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EXAMPLES OF NON-AMPLE NORMAL BUNDLES

Norman Goldstein

§1. Introduction

Let $Z = Gr(1, \mathbb{P}_\mathbb{C}^3)$ be the Grassmannian of 1 planes in \mathbb{P}^3 , embedded as a quadric hypersurface in \mathbb{P}^5 . In this note, I construct in Z two smooth surfaces, $X_3 = \mathbb{P}^2$ blown up at one point and $X_4 = \mathbb{P}^1 \times \mathbb{P}^1$, of degrees 3 and 4 respectively, such that for each, the normal bundle in Z is not ample. This is in apparent contradiction to a result of Papantonopoulou [4], stating that any smooth surface in Z , having a non-ample normal bundle, must be a linear \mathbb{P}^2 .

I do know of other examples, other than linear \mathbb{P}^2 's, although, I will not discuss these, here. The reader is referred to Hartshorne [3] for the definition of ampleness.

In a future paper, I will discuss the motivation for these constructions. It is described, briefly, below, and involves a geometric interpretation of the tangent and cotangent bundles of Z ; this generalizes to the r -dimensional quadric $Z^r \subset \mathbb{P}^{r+1}$. Also, the problem of which m -dimensional submanifolds $X^m \subset Z^r$ ($m \geq 2$) have non-ample normal bundles reduces, largely, via hyperplane sections, to the study of smooth surfaces in Z^{r-m+2} , whose normal bundles are not ample.

We consider the 4-dimensional quadric, Z . Let $\mathbb{P}(T^*Z) = (T^*Z \setminus Z)/\mathbb{C}^*$. The sections of TZ induce a map

$$\phi: \mathbb{P}(T^*Z) \rightarrow \mathbb{P}^n.$$

Let $X \subset Z$ be a surface, and NX the normal bundle of X . Let ϕ' denote the restriction of ϕ to NX . The normal bundle, NX , is ample when ϕ' is finite to 1. So, NX can fail to be ample in 3 ways:

- (a) $\dim \phi'(\mathbb{P}(N^*X)) = 2$
- (b) $\dim \phi'(\mathbb{P}(N^*X)) = 3$ and ϕ' has an infinity of positive dimensional fibres, or
- (c) $\dim \phi'(\mathbb{P}(N^*X)) = 3$ and ϕ' has only a finite number of positive dimensional fibres.

All these possibilities can occur. The possibility (a) happens when X is a linear \mathbb{P}^2 (cf [4]), and (b) when $X = X_4$, as in §3; in a future paper, we will see that for (a) and (b), that these are the only possibilities. Finally,

the example $X = X_3$ in §4 shows that (c) can occur with a *single* positive dimensional fibre.

I would like to thank Andrew Sommese for suggesting that I look at this topic, and for ways of viewing the problem. Towards completing the characterization of X_4 , I had helpful conversations with Gary Kennedy, Daniel Phillips and Avinash Sathaye. Also, the referee's suggestions have improved the presentation of this paper.

2. Notation and background material

(2.1) Let A be a submanifold of the manifold B . I denote the normal bundle of A in B as $N(B/A) := TB/TA$. Its dual, the conormal bundle, is

$$N^*(B/A) = \{\alpha \in T^*B : \alpha = 0 \text{ on } TA\}.$$

(2.2) Let $[z_0, \dots, z_5]$ be homogeneous coordinates in \mathbb{P}^5 , and let $\mathcal{O}(-1)$ be the tautological line bundle; it is the one for which the transition function from the patch $z_i \neq 0$ to the patch $z_j \neq 0$ is given by multiplication by $z_j z_i^{-1}$. We consider the well-known Euler sequence (see e.g. [2] p. 409).

$$0 \rightarrow T^*(\mathbb{P}^5) \rightarrow \mathcal{O}(-1)^{\oplus 6} \rightarrow \mathcal{O} \rightarrow 0.$$

If $\alpha = (\alpha_0, \dots, \alpha_5) \in T_z^*(\mathbb{P}^5)$ is represented in $z_i \neq 0$ by $a = (a_0, \dots, a_5) \in \mathbb{C}^6$, then in $z_j \neq 0$ α is represented by $z_j z_i^{-1} a$.

(2.3) Let Y be a projective subvariety of \mathbb{P}^5 and $E \rightarrow Y$ a holomorphic vector bundle spanned by global sections. Then, according to Gieseker ([1] Proposition 2.1), E is not ample precisely when there exists a curve $C \subset Y$ and a trivial bundle $\mathcal{O}_C \subset E^*|_C$.

A "line" in \mathbb{P}^5 denotes a linear \mathbb{P}^1 .

§3. The surface X_4

Let $\gamma_1, \gamma_2 : \mathbb{P}^1 \rightarrow \mathbb{P}^5$

$$\gamma_1(s, t) = (s^2, 2st, 2t^2, 0, 0, 0)$$

$$\gamma_2(s, t) = (0, 0, 0, s^2, 2st, 2t^2).$$

Let $X = X_4$ be the scroll surface in \mathbb{P}^5 determined by γ_1 and γ_2 , i.e. X is the union of the lines obtained by joining corresponding points of γ_1 and γ_2 . Since $\gamma_1(\mathbb{P}^1)$ and $\gamma_2(\mathbb{P}^1)$ are contained in disjoint linear spaces, X is smooth. Also, the equation for $Z \subset \mathbb{P}^5$ is $z_0 z_5 - z_1 z_4 + z_2 z_3 = 0$, so that $X \subset Z$, as is easily checked.

Consider the exact sequence of conormal spaces (see (2.1) for notation)

$$0 \rightarrow N^*(\mathbb{P}^5/Z) \rightarrow N^*(\mathbb{P}^5/X) \rightarrow N^*(Z/X) \rightarrow 0.$$

Let ℓ be any line of the ruling of X . We construct a “section”

$$\sigma: \ell \rightarrow N^*(\mathbb{P}^5/X)|_{\ell}$$

which is both well defined and nowhere zero, modulo $N^*(\mathbb{P}^5/Z)$. This, then, determines a nowhere zero section

$$\sigma: \ell \rightarrow N^*(Z/X)|_{\ell}$$

i.e. a trivial bundle $\mathcal{O}_{\ell} \subset N^*(Z/X)|_{\ell}$ so that, by (2.3), $N(Z/X)$ is not ample. We remark that $N(Z/X)$ is spanned by global sections since TZ is spanned by global sections (Z is a homogeneous space), and $N(Z/X)$ is a quotient bundle of TZ .

Construction of σ : Let $\mathbb{C} \times \mathbb{P}^1 \hookrightarrow X$ be the patch $(t, [\lambda, \mu]) \mapsto (\lambda, 2t\lambda, 2t^2\lambda, \mu, 2t\mu, 2t^2\mu)$. The tangent \mathbb{P}^2 to X at $(t, [\lambda, \mu])$ is spanned by $(1, 2t, 2t^2, 0, 0, 0)$, $(0, 0, 0, 1, 2t, 2t^2)$ and $(0, \lambda, 2t\lambda, 0, \mu, 2t\mu)$.

Consider the line ℓ corresponding to some fixed t . On the patch $\lambda \neq 0$, let $\sigma = (2t^2, -2t, 1, 0, 0, 0)$. Then, $\sigma \in N^*_{(t, [\lambda, \mu])}(\mathbb{P}^5/X)$ since σ vanishes on the tangent \mathbb{P}^2 to X . On $\mu \neq 0$, by (2.2), $\sigma = \mu\lambda^{-1}(2t^2, -2t, 1, 0, 0, 0)$. But, $N^*(\mathbb{P}^5/Z)$ is spanned by $(z_5, -z_4, z_3, z_2, -z_1, z_0) = (2t^2\mu, -2t\mu, \mu, 2t^2\lambda, -2t\lambda, \lambda)$. Thus $\sigma = -(0, 0, 0, 2t^2, -2t, 1)$ modulo $N^*(\mathbb{P}^5/Z)$ and is, therefore, a global nowhere zero section of $N^*(Z/X)$ over ℓ . Q.E.D.

Except for linear \mathbb{P}^2 's and up to an automorphism of Z , X_4 is the unique surface in Z with a non-ample normal bundle, and which contains a positive dimensional family of curves, along each of which $N(Z/X)$ is not ample. The proof of this will appear elsewhere.

§4. The surface X_3

Let $\delta_1, \delta_2: \mathbb{P}^1 \rightarrow \mathbb{P}^5$,

$$\delta_1(s, t) = (s, t, 0, 0, 0, 0)$$

$$\delta_2(s, t) = (0, 0, 0, s^2, st, t^2)$$

and $X = X_3 \subset Z$ the scroll surface determined by δ_1 and δ_2 . The line $t = 0$ is the only curve $l \subset X$ for which there is a trivial bundle $\mathcal{O}_{\ell} \subset N^*(Z/X)|_{\ell}$.

I defer the proof of this to a future paper. The construction of σ is similar to that in §3 for X_4 :

on $\lambda \neq 0$, $\sigma = (0, 0, 0, 0, 0, 1)$, and

on $\mu \neq 0$, $\sigma = -(0, 0, 1, 0, 0, 0)$.

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