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THE EXCEPTIONAL REPRESENTATIONS OF GL_2

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The purpose of this paper is to provide a characterization of the set of exceptional supercuspidal representations of $GL_2(F)$ where F is a local field of residual characteristic p and, in particular, to provide a proof for Lemma 4.2.2 of [5].

In §1, we describe the construction of a set of supercuspidal representations of $GL_2(F)$ by the method of Weil; supercuspidal representations which cannot be constructed in this way are said to be exceptional. In §2, we show that a “Weil representation” which belongs to a ramified quadratic extension of F may be constructed by induction from a one-dimensional representation of an open subgroup of $GL_2(F)$ and we show that the inducing representation must satisfy a certain condition ((3.01)). In §3, we show that, conversely, any supercuspidal representation which is induced from a representation satisfying (3.01) is a Weil representation. In §4, we show that condition (3.01) is equivalent to that given in Lemma 4.2.2 of [5]. In what follows we denote the ring of integers in F by \mathfrak{o}_F , the maximal ideal of \mathfrak{o}_F by \mathfrak{p}_F and we set $\mathfrak{q} = [\mathfrak{o}_F : \mathfrak{p}_F]$. Other notation used here is explained in [5].

Section 1

Let E/F be quadratic and separable, let τ be the nontrivial F -automorphism of E , denote by $N_{E/F}$ and $\text{Tr}_{E/F}$ the norm and trace maps of E/F and let $\omega_{E/F}$ be the nontrivial character of the multiplicative group, F^\times , of F which is trivial on $N_{E/F}E^\times$.

Let $C_c^\infty(E)$ be the space of compactly supported, locally constant, complex-valued functions on E , let ψ be a nontrivial character of the additive group, F^+ , of F and set $\psi_{E/F} = \psi \circ \text{Tr}_{E/F}$. Then there is a unique choice of Haar measure, μ_ψ , on E^+ for which *Fourier inversion* holds with respect to $\psi_{E/F}$; that is, if we define the map $f \mapsto \hat{f}$ on $C_c^\infty(E)$ by $\hat{f}(\beta) = \int_E f(\alpha) \psi_{E/F}(\alpha\beta) d\mu_\psi(\alpha)$ then we have $\hat{\hat{f}}(x) = f(-x)$.

Now it is a consequence of the work of Weil [7] on symplectic groups

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(see [2], p. 7) that there is a representation r of $\mathrm{Sl}_2(F)$ on $C_c^\infty(E)$ such that

$$r\left(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}\right)f(\beta) = \omega_{E/F}(x)|x|_E^{1/2}f(x\beta) \quad (1.01)$$

$$r\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right)f(\beta) = \psi(yN_{E/F}\beta)f(\beta) \quad (1.02)$$

$$r\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)f(\beta) = \gamma_{E/F}\hat{f}(\beta^\tau) \quad (1.03)$$

where $\gamma_{E/F}$ is a complex number whose value may be found in Lemma 1.2 of [2].

In [2] it is shown that this representation commutes with left translations by elements α of E for which $N_{E/F}\alpha = 1$ so that $C_c^\infty(E)$ may be decomposed into a sum of $\mathrm{Sl}_2(F)$ invariant subspaces which are parametrized by characters of the subgroup $\ker N_{E/F}$ of E^\times . It is then shown that the representations of $\mathrm{Sl}_2(F)$ thus obtained are irreducible and that those representations which are parametrized by nontrivial characters of $\ker N_{E/F}$ induce to supercuspidal representations of $\mathrm{Gl}_2(F)$ whose irreducible constituents will be referred to here as *Weil representations* of $\mathrm{Gl}_2(F)$ belonging to E/F .

Cartier has observed that the Weil representations belonging to E/F may also be obtained by first inducing the representation r to $\mathrm{Gl}_2(F)$ and then decomposing the resulting representation under a certain natural action of E^\times and it is this approach, summarized in the following two lemmas, which we will use. Since this approach has been described in detail elsewhere [N] we will omit proofs.

LEMMA 1.1: *There is a unique representation \tilde{r} on the space $C_c^\infty(F^\times \times E)$ for which*

$$\tilde{r}\left(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}\right)f(z, \beta) = \omega_{E/F}(x)|x|_E^{1/2}f(z, x\beta) \quad (1.04)$$

$$\tilde{r}\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right)f(z, \beta) = \psi(yzN_{E/F}\beta)f(z, \beta) \quad (1.05)$$

$$\tilde{r}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)f(z, \beta) = \gamma_{E/F}\omega_{E/F}(z)|z|_E^{1/2}\hat{f}(z, z\beta^\tau) \quad (1.06)$$

$$\tilde{r}\left(\begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}\right)f(z, \beta) = f(zw, \beta) \quad (1.07)$$

where $f \mapsto \hat{f}$ is the Fourier transform in the second variable.

LEMMA 1.2: Let θ be a character of E^\times and let C_θ be the subspace of functions f in $C_c^\infty(F^\times \times E)$ for which $f(xN_{E/F}\alpha, \beta\alpha^{-1}) = \theta(\alpha)|\alpha|_E^{1/2}f(x, \beta)$, α in E^\times . Then C_θ is stable under \bar{r} and if θ is not of the form $\chi \circ N_{E/F}$ then C_θ is an irreducible supercuspidal $\mathrm{Gl}_2(F)$ subspace of $C_c^\infty(F^\times \times E)$.

LEMMA 1.3: Denote by $W_\psi(\theta)$ the representation of $\mathrm{Gl}_2(F)$ on C_θ obtained as above. Then $W_\psi(\theta)$ is equivalent to the representation $\pi(\theta)$ defined on page 144 of [2]. In particular, $W_\psi(\theta) = \pi(\mathrm{Ind}_{W_E \uparrow W_F} \theta)$; that is, $W_\psi(\theta)$ corresponds in the sense of Langlands to the representation $\mathrm{Ind}_{W_E \uparrow W_F} \theta$ of the Weil group, W_F , of F .

PROOF: We recall that the representation $\pi(\theta)$ is induced from a representation $\pi(\theta, \psi)$ of the subgroup $G_{E/F}$ of $\mathrm{Gl}_2(F)$ consisting of elements g in $\mathrm{Gl}_2(F)$ for which $\det g$ lies in $N_{E/F}F^\times$. $\pi(\theta, \psi)$ acts on the subspace \bar{C}_θ of functions f in $C_c^\infty(E)$ which satisfy $f(\alpha\beta) = \theta^{-1}(\alpha)f(\beta)$ for α in $N_{E/F}$ and may be characterized by the following formulae ([2], p. 11):

$$\pi(\theta, \psi) \left(\begin{bmatrix} N_{E/F}\alpha & 0 \\ 0 & 1 \end{bmatrix} \right) f(\beta) = |\alpha|_E^{1/2} \theta(\alpha) f(\alpha\beta) \quad (1.08)$$

$$\pi(\theta, \psi)(g) = r(g) \quad \text{for } g \text{ in } \mathrm{Sl}_2(F). \quad (1.09)$$

(One should note that \bar{C}_θ is invariant under r .)

By Frobenius reciprocity, it will be enough to show that \bar{C}_θ is $G_{E/F}$ -isomorphic to a subspace of C_θ . In fact, one checks easily that if C_θ^+ is the subspace of C_θ consisting of functions $f(x, \beta)$ for which $f(x, \beta) = 0$ when x is not a norm from E then C_θ^+ is the required subspace and that $f \mapsto \bar{f}$ where $\bar{f}(\beta) = f(1, \beta)$ is the required $G_{E/F}$ -isomorphism from C_θ^+ to \bar{C}_θ .

COROLLARY 1.4: The equivalence class of $W_\psi(\theta)$ is independent of ψ . If θ_1, θ_2 are characters of E^\times then $W_\psi(\theta_1)$ is equivalent to $W_\psi(\theta_2)$ if and only if either $\theta_2 = \theta_1$ or $\theta_2 = \theta_1^\tau$.

We note that a Weil representation W may belong to more than one quadratic extension of F . If W belongs to the unramified quadratic extension of F , we say that W is an *unramified* Weil representation; otherwise we call W ramified. An irreducible supercuspidal representation of $\mathrm{Gl}_2(F)$ which is not a Weil representation will be called *exceptional*.

Section 2

The goal of this section is to describe a given Weil representation as an induced representation. To this end we need some preliminaries concern-

ing the construction of supercuspidal representations by induction from open subgroups. Further details and proofs are given in [5]. Let V be the standard plane over F ; i.e., $V = F \oplus F$. Then by a *lattice flag* in V we mean a sequence $L = \dots L_{-1}, L_0, L_1, \dots$ of free, rank two \mathcal{O}_F -sub-modules of V such that $L_k \supset L_{k+1}$, $P_F L_k = L_{k+2}$ and $\dim_{\mathcal{O}_F/P} L_k/L_{k+1} = 1$. There is a natural action of the ring, $M_2(F)$, of 2×2 matrices over F on the set of lattice flags which is, in fact, transitive; if we call two lattice flags L^1 and L^2 equivalent when there exists an integer m such that $L_k^2 = L_{k+m}^1$ for all k then $M_2(F)$ acts transitively on the set of classes of flags as well.

Given a lattice flag L , we denote by $\mathfrak{b}_m(L)$ the subset of elements g in $M_2(F)$ for which $gL_k \subset L_{k+m}$ for all k ; we set $\mathfrak{b}(L) = \mathfrak{b}_0(L)$ and note that for $k \geq 0$, $\mathfrak{b}_k(L)$ is a principal two-sided ideal in $\mathfrak{b}(L)$.

We set $B(L) = \mathfrak{b}^\times(L)$ and for $k \geq 1$ set $B_k(L) = 1 + \mathfrak{b}_k(L)$. We note that for $k \geq m/2 \geq 1$, the map $x \mapsto x - 1$ induces an isomorphism of abelian groups of $B_k(L)/B_m(L)$ and $\mathfrak{b}_k(L)/\mathfrak{b}_m(L)$. We note also that the pairing of $\mathfrak{b}_k(L)/\mathfrak{b}_m(L) \times \mathfrak{b}_{1-m}(L)/\mathfrak{b}_{1-k}(L)$ into F^+/P_F given by $(x, y) \mapsto \text{tr } xy$ is nondegenerate. It follows that if ψ is a character of F^+ of conductor P_F and if for b in $\mathfrak{b}_{1-m}(L)$ we define the character ψ_b on $B_k(L)$ by $\psi_b(x) = \psi(\text{tr } b(x - 1))$ then $b \mapsto \psi_b$ induces an isomorphism of $\mathfrak{b}_{1-m}/\mathfrak{b}_{1-k}$ with the complex dual, $\overline{B_k/B_m}$, of B_k/B_m whenever $k \geq m/2$.

Let, now, π be an irreducible supercuspidal representation of $\text{Gl}_2(F)$. Call π *unramified* if it may be c -induced (see [3] for the precise definition) from the subgroup $F^\times \cdot \text{Gl}_2(\mathcal{O}_F)$ and call π *ramified* otherwise. Then it is well known (see, e.g., [1]) that a Weil representation is unramified as a Weil representation if and only if it is unramified in the above sense.

On the other hand, [3], ramified supercuspidal representations may be characterized as representations which may be induced from the normalizer, $K(L)$, of some subgroup $B(L)$ of $\text{Gl}_2(F)$ (all such subgroups are, of course, conjugate).

To be precise, call an element b in $M_2(F) \setminus \mathfrak{b}(L)$ -*generic* of level $2k + 1$ if

1. $F[x]/F$ is quadratic ramified;
2. $F[x] \cap \mathfrak{b}(L) = \mathcal{O}_{F[x]}$;
3. $v_{F[x]}(x) = 2k + 1$.

It is easy to see that x lies in $\mathfrak{b}_{2k+1}(L)$ and that, in fact, the set of $\mathfrak{b}(L)$ -generic elements of level $2k + 1$ is precisely $\Pi_L^{2k+1} B(L)$ where Π_L is any generator of the ideal $\mathfrak{b}_1(L)$ of $\mathfrak{b}(L)$.

PROPOSITION 2.1: 1. *With notation as above, let n be a positive integer and let b be a $\mathfrak{b}(L)$ -generic element of level $1 - 2n$. Let θ be a character of the subgroup $T_b = (F[b])^\times$ of $\text{Gl}_2(F)$ such that $\theta(\beta) = \psi(\text{Tr}_{F[b]/F} b(\beta - 1))$ for β in $U_{F[b]}^n$. Then the complex-valued function $\theta\psi_b$ on $T_b B_n(L)$ defined*

by $\theta\psi_b(\beta k) = \theta(\beta)\psi_b(k)$, β in T_b , k in $B_n(L)$ is in fact a well-defined character of $T_b B_n(L)$ which induces an irreducible supercuspidal representation $\pi(L; \psi_b, \theta)$ of $\mathrm{Gl}_2(F)$. We have $\pi(L; \psi_b, \theta_1) \cong \pi(L; \psi_b, \theta_2)$ if and only if $\theta_1 = \theta_2$.

2. Given an irreducible ramified supercuspidal representation π of $\mathrm{Gl}_2(F)$ and a lattice flag L there exist n, b, θ as above and a character χ of F^\times so that $\pi \cong \pi(L; \psi_b, \theta) \otimes \chi \circ \det$. If $f(\chi) \leq n$ then χ may be taken to be trivial.

PROOF: This is Proposition 3.1.1 of [5].

In order to describe a given Weil representation W as an induced representation it will be helpful to write W as $W(\theta)$ where θ enjoys certain properties. Specifically, if we denote by $f(\theta)$ the exponent of the conductor of θ and by $d(E/F)$ the exponent of the different of the extension E/F then the existence of an appropriate character θ is given by the following lemma.

LEMMA 2.2: Let W be a ramified Weil representation of $\mathrm{Gl}_2(F)$. Then there exists an extension E/F , a character θ of E^\times such that $f(\theta) \geq 2d(E/F) - 1$ and $f(\theta) - d(E/F)$ is odd, and a character χ of F^\times so that W is equivalent to the representation $W(\theta) \otimes \chi \circ \det$. If there exist E', θ', χ' with the above properties and if $E' \neq E$ then $p = 2$, $f(\theta) = 2d(E/F) - 1 = 2d(E'/F) - 1 = f(\theta')$ and $f(\omega_{E'/F} \cdot \omega_{E/F}^{-1}) = d(E/F)$.

PROOF: This follows from Corollary 1.18 of [4] and the fact that $W(\theta) = \pi(\mathrm{Ind}_{W_E \uparrow W_F} \theta)$.

In what follows, we fix a ramified quadratic extension E/F and a character θ of E^\times for which $f(\theta) - d(E/F)$ is odd and $f(\theta) \geq 2d(E/F) - 1$; we set $n(\theta) = 1/2(f(\theta) + d(E/F) - 1)$. In addition we fix a character ψ of F^+ of conductor P_F which if $p = 2$ has the additional property that $\psi(x^2 + x) = 1$ for x in \mathcal{O}_F . We denote by $b = b_\psi(\theta)$ an element of E for which $\theta(\beta) = \psi(\mathrm{Tr}_{E/F} b(\beta - 1))$ for β in $U_E^{[(f(\theta)+1)/2]}$ and by $c_\psi = c_\psi(E/F)$ an element of F for which $\omega_{E/F}(x) = \psi(c_\psi(x - 1))$ for x in $U_F^{[k(E/F)+1/2]}$.

Finally, we fix a lattice flag L^n , $n = n(\theta)$, by setting $L_0^n = P_F^{1-n} \oplus \mathcal{O}_F$; $L_1^n = P_F^{1-n} \oplus P_F$. We note that then

$$\mathfrak{b}_{2k}(L^n) = P_F^k \begin{bmatrix} \mathcal{O}_F & P_F^{1-n} \\ P_F^n & \mathcal{O}_F \end{bmatrix}; \quad \mathfrak{b}_{2k+1}(L^n) = P_F^k \begin{bmatrix} P_F & P_F^{1-n} \\ P_F^n & P_F \end{bmatrix}.$$

PROPOSITION 2.3: With notation as above, define the function f_0 in the space C_θ by $f_0(x, \beta) = \theta^{-1}(\beta)|\beta|_E^{-1/2}$ if $xN_{E/F}\beta$ lies in $U_F^{[(n+1)/2]}$, $f_0(x,$

$\beta) = 0$ otherwise. Then for k in $B_n(L^n)$ we have that

$$W(\theta)(k)f_0 = \psi_{\bar{b}}(k)f_0$$

where

$$\bar{b} = \begin{bmatrix} 0 & -N_{E/F}b \\ 1 & \text{Tr}_{E/F}b + c_\psi \end{bmatrix}.$$

PROOF: It is a straightforward computation, using formulae (1.04), (1.05) and (1.07), that

$$W(\theta)(k)f_0 = \psi_{\bar{b}}(k)f_0$$

when k lies in $B_n(L^n)$ and is upper triangular. Our result will thus follow if we show that

$$W(\theta) \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} f_0 = \psi(-yN_{E/F}b)f_0$$

when b lies in $P_F^{n+[n/2]}$. Since

$$\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

it will suffice, by (1.05), (1.06), to show that if $\hat{f}_0(z, z\beta^\tau) \neq 0$ then $\psi(-yzN_{E/F}\beta) = \psi(-yN_{E/F}b)$; that is, to show that the support of the function $\hat{f}_0(z, z\beta^\tau)$ is contained in the set of (z, β) for which $zN_{E/F}(\beta b^{-1})$ lies in $U_F^{[(n+1)/2]}$.

Now we have that

$$\hat{f}_0(z, z\beta^\tau) = \int_Y \theta^{-1}(\alpha) |\alpha|_E^{-1/2} \psi_{E/F}(\alpha z\beta^\tau) d\mu_\psi(\alpha)$$

where Y is the set of α for which $N_{E/F}\alpha$ lies in $z^{-1}U^{[(n+1)/2]}$. Since $f(\theta) = 2n - d(E/F) + 1 \geq d(E/F)$ we have that $N_{E/F}(U_E^{[(f(\theta)+1)/2]}) \subset U_F^{[(n+1)/2]}$ and thus that $\hat{f}_0(z, z\beta^\tau)$ is a nonzero multiple of

$$\begin{aligned} & \int_{P_E^{[(f(\theta)+1)/2]}} \int_Y \theta^{-1}(\alpha(1+\gamma)) |\alpha|^{-1/2} \\ & \quad \times \psi_{E/F}(\alpha(1+\gamma)z\beta^\tau) d\mu_\psi(\alpha) d\mu_\psi(\gamma) \\ & = \int_Y \theta^{-1}(\alpha) |\alpha|_E^{-1/2} \psi_{E/F}(\alpha z\beta^\tau) \\ & \quad \times \int_{P_E^{[(f(\theta)+1)/2]}} \psi_{E/F}((\alpha z\beta^\tau - b)\gamma) d\mu_\psi(\gamma) d\mu_\psi(\alpha) \\ & = 0 \end{aligned}$$

unless $\alpha z \beta^\tau - b$ lies in $P_E^{2-d(E/F)-[(f(\theta)+1)/2]}$, that is, unless $\alpha z \beta^\tau b^{-1}$ lies in $U_E^{[(f(\theta)+1)/2]}$. (Here, one uses the fact that $\nu_E(b) = 1 - 2n$ so that $2 - d(E/F) - [(f(\theta) + 1)/2] - \nu_E(b) = f(\theta) - [(f(\theta) + 1)/2] = [f(\theta)/2]$.) Finally, since $zN_{E/F}\alpha$ lies in $U_F^{[(n+1)/2]}$ and since, in general, $N_{E/F}U_E^r \subset U_F^s$ where $s = \min([(r + d(E/F))/2], r)$ one checks that $\hat{f}_0(z, z\beta^\tau) = 0$ unless $zN_{E/F}(\beta b^{-1})$ lies in $U_F^{[(n+1)/2]}$.

COROLLARY 2.4: *With notation as above, there exists a character $\bar{\theta}$ of $T_{\bar{b}}$ such that $W(\theta)$ is equivalent with $\pi(L^n; \psi_{\bar{b}}, \bar{\theta})$.*

PROOF: We note first that \bar{b} is $\mathfrak{b}(L^n)$ -generic of level $1 - 2n$ since $\nu_F(\mathrm{Tr}_{E/F}b + c_\psi) \geq \min(1 - n, 1 - d(E/F)) = 1 - n$. Next, since $\psi_{\bar{b}}$ is stable under $T_{\bar{b}}B_n(L^n)$, the span under $T_{\bar{b}}B_n(L^n)$ of f_0 decomposes into a sum of the form $\oplus \langle f_{\bar{\theta}_j} \rangle$ where $\bar{\theta}_j$ is a character of $T_{\bar{b}}$ of the form described in Proposition 2.1 and where $W(\theta)(h)f_{\bar{\theta}_j} = \theta_j \psi_{\bar{b}}(h)f_{\bar{\theta}_j}$ for h in $T_{\bar{b}}B_n(L^n)$. Finally, since distinct characters $\theta_j \psi_{\bar{b}}$ induce distinct irreducible supercuspidal representations of $\mathrm{Gl}_2(F)$, we see that the span under $T_{\bar{b}}B_n(L^n)$ of f_0 is one-dimensional, that we may set $\bar{\theta} = \bar{\theta}_1$ whence $f_{\bar{\theta}_1} = f_0$, and $W(\theta)$ is equivalent to $\pi(L^n; \psi_{\bar{b}}, \bar{\theta})$.

Section 3

In this section we fix, once and for all, an integer $n \geq 1$ and a $\mathfrak{b}(L^n)$ -generic element, \bar{b} , of level $1 - 2n$. Our goal is to determine whether some or all of the representations $\pi(L^n; \psi_{\bar{b}}, \theta)$ are Weil representations. From Proposition 2.3, it is clear that in order that some representation $\pi(L^n; \psi_{\bar{b}}, \theta)$ be Weil it is necessary that there exist a ramified quadratic extension E/F with $3d(E/F) \leq 2(n+1)$ and an element b in E with $\nu_E(b) = 1 - 2n$ such that

$$\begin{aligned} i. \quad \mathrm{tr} \bar{b} &\equiv \mathrm{Tr}_{E/F}b + c_\psi(E/F) \pmod{P_F^{-(n-1)/2}} \\ ii. \quad (\det \bar{b})/N_{E/F} &\equiv 1 \pmod{P_F^{[(n+1)/2]}}. \end{aligned} \tag{3.01}$$

We will say that such an element \bar{b} is *Weil-generic*. Our main result in this section is

PROPOSITION 3.1: *The representation $\pi(L^n; \psi_{\bar{b}}, \theta)$ is Weil if and only if \bar{b} is Weil-generic.*

We will need several lemmas.

LEMMA 3.2: *Suppose that the pair (E, b) satisfies condition (3.01). Let E_1/F be ramified quadratic and suppose for some b_1 in E_1 we have*

$\mathrm{Tr}_{E_1/F} b_1 \equiv \mathrm{Tr}_{E/F} b \pmod{P_F^{-(n-1)/2}}$, $N_{E_1/F} b_1 / N_{E/F} b \equiv 1 \pmod{P_F^{[(n+1)/2]}}$. Then the pair (E_1, b_1) satisfies condition (3.01).

PROOF: We must show that $c_\psi(E/F) \equiv c_\psi(E_1/F) \pmod{P_F^{-(n-1)/2}}$. To begin with, we note that since $2(n+1) \geq 3d(E/F)$ it follows that $-[(n-1)/2] > \frac{1}{2}d(E/F) - n$. In addition, we have that

$$d(E/F) = \min\left(2\left(\nu_F(\mathrm{Tr}_{E/F} b) + n\right), 2\nu_F(2) + 1\right),$$

$$d(E_1/F) = \min\left(2\left(\nu_F(\mathrm{Tr}_{E_1/F} b_1) + n\right), 2\nu_F(2) + 1\right).$$

One may then deduce from the congruence $\mathrm{Tr}_{E_1/F} b_1 \equiv \mathrm{Tr}_{E/F} b \pmod{P_F^{-(n-1)/2}}$ that $d(E_1/F) = d(E/F)$.

Now since $-[(n-1)/2] \leq 1 - [(d(E/F) + 1)/2]$, the congruence $c_\psi(E/F) \equiv c_\psi(E_1/F) \pmod{P_F^{-(n-1)/2}}$ is equivalent to the statement that the restrictions of $\omega_{E/F}$ and $\omega_{E_1/F}$ to $U_F^{[(n+1)/2]}$ coincide. However $\omega_{E/F}|_{U_F^{[(n+1)/2]}}$ is determined by the data $f(\omega_{E/F}) = d(E/F)$, $\omega_{E/F}^2 = 1$, $\omega_{E/F}(1+x \mathrm{Tr}_{E/F} b + x^2 N_{E/F} b) = 1$ for x with $2\nu_F(x) \geq 2n - 1 + [(n+1)/2]$. Since

$$\begin{aligned} & \frac{1}{2}(2n-1) + [(n+1)/2] + \nu_F\left(\mathrm{Tr}_{E_1/F} b_1 - \nu_F(\mathrm{Tr}_{E/F} b)\right) \\ & \geq d(E/F); \end{aligned}$$

$$2n - 1 + [(n+1)/2] + \nu_F\left(N_{E_1/F} b_1 - N_{E/F} b\right) \geq d(E/F),$$

we see that $\omega_{E_1/F}$ satisfies the above data, whence our result.

Let E/F be quadratic ramified with $3d(E/F) \leq 2(n+1)$ and let b be an element of E with $\nu_E(b) = 1 - 2n$. Denote by $W(E; b)$ the set of representations $W(\theta)$ where θ is a character of E^\times such that $\theta(\beta) = \psi(\mathrm{Tr}_{E/F} b(\beta - 1))$ for β in $U_F^{[(2n-d(E/F)+2)/2]}$ and $\theta(\tilde{\omega}_F)\omega_{E/F}(\pi_F) = 1$ for some fixed prime element $\tilde{\omega}_F$ of F .

LEMMA 3.3: Let $m = \lfloor \frac{1}{2}(2n - d(E/F) + 2) \rfloor$. Then $W(E; b)$ consists of $(q-1)q^{m-1}$ distinct representations if $3d(E/F) < 2(n+1)$ and $\frac{1}{2}(q-1)q^{m-1}$ distinct representations if $3d(E/F) = 2(n+1)$.

PROOF: This follows from the fact that $[U_E : U_E^m] = (q-1)q^{m-1}$ together with Corollary 1.4 and the fact that $b \equiv b^\tau \pmod{P_E^{2-d(E/F)-m}}$ if and only if $-2n + d(E/F) \geq 2 - d(E/F) - m$, that is, if and only if $3d(E/F) \geq 2(n+1)$.

LEMMA 3.4: Let S be the subgroup of $F \times F^\times$ consisting of pairs (x, y) with x in $\mathrm{Tr}_{E/F} P_E^{1-n-[(d(E/F)/2]}$ and y in $N_{E/F} U_E^{n-[(d(E/F)/2]}$. Suppose E_1, E_2 are

ramified quadratic extensions of F , b_i lies in E_i and $\text{Tr}_{E_i/F} b_i = \text{Tr}_{E/F} b$ (mod $P_F^{-(n-1)/2}$); $N_{E_i/F} b_i / N_{E/F} b \equiv 1 \pmod{P_F^{(n+1)/2}}$. Suppose further that $(\text{Tr}_{E_1/F} b_1, N_{E_1/F} b_1) \not\equiv (\text{Tr}_{E_2/F} b_2, N_{E_2/F} b_2) \pmod{S}$. Then $W(E_1, b_1)$ and $W(E_2, b_2)$ are disjoint sets.

PROOF: It was shown in Lemma 3.2 that $d(E_1/F) = d(E_2/F) \geq \frac{2}{3}(n+1)$ and that if $d(E_i/F) = \frac{2}{3}(n+1)$ then $f(\omega_{E_i/F} \omega_{E_2/F}^{-1}) < d(E_1/F)$. It follows by Lemma 2.2 that $W(E_1, b_1)$ and $W(E_2, b_2)$ are disjoint unless $E_1 = E_2$.

Suppose now that $E_1 = E_2$, that $W(\theta_1)$ lies in $W(E_i, b_i)$ and that $W(\theta_1)$ is equivalent with $W(\theta_2)$. By Corollary 1.4, there exists an element ν in the galois group of E_1/F such that

$$b_1 \equiv b_2^\nu \pmod{P_E^{1-n-d(E/F)/2}}$$

which contradicts our hypothesis.

LEMMA 3.5: $[P_E^{-(n-1)/2} \times U_F^{[(n-1)/2]}; S] = q^{\lfloor 1/2(d(E/F)-1) \rfloor}$ if $2(n+1) > 3d(E/F)$; $[P_E^{-(n-1)/2} \times U_F^{[(n+1)/2]}; S] = 2q^{\lfloor 1/2(d(E/F)-1) \rfloor}$ if $2(n+1) = 3d(E/F)$.

PROOF: Straightforward.

PROOF OF PROPOSITION 3.1: Suppose that \bar{b} is Weil-generic. Then \bar{b} is $K(L^n)$ conjugate to

$$\begin{bmatrix} 0 & -\det \bar{b} \\ 1 & \text{tr } \bar{b} \end{bmatrix}$$

and we have thus produced, by Lemmas 3.3, 3.4, 3.5, $(q-1)q^{n-1}$ distinct irreducible Weil summands of $\text{Ind}_{B_n(L^n) \uparrow_{G_{12}(F)}} \psi_{\bar{b}}$ each having central character which is trivial at $\tilde{\omega}_F$. On the other hand, the total number of such summands is

$$[U_{F[\bar{b}]} B_n(L^n) : B_n(L^n)] = [U_{f[\bar{b}]} : U_{f[\bar{b}]}] = (q-1)q^{n-1}.$$

Since given any representation $\pi(L^n; \psi_{\bar{b}}, \theta)$ we may find a character χ of F^\times such that $f(\chi) = 0$ and $\pi(L^n; \psi_{\bar{b}}, \theta) \otimes \chi \circ \det$ has a central character trivial on $\tilde{\omega}_F$ we have shown that all representations $\pi(L^n; \psi_{\bar{b}}, \theta)$ are Weil representations.

Section 4

The purpose of this section is to prove the following proposition which gives a simple characterization of the property of being Weil-generic.

PROPOSITION 4.1: Fix $n \geq 1$ and let L^n be the lattice flag described in §3. Let \bar{b} be $\mathfrak{b}(L^n)$ -generic of level $1 - 2n$ and set $\bar{E} = F(\bar{b})$. Then the following are equivalent.

1. \bar{b} is Weil-generic.
2. Either $2(n+1) > 3d(\bar{E}/F)$ or the polynomial $X^3 - (\text{tr } \bar{b})X^2 + \det \bar{b}$ has a root in F .
3. There exists a ramified quadratic extension E/F with $3d(E/F) \leq 2(n+1)$ and an element b in E with $N_{E/F}b = \det \bar{b}$ and $\text{Tr}_{E/F}b + c_\psi(E/F) \equiv \text{tr } \bar{b} \pmod{P_F^{d(E/F)/2+1-n}}$.

PROOF: $1 \Rightarrow 2$. Suppose that \bar{b} is Weil-generic and that $2(n+1) \leq 3d(\bar{E}/F)$. Pick b in E satisfying (3.01). We show first that $3d(E/F) = 2(n+1)$. Suppose that $d(E/F)$ is odd. Then since, by assumption, $3d(E/F) \leq 2(n+1)$ we must have $3d(E/F) < 2(n+1)$. Now (see Lemma 3.2), $d(E/F) = \min(2(\nu_F(\text{Tr}_{E/F}b) + n), 2\nu_F(2) + 1)$ so that $2\nu_F(2) + 1 = d(E/F) < 2(n+1)/3$ and also $\nu_F(\text{Tr}_{E/F}b) \geq \nu_F(2) + 1 - n$. By (3.01) - i ,

$$\nu_F(\text{tr } \bar{b}) \geq \min(\nu_F(2) + 1 - n, -2\nu_F(2), -[(n-1)/2]).$$

However, from $2\nu_F(2) + 1 < 2(n+1)/3$ we obtain that $\nu_F(2) + 1 - n \leq -2\nu_F(2)$ while $\nu_F(2) + 1 - n \leq -[(n-1)/2]$ since $n \geq d(E/F) = 2\nu_F(2) + 1$. Thus $\nu_F(\text{tr } \bar{b}) \geq \nu_F(2) + 1 - n$ whence $d(\bar{E}/F) = 2\nu_F(2) + 1 = d(E/F)$. Therefore $3d(\bar{E}/F) < 2(n+1)$ which is false.

Now suppose that $d(E/F)$ is even so that $d(E/F) = 2(\nu_F(\text{Tr}_{E/F}b) + n) \leq 2\nu_F(2)$. Then if $3d(E/F) < 2(n+1)$, we would have $\nu_F(\text{Tr}_{E/F}b) = \frac{1}{2}d(E/F) - n < 1 - d(E/F) = \nu_F(c_\psi(E/F))$. Since $\frac{1}{2}d(E/F) - n \leq -[(n-1)/2]$ it would follow that $\nu_F(\text{Tr}_{\bar{E}/F}\bar{b}) = \frac{1}{2}d(E/F) - n$ whence $d(\bar{E}/F) = d(E/F) < \frac{2}{3}(n+1)$. Thus we have shown that $3d(E/F) = 2(n+1)$ and we note that $d(E/F)$ is even.

Now by definition,

$$\begin{aligned} 1 &= \psi\left(c_\psi(E/F)(N_{E/F}(1+xb) - 1)\right) \\ &= \psi\left(c_\psi(E/F)(x \text{Tr}_{E/F}b + x^2N_{E/F}b)\right) \end{aligned}$$

for x in $P_F^{d/2+n-1}$. Setting $x = y \text{Tr}_{E/F}b / N_{E/F}b$ and noting that $\nu_F(\text{Tr}_{E/F}b) = \frac{1}{2}d(E/F) - n$ while $\nu_F(N_{E/F}b) = 1 - 2n$ we see that

$$\psi\left(\left(c_\psi(E/F)(\text{Tr}_{E/F}b)^2 / N_{E/F}b\right)(y + y^2)\right) = 1$$

for y in \mathcal{O}_F . Since ψ has been picked so that $\psi(y + y^2) = 1$ for y in \mathcal{O}_F ($p=2$ here since $d(E/F)$ is even) we see that $c_\psi(E/F)(\text{Tr}_{E/F}b)^2 /$

$N_{E/F}b \equiv 1 \pmod{P_F}$. By (3.01) it follows that $X = \mathrm{Tr}_{E/F}b$ satisfies the congruence $X^3 - \mathrm{tr} \bar{b} X^2 + \det \bar{b} \equiv 0 \pmod{P_F^{1-2n}}$. A Hensel's lemma argument now shows that the polynomial $X^3 - \mathrm{tr} \bar{b} X^2 + \det \bar{b}$ has a root in F .

$2 \Rightarrow 3$. If $2(n+1) > 3d(\bar{E}/F)$ then $v_F(c_\psi(\bar{E}/F)) = 1 - d(\bar{E}/F) \geq [d(\bar{E}/F)/2] + 1 - n$ and so we may take $E = \bar{E}$, $b = \bar{b}$.

Now suppose that $2(n+1) \leq 3d(\bar{E}/F)$ and let s be a root in F of the polynomial $X^3 - (\mathrm{tr} \bar{b})X^2 + \det \bar{b}$. Then since $v_F(\mathrm{tr} \bar{b}) \geq [(d(\bar{E}/F) + 1)/2] - n$ while $v_F(\det \bar{b}) = 1 - 2n$, a standard argument shows that $v_F(s) = \frac{1}{3}(1 - 2n) \leq v_F(2) - n$. It follows that the polynomial $X^2 - sX + \det \bar{b}$ is irreducible over F and that if E/F is a splitting field then $3d(E/F) = 2(n+1)$. Let b be a root in E of the polynomial $X^2 - sX + \det \bar{b}$. Then since $d(E/F)$ is even we obtain, as above, that $c_\psi(E/F)(\mathrm{Tr}_{E/F}b)^2/N_{E/F}b \equiv 1 \pmod{P_F}$ whence $c_\psi(E/F) \equiv N_{E/F}b/(\mathrm{Tr}_{E/F}b)^2 \pmod{P_F^{1+(1/3)(1-2n)}}$. Finally, $N_{E/F}b = \det \bar{b}$ while $\mathrm{Tr}_{E/F}b$ satisfies $X^3 - \mathrm{Tr} \bar{b} X^2 + \det \bar{b} = 0$ so that $N_{E/F}b/(\mathrm{Tr}_{E/F}b)^2 = \mathrm{tr} \bar{b} - \mathrm{Tr}_{E/F}b$. Combining this last equation with the congruence preceding it and noting that $1 + \frac{1}{3}(1 - 2n) = [d(E/F)/2] + 1 - n$ yields our result.

$3 \Rightarrow 1$. Set $d = d(E/F)$ and suppose by induction that for $1 \leq j \leq k$ we have picked quadratic extensions E_j/F and elements b_j in E_j such that $d(E_j/F) = d$, $N_{E_j/F}b_j = \det b$ and $\mathrm{Tr}_{E_j/F}b_j + c_\psi(E_j/F) \equiv \mathrm{tr} \bar{b} \pmod{P_F^{[d/2]-n+j}}$. Set $\bar{s} = \mathrm{tr} \bar{b}$, $s_k = \mathrm{Tr}_{E_k/F}b_k$, $\Delta = \det b$, let a be an element of $P_F^{[d/2]-n+k}$ and set $s_a = s_k + a$. Let E_a be a splitting field of $X^2 - s_a X + \Delta$ over F and pick a root, b_a , of this polynomial in E_a .

Now since $v_F(s_a - s_k) \geq [d/2] - n + k$ and since $d(E_k/F) = \min(2(v_F(s_k) + n), 2v_F(2) + 1)$ while $d(E_a/F) = \min(2(v_F(s_a) + n), 2v_F(2) + 1)$ it follows that $d(E_a/F) = d(E_k/F) = d$. Since $v_F(s_a - s_k) \geq [d/2] - n + k$ while $N_{E_k/F}b_k = N_{E_a/F}b_a$ it follows that $U_F^l \cap N_{E_k/F}E_k^x = U_F^l \cap N_{E_a/F}E_a^x$ where $l = \max 2[(d+1)/2] - 2k, [(d+1)/2]$ and thus that $c_\psi(E_a/F) \equiv c_\psi(E_k/F) \pmod{P_F^{1-l}}$.

Since $1 + 2k - 2[(d+1)/2] \geq [d/2] - n + k + 1$ when $k \geq 1$ while $1 - [(d+1)/2] \geq [d/2] - n + k + 1$ when $k \leq n - d$ we see that if we set $a = \bar{s} - c_\psi(E_k/F) - s_k$, $b_{k+1} = b_a$, $E_{k+1} = E_a$ then the pair (b_{k+1}, E_{k+1}) satisfies our inductive hypothesis whenever $k \leq n - d$. Finally since $-[(n-1)/2] + n - [(d)/2] \leq n - d + 1$ we see that we may find (b_k, E_k) as above for $k = [(n-1)/2] + n - [(d)/2]$. The pair (b_k, E_k) then satisfies (3.01) whence \bar{b} is Weil-generic.

We may now state our main result.

THEOREM 4.2: *Let π be an irreducible ramified supercuspidal representation of $\mathrm{Gl}_2(F)$ and let L be a lattice flag. Pick n, b, θ as in Proposition 2.1 so that $\pi \cong \pi(L; \psi_b, \theta) \otimes \chi \circ \det$ and set $E = F(b)$. Then π is an exceptional representation of $\mathrm{Gl}_2(F)$ if and only if $2(n+1) \leq 3d(E/F)$ and the polynomial $X^3 - (\mathrm{tr} b)X^2 + \det b$ is irreducible over F .*

PROOF: Propositions 3.1, 4.1.

We note, in conclusion, that we obtain as a consequence

COROLLARY 4.3: $GL_2(F)$ has no exceptional representations unless $p = 2$.

PROOF: If $p \neq 2$, then we have $d(E/F) = 1$ for all quadratic ramified extensions E/F . Since $n \geq 1$ in all cases we have that $2(n + 1) > 3d(E/F)$.

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