COMPOSITIO MATHEMATICA

XAVIER BENVENISTE

On the fixed part of certain linear systems on surfaces

Compositio Mathematica, tome 51, nº 2 (1984), p. 237-242
<http://www.numdam.org/item?id=CM_1984__51_2_237_0>

Compositio Mathematica 51 (1984) 237–242

ON THE FIXED PART OF CERTAIN LINEAR SYSTEMS ON SURFACES

Xavier Benveniste

First we recall what is a numerically positive divisor:

DEFINITION: Let V be a smooth projective variety and D be a divisor on V. We say that D is numerically positive if for any curve C on V we have $D.C \geqslant 0$.

Let R be a numerically positive divisor on a smooth projective surface over an algebraically closed field of any characteristic such that $R^2 > 0$; let $\mathscr{A}$ be the set of irreducible curves C such that $R.C = 0$ (we will show that $\mathscr{A}$ is finite). The aim of this paper is to prove the following result:

PROPOSITION: *Let ξ be a connected component of $\bigcup_{C \in \mathscr{A}} C$; let Z be the fundamental cycle associated to ξ. If $H^1(Z, \mathcal{O}_Z) = 0$, then ξ is not a fixed component of $|nR|$ for sufficiently large n.*

Let us make a few comments. The first one is that the condition $H^1(Z, \mathcal{O}_Z) = 0$ characterizes rational singularities. The second is that this proposition generalizes slightly a well known theorem of Zariski ([1] theorems 6-1, 6-2).

PROOF: Let ξ be as in the proposition and let $(C_i)_{i \in \{1,\dots,m\}}$ the set of irreducible components of ξ. Then:

LEMMA 1: *The set $\mathscr{A}$ is finite and the bilinear symmetric form defined by the matrix $(C_i \cdot C_j)_{i,j \in \{1,\dots,m\}}$ is negative definite.*

PROOF: We shall show that if $(E_i)_{i \in \{1,\dots,n\}}$ is a finite family of elements of $\mathscr{A}$, then the classes $[E_i]$ of the E_i are $\mathbf{Q}$-linearly independent in $NS(S) \otimes_{\mathbf{Z}} \mathbf{Q}$. Assume the contrary. Any relation of dependence between the $[E_i]$ in $NS(S) \otimes_{\mathbf{Z}} \mathbf{Q}$ can be written:

$$\sum_{i \in I} a_i [E_i] \sim \sum_{i \in I'} a_i [E_i],$$

where I and I' are two non-void disjoint subsets of $\{1,\dots,n\}$ and, for any $i \in I \cup I'$, $a_i \in \mathbb{N} - \{0\}$. Because $R \cdot \sum_{i \in I} a_i E_i = 0$, the index theorem on S implies:

$$\left(\sum_{i \in I} a_i E_i \right)^2 \leqslant 0.$$

But

$$\left(\sum_{i \in I} a_i E_i \right)^2 = \left(\sum_{i \in I} a_i E_i \right) \cdot \left(\sum_{i \in I'} a_i E_i \right) \geqslant 0.$$

Again by the index theorem on S we have:

$$\sum_{i \in I} a_i [E_i] \sim 0.$$

So

$$\forall i \in I \cup I', \qquad a_i = 0.$$

If we observe that $rk_{\mathbf{Z}}\,(NS(S))$ is finite we have the result. The fact that the bilinear form defined by the matrix $(C_i \cdot C_j)_{i,j \in \{1,\ldots,m\}}$ is definite negative, is because the intersection form on $NS(S)$ is negative definite on the orthogonal of R. □

We recall now that the fundamental cycle Z associated to ξ is defined by the following condition:

It is the "smallest" effective divisor with support in ξ such that:

$$\forall i \in \{i,\ldots,m\}, \qquad Z \cdot C_i \leqslant 0.$$

This is the definition of Artin in [2].

Lemma 2: *Let $D = \sum_{i=1}^{n} a_i C_i$ and $\mathscr{L}$ an invertible sheaf on S such that for any $i \in \{1,\ldots,m\}$, $\deg_{C_i}(\mathscr{L}) \geqslant 0$. Then*

$$H^1(D, \mathscr{O}_D \otimes \mathscr{L}) = 0.$$

Proof: The proof can be found in [2] Lemma 5, but here we give an elementary proof. First of all we observe that all C_i are rational smooth curves because we have a surjective map:

$$\mathscr{O}_Z \to \mathscr{O}_{C_i} \to 0$$

for each $i \in \{1,\ldots,m\}$, and H^1 is a right exact functor in this case. Now we distinguish two cases.

1st case: Assume that we have proved the result for all divisors of the form nZ with $n \in \mathbb{N} - \{0\}$; because there exists an integer n such that

$$nZ \geqslant D,$$

we have a surjective map $\mathcal{O}_{nZ} \to \mathcal{O}_D \to 0$, we get a surjective map

$$\mathcal{O}_{nZ} \otimes \mathcal{L} \to \mathcal{O}_D \otimes \mathcal{L} \to 0.$$

This gives the result because H^1 is right exact.

2nd case: We can assume $D = nZ$ and shall prove the result by induction on n. If $n = 1$ it is very easy to see that there exists a sheaf of finite length $\mathcal{F}$ and an exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z \otimes \mathcal{L} \to \mathcal{F} \to 0.$$

Because $H^1(Z, \mathcal{O}_Z) = H^1(Z, \mathcal{F}) = 0$ we have the result. Assume we have the result for n; we shall prove it for $n + 1$. We have the exact sequence:

$$0 \to \mathcal{O}_Z \otimes \mathcal{L}(-nZ) \to \mathcal{O}_{(n+1)Z} \otimes \mathcal{L} \to \mathcal{O}_{nZ} \otimes \mathcal{L} \to 0.$$

We observe that:

$$\forall i \in \{i, \ldots, m\}, \deg_{C_i}(\mathcal{L}(-nZ)) = \deg_{C_i}(\mathcal{L}) - nZ \cdot C_i \geqslant 0.$$

This implies that:

$$H^1(\mathcal{O}_Z \otimes \mathcal{L}(-nZ)) = H^1(\mathcal{O}_{nZ} \otimes \mathcal{L}) = 0. \quad \square$$

LEMMA 3: *There exists an effective divisor L such that the rational map φ_L defined by $|L|$ is a birational morphism on its image from S to a surface Y such that $\varphi_L : S - \xi \xrightarrow{\sim} Y - \{P\}$, where P is a closed point of Y and*:

$$\forall n \in \mathbb{N} - \{0\}, \qquad H^1(S, \mathcal{O}_S(nL)) = 0.$$

PROOF: This lemma can also be found in [2] minus the last global condition which could be easily obtained. Here we prefer to give another proof. Let H be a very ample divisor on S such that:

$$\forall n \in \mathbb{N} - \{0\}, \qquad H^1(S, \mathcal{O}_S(nH)) = 0,$$
$$\forall i \in \{i, \ldots, m\}, \qquad H \cdot C_i = db_i,$$

where $b_i \in \mathbb{N}$ and $d = |\det(C_i \cdot C_j)|$. Because the bilinear form defined by the matrix $(C_i \cdot C_j)_{i,j \in \{1, \ldots, m\}}$ is definite negative by Lemma 1, there exists an effective divisor $D=\sum_{i=1}^m a_i C_i$ with $a_i \in \mathbb{N}$ such that:

$$\forall i \in \{1,\ldots,m\}, \qquad H \cdot C_i = -D \cdot C_i.$$

We consider the exact sequence

$$0 \to \mathcal{O}_S(H) \to \mathcal{O}_S(H+D) \to \mathcal{O}_D(H+D) \to 0.$$

Observe that $\mathcal{O}_D(H+D)=\mathcal{O}_D$. Hence we have the exact sequences:

$$H^0(\mathcal{O}_S(H+D)) \to H^0(\mathcal{O}_D) \to 0,$$

$$0 \to H^1(\mathcal{O}_S(H+D)) \to H^1(\mathcal{O}_D).$$

This implies by Lemma 2 that

$$H^1(\mathcal{O}_S(H+D))=0.$$

On the other hand the constant function equal to 1 belongs to $H^0(\mathcal{O}_D)$ so the linear system $|H+D|$ does not have any fixed component in ξ. Because H is very ample it is clear that $L=H+D$ satisfies the conditions of Lemma 3. □

LEMMA 4: *There exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ there exists $r(n) \in \mathbb{N}$ such that for $r \geqslant r(n)$, $|rL+nR|$ doesn't have ξ as fixed component.*

PROOF: Because we have $R^2>0$, if we replace R by a suitable multiple, we can assume R to be an effective divisor. So we can write $R=F+G$ with F and G effective divisors such that all the irreducible components of F are contained in ξ and none of the irreducible components of G are in ξ. But we have the exact sequences for $r \in \mathbb{N}-\{0\}$,

$$0 \to \mathcal{O}_S(rL) \to \mathcal{O}_S(rL+R) \to \mathcal{O}_R(rL+R) \to 0,$$

$$0 \to \mathcal{O}_G(rL+G) \to \mathcal{O}_R(rL+R) \to \mathcal{O}_F \to 0.$$

The exact sequence of cohomology gives:

$$H^0(\mathcal{O}_S(rL+R)) \to H^0(\mathcal{O}_R(rL+R)) \to 0.$$

If r is big enough, because of the vanishing theorem of Serre, we have:

$$H^1(\mathcal{O}_G(rL+G)=0,$$

so we have a surjective map:

$$H^0(\mathcal{O}_R(rL+R)) \to H^0(\mathcal{O}_F) \to 0.$$

But the constant function equal to 1 belongs to $H^0(\mathcal{O}_F)$, so we get the result. □

Now we show the proposition; we assume that there exist infinitely many n such that $|nR|$ has ξ as a fixed component. For any $n \in \mathbb{N} - \{0\}$ we write

$$nR \equiv M_n + F_n,$$

where M_n is the moving part of $|nR|$, F_n its fixed part.

By Theorem 9-1 of [1] we have that for sufficiently large n, $R \cdot C = 0$ for any irreducible component C of F_n, because R is numerically positive. Replacing R by a suitable multiple we can assume that

$$R \equiv \Delta + F + G,$$

where Δ is the moving part of $|R|$, $F + G$ its fixed part such that $R \cdot F = R \cdot G = 0$, and all the irreducible components of F are contained in ξ, while none of the irreducible components of G are contained in ξ. By the hypothesis for any $i \in \{1,\dots,m\}$ we have $C_i \cdot G = 0$.

Because $nR \equiv n\Delta + nF + nG$, we can assume that ξ is a fixed component of $|n(\Delta + F)|$ for infinitely many n. But if $R' = \Delta + F$ we have

$$\forall i \in \{1,\dots,m\}, \quad R' \cdot C_i = (R - G) \cdot C_i = 0.$$

By the definition of ξ and applying the proof of Lemma 4, there exists r_0 such that for $r \geqslant r_0$, $|rL + R'|$ doesn't have ξ as a fixed component.

Let $\{P_1,\dots,P_s\}$ be the base points of $|\Delta|$. Then there exists a smooth irreducible curve $E \in |L|$ such that:

$$\forall i \in \{1,\dots,s\}, \qquad P_i \notin E,$$

$$E \cap \xi = \emptyset,$$

For any $i, j \in \mathbb{N}$ we denote by H_{ij} the trace on E of the linear systems $|iL + jR'|$ on S. Then it follows from the choice of E (recall that $|R'|$ has only fixed components in ξ) that:

(A) $H_{i_1 j_1} + H_{i_2 j_2} \subset H_{i_1 + i_2, j_1 + j_2}$ for any $i_1, i_2, j_1, j_2 \in \mathbb{N}$

(B) H_{10} and H_{01} are free from base points.

By the Theorem 4-2 (Relation 24) of [1] we have for a suitable integer N,

$$\forall i, j \in \mathbb{N}, \qquad j \geqslant N, \quad H_{ij} = H_{ij-1} + H_{01}$$

It follows that $|iL + jR'|$ is spanned by the following two subsystems

$$(1)\ |iL + (j-1)R'| + |R'|, \qquad |(i-1)L + jR'| + E,$$

where the second system has E as a fixed component. For fixed j the linear system $|iL + jR'|$ has no base point if i is sufficiently large by the proof of Lemma 4. Let then j a fixed integer $\geqslant N$ and let i be such that $|iL + jR'|$ has no base points. If P is any base point of $|R'|$ then P is also a base point of the first of the two linear systems (1).

Therefore P is not a base point of $|(i-1)L + jR'|$. Applying the same argument to this last system (if $i - 1 > 0$), we find that P is not a base point of $|(i-2)L + jR'|$. Ultimately we reach the conclusion that P is not a base point of $|jR'|$ (if $j \geqslant N$). □

References

[1] O. ZARISKI: The theorem of Riemann Roch for high multiples of an effective divisor on an algebraic surface. *Ann. Math. 76* (3) (1962) 560–615.

[2] M. ARTIN: On isolated rational singularities of surfaces. *Amer. J. Math. 88* (1966) 129–136.

(Oblatum 20-II-1982 & 21-III-1983)

Mathematics Department
Centro de Investigación y de Estudios Avanzados del IPN.
México, D.F. 07000