergodic if and only if H/N is infinite. But the latter action is canonically equivalent to the Γ-action on $G/H \cap P^-$, $H \cap P^-$ being the isotropy subgroup of a suitable point in Ω. The Γ-action on $G/H \cap P^-$ is ergodic if and only if the $H \cap P^{-1}$ action on G/Γ is ergodic (cf. [4] Proposition 6). Since $G = SL(n, \mathbb{R})$ is a simple non-compact Lie group with finite center, by Moore's ergodicity theorem (cf. [4], Theorem 4) the $H \cap P^{-1}$ action on G/Γ is ergodic if and only if $H \cap P^-$ is non-compact. Combining the steps, we conclude that the Γ-action on $G/H \times G/P$ is ergodic if and only if $H \cap P^-$ is non-compact. The lemma now follows from the following Lie group theoretic Lemma which we separate.

3.3. Lemma. *Let G, N, P, P^- and H be as above. Then $H \cap P^-$ is non-compact if and only if H/N is infinite.*

Proof: The Lie algebra $\mathfrak{G}$ of G consists of all $n \times n$ matrices of trace 0. The Lie sub-algebras corresponding to N and P^- consist respectively of the subalgebra $\mathfrak{n}$ of upper triangular nilpotent matrices and the subalgebra $\mathfrak{p}^-$ of all lower triangular matrices. First suppose that the Lie subalgebra $\mathfrak{h}$ of H contains $\mathfrak{n}$ as a proper subspace. Then clearly $\mathfrak{h} \cap \mathfrak{p}^-$ is non-zero. Therefore $H \cap P^-$ contains a non-trivial 1-parameter subgroup. Any non-trivial 1-parameter subgroup of P^- is a closed non-compact subgroup. Hence, in the case at hand, $H \cap P^-$ is non-compact. Next suppose that $\mathfrak{h} = \mathfrak{n}$. Then N is the connected component of the identity in H. In particular, H normalises N. It is well-known (and easy to verify) that P is the normaliser of N in G. Hence $H \subset P$. Let D be the subgroup of G consisting of diagonal matrices. Then $P = D \cdot N$ (semi-direct product). Since $P \supset H \supset N$ we have $H = (H \cap D) \cdot N$. But $H \cap D$ is contained in P^-. Hence if $H \cap D$ is non-compact then so is $H \cap P^-$. On the other hand, if $H \cap D$ is compact then obviously it is finite and so is H/N. Thus if H/N is infinite then $H \cap P^-$ is non-compact. The converse is obvious.

§4. Main results and questions

After recalling a result of Guivarch we shall present, in this section, proofs of the main results.

Let $G = SL(n, \mathbb{R})$ and Γ be a uniform lattice in G; that is, Γ is a discrete subgroup such that G/Γ is compact. Let P be the subgroup of G consisting of all upper triangular matrices in G. The compact subgroup K consisting of all orthogonal matrices in G acts transitively on G/P. Therefore there exists a unique K-invariant probability measure on G/P which we shall denote by ρ. The measure ρ is quasi-invariant under the G-action. In the terminology introduced in §2, a result of Guivarch asserts the following.

4.1. THEOREM (*cf* [3], *Corollary* 2 *on page* 244): *There exists a probability measure μ supported by Γ such that $(G/P, \rho, \rho)$ is a Poisson-Furstenberg representation space for $\mathcal{H}_\mu(\Gamma)$.*

As in §3 let N be the subgroup of G consisting of all upper triangular unipotent matrices and P^- be the subgroup consisting of all lower triangular matrices. We now deduce the following result on μ-harmonic ergodicity.

4.2. THEOREM: *There exists a probability measure μ supported by Γ such that the following holds: let H be a closed subgroup of G containing N and let σ_H be a G-quasi-invariant probability measure on $H \backslash G$ (action on the right). Then the non-singular right Γ-action on $(H \backslash G, \sigma_H)$ is μ-harmonically ergodic if and only if H/N is infinite.*

PROOF: In view of Theorem 4.1, Corollary 2.3, and Proposition 2.4 the Γ-action on $(H \backslash G, \sigma_H)$ is μ-harmonically ergodic if and only if the product Γ-action on $G/H \times G/P$ is ergodic. By Lemma 3.2, the latter holds if and only if H/N is infinite. This proves the theorem.

4.3. COROLLARY: *Let $G = SL(n, \mathbb{R})$ and Γ be a uniform lattice in G. There exists a probability measure μ supported by Γ such that the natural action of Γ on $\mathbb{R}^n$ is μ-harmonically ergodic (with respect to the Lebesgue measure) if and only if $n \geqslant 3$.*

PROOF: Let μ be the probability measure on Γ as in Theorem 4.2. It is enough to prove that the Γ-action on $\mathbb{R}^n - (0)$ is μ-harmonically ergodic if and only if $n \geqslant 3$. The G-action on $\mathbb{R}^n - (0)$ is transitive. Hence the Γ-action on $\mathbb{R}^n - (0)$ can be identified with the action of Γ on $H \backslash G$ where H is the isotropy subgroup of any point. Now, choosing the point to be a fixed point for the N-action we may assume that H contains N. It is easy to verify (indeed just from dimension considerations) that if $n \geqslant 3$,

H/N is infinite. On the other hand for $n = 2$, $H = N$. In view of Theorem 4.2 this implies the Corollary.

4.4. COROLLARY: *Let the notations* G, Γ, H *etc. be as in Theorem* 4.2. *Assume further that* H/N *is infinite. Then the* Γ*-action on* $N\backslash G$ *is not a factor of the* Γ*-action on* $H\backslash G$. *In particular, the actions are not isomorphic.*

PROOF is obvious (as noted in the introduction).

It is conceivable that analogues of the above results are true for any semisimple Lie group in the place of $SL(n, \mathbb{R})$. For obtaining such results it would be necessary to generalise the theorem of Guivarch. It would also be interesting to know whether Theorem 4.2 is true for a non-uniform lattice Γ and in particular for the lattice $SL(n, \mathbb{Z})$. Finally, in each of the cases including the present one, it would be of interest to analyse the class of measures μ for which μ-harmonic ergodicity holds.

References

[1] R. AZENCOTT: Espaces de Poisson des Groupes Localement Compacts. Springer-Verlag, Berlin, Heidelberg, New York (1970).

[2] H. FURSTENBERG: Random walks and discrete subgroups of Lie groups. *Advances in Probability and Related Topics I* (1971).

[3] Y. GUIVARCH: Quelques proprietes asymptotiques des produits de matrices aleatoires. In: Ecole d'été de probabilitès de Saint-Flour VIII-1978. Springer-Verlag, Berlin, Heidelberg, New York (1980).

[4] C.C. MOORE: Ergodicity of flows on homogeneous spaces. *Amer J. Math. 88* (1966) 154–178.

[5] G. WARNER: Harmonic analysis of semisimple Lie groups I. Springer-Verlag, Berlin, Heidelberg, New York (1972).

[6] R.J. ZIMMER: Induced and amenable ergodic actions of Lie groups. *Ann. Sci. Ecole Norm. Sup. (4) 11* (1978) 407–428.

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