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# DESINGULARISATION OF THE TRIPLE POINTS AND OF THE STATIONARY POINTS OF A MAP

Felice Ronga

## §1. Introduction

Let $f$: $V^n \to W^{n+r}$ be a proper map between manifolds of dimension $n$ and $n+r$ respectively, with $r \geqslant 1$. We will consider simultaneously the $C^\infty$ case and the complex analytic or algebraic case (that we will refer to as the real or complex case), indicating when necessary the peculiarities of each case. Define the following subsets of $V$:

$\Sigma(f) = \{x \in V | \mathrm{d}f_x$ is not injective$\}$, the set of singular points;

$M_k(f) = \{x \in V | \sharp(f^{-1}f(x)) = k\}$, the set of (set-theoretical) *k-tuple points in the source of f*;

$M_{0,1}(f) = \{x \in V - \Sigma(f) | \exists x' \in \Sigma(f), f(x) = f(x')\}$, the set of stationary points of f;

$M_{1,0}(f) = \{x \in \Sigma(f) | \exists x' \in V - \Sigma(f), f(x) = f(x')\}$, that we call the set of costationary points of f;

$M_{1,1}(f) = \{x \in \Sigma(f) | \exists x' \in \Sigma(f), x \neq x', f(x) = f(x')\}$, for which we provide no name.

We shall work with maps satisfying certain conditions, that we call excellent maps; in many cases the set of excellent maps is dense in the set of all proper maps.

Our goal is to construct explicit desingularisations of the closures of $M_3(f)$, $M_{0,1}(f)$ and $M_{1,0}(f)$ in $V$ (see §4). By desingularisation of a singular variety $\Sigma$ we mean a proper map $\tilde{\Sigma} \to \Sigma$ where $\tilde{\Sigma}$ is non-singular and the map is an isomorphism from an open dense subset of $\tilde{\Sigma}$ onto the regular part of $\Sigma$. Actually we will have inclusions of codimension one submanifolds:

$$\tilde{\Sigma}^{1,1}(f) \subset \tilde{M}_{0,1}(f) \subset \tilde{M}_3(f) \quad \text{and} \quad \tilde{\Sigma}^{1,1}(f) \subset \tilde{M}_{1,0}(f) \subset \tilde{M}_3(f)$$

and a projection $\pi$: $\tilde{M}_3(f) \to V$ which desingularises $\overline{M}_3(f)$; also $\pi|\tilde{M}_{0,1}(f)$ and $\pi|\tilde{M}_{1,0}(f)$ desingularise $\overline{M}_{0,1}(f)$ and $\overline{M}_{1,0}(f)$ respectively and $\pi|\tilde{\Sigma}^{1,1}(f)$ coincides with the desingularisation of $\overline{\Sigma}^{1,1}(f)$ found in [10]. In the real case $\tilde{M}_3(f)$ will be a manifold with boundary $\tilde{M}_{0,1}(f)$.

The reason for studying these desingularisations is both to have local and global informations on the triple points and stationary points of $f$. Locally, we can see which singular points are in the closure of $M_3(f)$, $M_{0,1}(f)$ and $M_{1,0}(f)$ and how (see also Cor. 4.4). Globally, in the complex case one can relate the cohomology classes (with integer coefficients) $m_3(f) \in H^{2r}(V)$, $m_{0,1}(f)$ and $m_{1,0}(f) \in H^{2r+1}(V)$ that are Poincare dual respectively to $\overline{M}_3(f)$, $\overline{M}_{0,1}(f)$ and $\overline{M}_{1,0}(f)$ in $V$ to the Chern classes of $V$ and $W$ and multiple points of lower order (see §5; in the real case, $\overline{M}_3(f)$ can be thought of as a chain whose boundary is $\overline{M}_{0,1}(f)$ and therefore $m_3(f)$ doesn't exist and $m_{0,1}(f) = 0$).

The formulas for $m_3(f)$ and $m_{0,1}(f)$ are not new. In the algebraic case, working with a very general $f$: $V \to W$, S. Kleiman ([3]; see also there for more references) obtains the formula for $2m_3(f)$ (that he denotes by $m_3$) in the Chow ring of $V$ using the iteration method. He also announces there the formula for our $m_3(f)$ (that he denotes by $v_3$). In [9], using the iteration method J. Roberts obtains the formula for $m_{0,1}(f)$, and the same method would give $m_{1,0}(f)$.

We will pursue and improve the methods of [11], where the double points of a map were considered. In order to extend our method to 4-tuple points, we would need a desingularisation of $\overline{N}_3(f)$, which we aren't able to provide yet. A desingularisation of $\overline{M}_{1,1}(f)$ would be part of a desingularisation of $\overline{M}_4(f)$.

We will adopt the following notations and conventions. A manifold will always be smooth. By a map we mean either a $C^\infty$ map between $C^\infty$ manifolds or a holomorphic map between complex analytic manifolds. By isomorphism we mean an equivalence in the category of $C^\infty$ or holomorphic maps. If $X$ is a manifold and $Z$ a submanifold, $B_Z(X)$ will denote the blow-up of $X$ along $Z$ and $\gamma$ the line bundle associated with the blown-up $Z$. By $TX$ we denote the tangent bundle to $X$ and by $N(Z, X) = (TX|Z)/TZ$ the normal bundle of $Z$ in $X$; $D(Z, X) \in H^*(X)$ will denote the Poincare dual of $Z$ in $X$, where $H^*(\ )$ denotes the cohomology with integer coefficients in the complex case, the integers mod two in the real case. If $f$: $X \to Y$, $N_f = f^*TY - TX$ will be the virtual normal bundle of $f$ and $df$: $TX \to f^*TY$ the derivative of $f$. We let $X^{(k)}$ denote the $k$-fold product of $X$, $\Delta_X(k) = \{(x_1, \dots, x_k) \in X^{(k)} | \exists i \neq j$ with $x_i = x_j\}$ and $\delta_X(k) = \{(x, \dots, x) \in X^{(k)}\}$. By $J^k(X, Y)$ we denote the bundle of $k$-jets of maps from $X$ to $Y$ and by $j^k(f)$ the $k$-th jet extension of $f$: $X \to Y$.

If $\xi = (E \to X)$ is a smooth vector bundle (either real or complex) of rank $n$, $E_x$ will be its fiber over $x \in X$ and $e(\xi) \in H^n(X)$ its Euler class; if $\eta = (F \to X)$ is another such bundle, $\mathrm{HOM}(\xi, \eta)$ will denote the vector bundle with fibers $\mathrm{Hom}(E_x, F_x)$, i.e. the vector space of linear maps from $E_x$ to $F_x$. $P(\xi) = (P(E) \to X)$ will be the projective bundle associated to $\xi$. We shall write $c_i(\xi)$ for the $i$-th Chern class if $\xi$ is a complex vector bundle, the $i$-th Stiefel-Whitney class if $\xi$ is a real vector bundle. We will sometimes use the same notation for a bundle and his various pull-backs.

If $a$: $\xi \to \eta$ is a vector bundle morphism, we shall say that it is generic if the corresponding section $a$: $X \to \mathrm{HOM}(\xi, \eta)$ is transversal to the subbundle $\Sigma^i(\xi, \eta) = \cup_x \{A \in \mathrm{Hom}(E_x, F_x) | \dim(\ker(A)) = i\}$, for all $i$. Letting $\xi_1$ denote the canonical vector bundle of rank one on $P(\xi)$, we recall that if $a$: $\xi \to \eta$ is generic then the section of $\mathrm{HOM}(\xi_1, \eta)$ naturally associated to $a$ is transversal to the zero section (see [11], Prop. 2.2). The projection on $X$ of $\tilde{\Sigma}^1(a) = \{(x, d) | d \in P(E_x),\ a|d = 0\}$ is a desingularisation of $\bar{\Sigma}^1(a) = \cup_{i \geqslant 1} \Sigma^i(a)$, where $\Sigma^i(a) = a^{-1}(\Sigma^i(\xi, \eta))$ (see e.g. [10], Prop. $I$-1.1 or [8], Prop. 1.1).

## §2. A basic blow-up

Let $\xi = (E \to X)$ be a smooth vector bundle and $s$: $X \to E$ a section. Letting (0) denote the zero section of $\xi$, we set $Z = s^{-1}(0)$. If $x \in X$ and $U_x$ is an open neighborhood of $x$ on which $\xi$ is trivial, using the trivialisation we can write $s|U_x$ as a map $s'$: $U_x \to E_x$. The derivative $\mathrm{d}s'_x$: $TX_x \to E_x$ usually depends on the trivialisation of $\xi|U_x$, but if $s(x) = 0$ it doesn't (this is a special case of [1], Lemma 7.4); this yields a section $\mathrm{d}s$: $Z \to \mathrm{HOM}(TX, \xi)$ called Porteous' intrinsic derivative. If $Z$ is smooth, from $\mathrm{d}s$ we deduce a section $\mathrm{d}\bar{s}$: $Z \to \mathrm{HOM}(N(Z, X), \xi)$.

The aim of this paragraph is the following proposition, which generalises cor. 2.3 of [11].

2.1. PROPOSITION: *Let $s$: $X \to E$ be a section of the bundle $\xi = (E \to X)$ of rank $n$. Assume that $s^{-1}(0) = Z = Z_1 \cup Z_2$ with $Z_1 = (\overline{Z_1 - Z_2})$, $Z_2 = (\overline{Z_2 - Z_1})$ and that*:

(i) *$Z_2$ is smooth*;

(ii) *$Z_1 - Z_2$ is smooth and $s$ is transversal to* (0) *on $Z_1 - Z_2$*;

(iii) *$\mathrm{d}\bar{s}$: $Z_2 \to \mathrm{HOM}(N(Z_2, X), \xi)$ is generic.*

*Let $\tilde{X} = B_{Z_2}(X)$, $\sigma$: $\tilde{X} \to X$ the natural projection, $\gamma$ the rank one bundle over $\tilde{X}$ associated to $\tilde{Z}_2 = \sigma^{-1}(Z_2)$ and $t = e(\gamma) = D(\tilde{Z}_2, \tilde{X})$. We have*:

(i) *let $\tilde{Z}_1$ = closure of $\sigma^{-1}(Z_1 - Z_2)$ in $\tilde{X}$; $\sigma|\tilde{Z}_1$ is a desingularisation of $Z_1$*;

(ii) *the section $(s|X - Z_2) \cdot (\sigma|\tilde{X} - \tilde{Z}_2)$: $\tilde{X} - \tilde{Z}_2 \to E$ extends to a section*

$\tilde{s}$: $\tilde{X} \to \mathrm{HOM}(\gamma, \xi)$ *transversal to the zero section and* $\tilde{s}^{-1}(0) = \tilde{Z}_1$;

(iii) $\tilde{Z}_1$ *is transversal to* $\tilde{Z}_2$. *More precisely,* $\tilde{s}|\tilde{Z}_2$ *is transversal to the zero section of* $\mathrm{HOM}(\gamma, \xi)$, $Z_1 \cap Z_2 = \bar{\Sigma}^1(\mathrm{d}\bar{s})$ *and* $\tilde{Z}_1 \cap \tilde{Z}_2 = \tilde{\Sigma}^1(\mathrm{d}\bar{s})$:

(iv) *let* $s'$: $X \to E$ *be a* $C^\infty$ *section (in both the real and complex case) approximating s which is transversal to* (0) *everywhere. Setting* $Z' = s'^{-1}(0)$ *and denoting by* $i$: $\tilde{Z}_2 \to \tilde{X}$ *the inclusion, we have*:

$$D(\tilde{Z}_1, \tilde{X}) = \sigma^*\big(D(Z', X)\big) - i_!\left(\sum_{h=0,\ldots,n-1} (-1)^h \cdot t^h \cdot c_{n-1-h}(\xi)\right).$$

PROOF: We may work locally and hence assume that $X$ is open and convex in $H \times H'$, $Z_2 = X \cap (H \times \{0\})$ and $s$: $X \to E_0$, where $H$, $H'$ and $E_0$ are finite dimensional vector spaces. Writing elements of $X$ as pairs $(x, v)$ with $x \in H$ and $v \in H'$ we have $B_{Z_2}(X) = \{(x, v, d) \in X \times P(H') | v \in d\}$. Define:

$$\tilde{s}(x, v, d) = \left(\int_0^1 \mathrm{d}s_{(x, t.v)} \cdot \mathrm{d}t\right) |0 \times d$$

where the integral takes place in $\mathrm{Hom}(H \times H', E_0)$. It is elementary to verify that $\tilde{s}(x, v, d)_{(0, v)} = s(x, v) - s(x, 0) = s(x, v)$. From this (i) and (ii) follow easily.

Since $\tilde{s}(x, 0, d) = \mathrm{d}s_{(x,0)}|0 \times d$, assumption (iii) implies that $\tilde{s}|\tilde{Z}_2$ is transversal to the zero section of $\mathrm{HOM}(\gamma, \xi)$ and assertion (iii) follows.

Since $\tilde{Z}_1 = \tilde{s}^{-1}(0)$, $D(\tilde{Z}_1, \tilde{X}) = e(\gamma^* \otimes \xi) = \sigma^*(e(\xi)) + \Sigma_{h=1,\ldots,n}(-1)^h \cdot t^h \cdot c_{n-h}(\xi)$. Since $e(\xi) = D(Z', X)$ and $i_!(1) = t$, the stated formula follows.

In §5 we will use a slightly generalised version of 2.1. Namely, we will replace $X$ by a closed neighborhood $\Omega$ of $Z_1 \cup Z_2$ in $X$, that we choose to be a $C^\infty$ manifold with boundary $\partial\Omega$ in both the real and complex case. 2.1. remains valid, provided we define suitably $e(\xi) \in H^n(\Omega, \partial\Omega)$ (see [11], proof of 2.6).

## §3. Some preliminary constructions

Let $V$ be a manifold and set $V \tilde{\times} V = B_{\Delta(2)}(V \times V)$ (this space was called the fat square of $V$ in [11] and denoted by $F_2(V)$). The blown-up diagonal will be denoted by $\tilde{\Delta}(2)$. The antidiagonal inclusion $TV_x \subset TV_x \oplus TV_x$ sending $v \in TV_x$ to $(v, -v)$ induces an isomorphism of $TV$ with $N(\Delta(2), V \times V)$; using this we can write $V \tilde{\times} V = (V \times V - \Delta(2)) \cup PTV$. Hence elements of $V \tilde{\times} V$ can be written $(x_1, x_2)$, $x_1 \neq x_2$ or $(x, d)$, where $x_1$, $x_2$ and $x$ are in $V$ and $d$ is a line in $TV_x$.

The group $S_2$ of permutations of two objects acts obviously on $V\times V$ and this action extends to an action of $V\tilde{\times} V$, leaving $\tilde{\Delta}(2)$ pointwise fixed. The quotient $V\tilde{\times} V/S_2$ is smooth and contains $\tilde{\Delta}(2)$ as submanifold (in the real case $V\tilde{\times} V/S_2$ is a manifold with boundary $\tilde{\Delta}(2)$). We can write:

$$V\tilde{\times} V/S_2=\{[x_1, x_2]|x_1\neq x_2\}\cup PTV$$

where $[x_1, x_2]$ denotes the unordered pair.

Consider the product $V\times(V\tilde{\times} V)/S_2$; it contains the subset:

$$\Delta'(3)=\text{closure of }\{(x_1, [x_2, x_3])|x_2\neq x_3,\ x_1=x_2 \text{ or } x_1=x_3\}$$

and also $\delta'(3)=\{(x, x, d)\in V\times\tilde{\Delta}(2)\}$.

3.1. PROPOSITION:

(i) $\Delta'(3)$ *is a submanifold* (*of codimension* $n$) *of* $V\times(V\tilde{\times} V/S_2)$;

(ii) $\delta'(3)$ *is a submanifold* (*of codimension* 1) *of* $\Delta'(3)$;

(iii) *the map* $g^0$: $\Delta'(3)-V\times\tilde{\Delta}(2)\to V\times V-\Delta(2)$ *defined by*

$$g^0(x_1, x_2, x_3)=\begin{cases}(x_1, x_3) & \textit{if}\quad x_1=x_2\\ (x_1, x_2) & \textit{if}\quad x_1=x_3\end{cases}$$

*extends to an isomorphism* $g$: $\Delta'(3)\to V\tilde{\times} V$ *sending* $\tilde{\delta}'(3)$ *isomorphically onto* $\tilde{\Delta}(2)$.

PROOF: It is quite clear that $\Delta'(3)-V\times\tilde{\Delta}(2)$ is smooth. We replace now $V$ by an open convex subset $U$ of the linear space $E$. First we give local descriptions of the various spaces involved, using the canonical bundle $\gamma$ over $PE$. Let $O_2=\{(x, d, v)\in U\times PE\times E|v\in d,\ x+v \text{ and } x-v\in U\}$, an open neighborhood of the zero section of the pull-back $\gamma'$ of $\gamma$ on $U\times PE$, and define $\theta_2$: $O_2\to U\times U$ by:

$$\theta_2(x, d, v)=\begin{cases}(x+v, x-v) & \text{if}\quad v\neq 0\\ (x, d) & \text{if}\quad v=0\end{cases}$$

Any element of the tensor product $d\circ d$ can be written as $v\circ v$, where $v$ is determined up to $a+or-$ sign. Let $\overline{O}_2=\{(x, d, v\circ v)\in U\times PE\times(d\circ d)|x\pm v\in U\}$, an open neighborhood of the zero section of $\gamma'\circ\gamma'$, and define $\bar{\theta}_2$: $\overline{O}_2\to U\tilde{\times} U/S_2$ by:

$$\bar{\theta}_2(x, d, v\circ v)=\begin{cases}[x+v, x-v] & \text{if}\quad v\neq 0\\ (x, d) & \text{if}\quad v=0\end{cases}$$

Finally, let $O_3 = \{(x, d, w, v \circ v) \in U \times PE \times E \times (d \circ d) | x + v,\ x - v$ and $x + w \in U\}$, an open neighborhood of the zero section of $E \oplus \gamma' \circ \gamma'$, and define $\theta_3$: $O_3 \to U \times (U \tilde{\times} U/S_2)$ by:

$$\theta_3(x, d, w, v \circ v) = \begin{cases} (x + w, [x + v, x - v]) & \text{if} \quad v \neq 0 \\ (x + w, x, d) & \text{if} \quad v = 0 \end{cases}$$

$\theta_2$, $\bar{\theta}_2$ and $\theta_3$ are isomorphisms. We have:

$$\theta_3^{-1}(\Delta'(3)) = \{(v, x, d, v \circ v) \in O_3\} \quad \text{and}$$

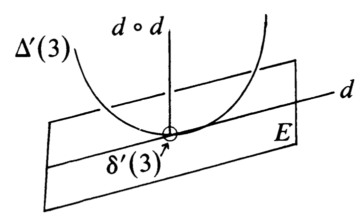
$$\theta_3^{-1}(\delta'(3)) = \{(0, x, d, 0) \in O_3\}.$$

Assertions (i) and (ii) follows at once. As for (iii), we have:

$$\theta_2^{-1} \cdot g \cdot \theta_3(x, d, v, v \circ v) = (x, d, v)$$

which is clearly an isomorphism.

Here is the picture in a fiber of $E \oplus \gamma' \circ \gamma'$ over some $(x, d) \in U \times PE$:




We set $F_3(V) = B_{\Delta'(3)}(V \times (V \tilde{\times} V/S_2))$, call $\sigma$: $F_3(V) \to V \times (V \tilde{\times} V/S_2)$ the blow-up map and $\tilde{\Delta}'(3) = \sigma^{-1}(\Delta'(3))$, $\tilde{\delta}'(3) = \sigma^{-1}(\delta'(3))$. We let $\pi$: $F_3(V) \to V$ be the composite of $\sigma$ with the projection on the factor $V$.

On $V \tilde{\times} V$ we have the line bundle $\gamma_2$ associated with $\tilde{\Delta}(2)$. Its restriction to $\tilde{\Delta}(2)$ coincides with the canonical bundle over $PTV$, a sub-bundle of the pull-back of $TV$ on $PTV$.

3.2. Proposition: *We have vector bundle isomorphisms*:

(i) $N(\Delta'(3), F_3(V)) \oplus g^*(\gamma_2) \simeq \pi^*TV \oplus g^*(\gamma_2 \circ \gamma_2)$

(ii) $N(\Delta'(3), F_3(V)) | \delta'(3) \simeq \pi^*TV/g^*\gamma_2 \oplus g^*(\gamma_2 \circ \gamma_2) | \delta'(3)$ *and* $N(\Delta'(3), F_3(V)) | \Delta'(3) - \delta'(3) \simeq \pi^*TV | \Delta'(3) - \delta'(3)$

(iii) $N(\delta'(3), \Delta'(3)) \simeq g^*\gamma_2 | \delta(3)$

This proposition follows immediately from the local description of $\Delta'(3)$ and $\delta'(3)$ given in the proof of 3.1. From assertion (ii) we deduce that $\tilde{\delta}'(3)$ is isomorphic to $P(\pi^*TV/g^*\gamma_2 \oplus g^*(\gamma_2 \circ \gamma_2)) | \delta'(3)$.

Let now $f\colon V^n \to W^{n+r}$ be a map which is transversal for double points, namely (cf. [11], cor. 2.3):

(i) $f^{(k)}\colon V^{(k)} \to W^{(k)}$ is transversal to $\delta_W^{(k)}$ outside $\Delta_V^{(k)}$, $k \leqslant 3$.

(ii) $\mathrm{d}f\colon TV \to f^*TW$ is generic.

The action of $S_2$ on $V \tilde{\times} V$ leaves $\tilde{M}_2(f) =$ closure of $\{(x_1, x_2) | x_1 \neq x_2, f(x_1) = f(x_2)\}$ invariant. It was proved in [11], theorem 2.5, that $\tilde{M}_2(f)$ is smooth and transversal to $\tilde{\Delta}(2)$; therefore the quotient $\tilde{N}_2(f) = \tilde{M}_2(f)/S_2$ is a submanifold of $V \tilde{\times} V/S_2$ (with boundary $\tilde{M}_2(f) \cap \tilde{\Delta}(2) = \tilde{\Sigma}^1(f)$ in the real case).

Set $\Delta'(f, 3) = \Delta'(3) \cap (V \times \tilde{N}_2(f))$ and $\delta'(f, 3) = \delta'(3) \cap (V \times \tilde{N}_2(f))$. Since the projection $\Delta'(3) \to V \tilde{\times} V/S_2$ is a submersion, $\Delta'(f, 3)$ and $\delta'(f, 3)$ are submanifolds of $V \times \tilde{N}_2(f)$. Setting $V \tilde{\times} \tilde{N}_2(f) = B_{\Delta'(f,3)}(V \times \tilde{N}_2(f))$, we have natural identifications $V \tilde{\times} \tilde{N}_2(f) = \sigma^{-1}(V \times \tilde{N}_2(f))$, $\tilde{\Delta}'(f, 3) = (V \tilde{\times} \tilde{N}_2(f)) \cap \tilde{\Delta}'(3)$, $\tilde{\delta}'(f, 3) = (V \tilde{\times} \tilde{N}_2(f)) \cap \tilde{\delta}'(3)$, all the intersections being transversal. The expressions of the various normal bundles given in 3.2 remain valid if $\Delta'(3)$ and $\delta'(3)$ are replaced by $\Delta'(f, 3)$ and $\delta'(f, 3)$, provided that the various bundles are suitably restricted. The map $g\colon \Delta'(3) \simeq V \tilde{\times} V$ restricts to an isomorphism $\bar{g}\colon \Delta'(f, 3) \xrightarrow{\simeq} \tilde{M}_2(f)$.

Let's recall on a diagram the various spaces that we have defined:

$$
\begin{tikzcd}
 & \tilde{\delta}'(f,3) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \tilde{\delta}'(3) \arrow[d, phantom, "\cap"] & \\
 & \tilde{\Delta}'(f,3) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \tilde{\Delta}'(3) \arrow[d, phantom, "\cap"] & \\
\gamma_3 \arrow[r] & V \tilde{\times} \tilde{N}_2(f) \arrow[r, phantom, "\subset"] \arrow[d, "\sigma|\ldots"] & F_3(V) \arrow[r, "\pi"] \arrow[d, "\sigma"] & V \\
 & V \times \tilde{N}_2(f) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] & V \times (V \tilde{\times} V/S_2) \arrow[ur] \arrow[d, phantom, "\cup"] & \\
\gamma_2 \arrow[r] & \Delta'(f,3) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] \arrow[ddr, bend right, "\simeq \bar{g}"'] & \Delta'(3) \arrow[d, phantom, "\cup"] \arrow[dd, bend left, "g"] & \\
 & \delta'(f,3) \arrow[r, phantom, "\subset"] & \delta'(3) \simeq & \\
 & & \tilde{M}_2(f) \subset V \tilde{\times} V &
\end{tikzcd}
$$

We have denoted by $\gamma_3$ the line bundle associated to $\tilde{\Delta}'(f, 3)$.

## §4. Desingularisation of $\overline{M}_3(f)$, $\overline{M}_{0,1}(f)$ and $\overline{M}_{1,0}(f)$

Let $f\colon V \to W$ be proper.

4.1. Definition: We say that $f$ is excellent if:

(i) $f^{(k)}\colon V^{(k)} \to W^{(k)}$ is transversal to $\delta_W(k)$ outside $\Delta_V(k)$ for $k \leqslant 4$.

(ii) $\mathrm{d}f\colon TV \to f^*(TW)$ is generic.

(iii) $f^{(2)})|\Sigma^i(f)\times V$: $\Sigma^i(f)\times V\to W\times W$ is transversal to $\delta_W(2)$ outside $\Delta_V(2)$ for all i $\geqslant 1$.

(iv) $j^2(f)$ is transversal to $\bar{\Sigma}^{1,1}(f)$. More precisely, consider the case of an $f$: $U\to F$, $E$ and $F$ vector spaces and $U$ open in $E$. Let $\tilde{\Sigma}^1(f)=\tilde{\Sigma}^1(df)=\{(x, d)\in U\times PE|\mathrm{d}f_x|d=0\}$ and $\gamma$ the canonical bundle over $PE$; the morphism $\Psi_2$: $\tilde{\Sigma}^1(f)\to \mathrm{HOM}(E/\gamma\oplus\gamma\circ\gamma, F)$ defined to be the map induced by $\mathrm{d}f$ on $E/\gamma$ and by the restriction of $\mathrm{d}^2f$ on $\gamma\circ\gamma$ should be generic (cf. [10], I§2).

It is readily verified that condition iv) above, though not $\Psi_2$, is invariant by coordinate change. It says essentially that $j^2(f)$ must be transversal to an appropriate stratification of $\bar{\Sigma}^{1,1}$.

It follows from the usual transversality theorems that the set of excellent maps is dense in the set of all proper maps in the $C^\infty$ case, in the complex analytic if $V$ is Stein and $W=\mathbb{C}^{n+r}$ and in the case of a projective variety $V$ and a linear projection $f$: $V\to PC^{n-r}$ (see [6] for the latter).

Recall $M_k(f)$, $M_{0,1}(f)$, $M_{1,0}(f)$ and $M_{1,1}(f)$ from §1 and define:
$\tilde{M}_3^0(f)=\{(x_1, [x_2, x_3])\in F_3(V)|f(x_1)=f(x_2)=f(x_3)$, $x_1$, $x_2$ and $x_3$ all different and not in $\Sigma(f)$ and $f^{-1}(f(x_1))=\{x_1, x_2, x_3\}\}$,
$\tilde{M}_3(f)=$ closure of $\tilde{M}_3^0(f)$ in $F_3(V)$,
$M_3^0(f)=\pi(\tilde{M}_3^0(f))=\{x\in V|f^{-1}f(x)=\{x, x', x''\}$, $x, x', x''$ all different and not in $\Sigma(f)\}$,
$\tilde{M}_{1,0}(f)=\tilde{M}_3(f)\cap\tilde{\Delta}'(f, 3)$
$\tilde{M}_{0,1}(f)=\tau(\tilde{M}_{1,0}(f))$, where $\tau$ is the extension to $\tilde{\Delta}'(f, 3)$ of the action on $\tilde{M}_2(f)$ of the non-trivial element of $S_2$,
$\tilde{\Sigma}^{1,1}(f)=\tilde{M}_3(f)\cap\tilde{\delta}'(f, 3)$.
We will drop the mention $(f)$ when not needed.

4.2. Theorem: *Let $f$: $V\to W$ be excellent. We have:*

(i) *$\tilde{M}_3$ is a submanifold of $F_3(V)$ transversal to $\tilde{\Delta}'(f, 3)$ and $\tilde{\delta}'(f, 3)$.*

(ii) *$\tilde{M}_3^0$ is open dense in $\tilde{M}_3$ and $\pi|\tilde{M}_3^0$ is an isomorphism on $M_3^0$.*

(iii) *$\pi|\tilde{M}_3$ is proper*

(iv) $\pi(\tilde{M}_3)=(\bigcup_{k\geqslant 3}M_k)\cup M_{0,1}\cup M_{1,0}\cup M_{1,1}\cup\bar{\Sigma}^{1,1}$

(v) *$\pi|\tilde{M}_{0,1}$ and $\pi|\tilde{M}_{1,0}$ are desingularisations of $\overline{M}_{0,1}$ and $\overline{M}_{1,0}$ respectively and $\pi|\tilde{\Sigma}^{1,1}$ coincides with the desingularisation of $\bar{\Sigma}^{1,1}(f)$ found in* [10].

The decomposition of $\overline{M}_3$ given in iv) could be further elaborated by stratifying the map $\pi|\tilde{M}_3$.

Consider first a map $f$: $U\to F$, where $U$ is convex, open in $E$ and $E$ and $F$ are finite dimensional vector spaces. Define $S_3(f)$: $U\times\tilde{N}_2(f)\to F$ by $S_3(f)_{x_1[x_2,x_3])}=f(x_1)-\frac{1}{2}(f(x_2)+f(x_3))$ if $x_2\neq x_3$ and $S_3(f)_{(x_1,x_2,d)}=f(x_1)-f(x_2)$.

4.3. Proposition: *If $f$: $U\to F$ is excellent, $S_3(f)$ is transversal to $0\in F$ on*

*$U \times \tilde{N}_2(f) - \Delta'(f, 3)$ and its derivative induces a generic morphism $\mathrm{d}\bar{S}_3$: $\Delta'(f, 3) \to \mathrm{HOM}(N, F)$, where $N = N(\Delta'(f, 3), U \times \tilde{N}_2(f))$. Moreover, $\mathrm{d}\bar{S}_3|\delta'(f, 3)$ is also generic.*

PROOF: That $S_3(f)$ is transversal to $0 \in F$ on $U \times \tilde{N}_2(f) - \Delta'(f, 3)$ follows from Assumption 4.1 (i) on $f$ at points $(x_1, [x_2, x_3])$ with $x_1$, $x_2$, $x_3$ all different; it follows from 4.2 (iii) and Lemma 1.3 of [11] at points $(x_1, x_2, d)$.

If $x_1 = x_2 \neq x_3, f(x_2) = f(x_3)$, we can use the isomorphism $\theta_3$: $O_3 \to U \times (U \times U/S_2)$ of the proof of 3.1. Assume that $(x_1, [x_1, x_3]) = \theta_3(x, d, v, v \circ v)$. Sending $w \in E$ to $(x, d, v + w, v \circ v)$ injects $E$ as a normal space to $\theta_3^{-1}(\Delta'(f, 3))$ in $\theta_3^{-1}(U \times \tilde{N}_2(f))$. Restricting $S_3(f)$ to this normal space gives the map $E \ni w \to f(x + w + v) - f(x - v)$, whose derivative is the derivative of $f$. It follows from condition 4.1 iii) that this is a generic morphism. This shows that $\mathrm{d}\bar{S}_3(f)|\Delta'(f, 3) - \delta'(f, 3)$ is a generic morphism.

If $d \in PE$ and $\mathrm{d}f_x|d = 0$, take $E'$ to be a supplementary subspace of $d$ in $E$ and map $E' \oplus d \circ d \to E \oplus d \circ d$ by the natural inclusion. This injects $E' \oplus d \circ d$ as a normal space to $\theta_3^{-1}(\Delta'(f, 3))$ in $\theta_3^{-1}(U \times \tilde{N}_2(f))$, the restriction of $S_3(f) \cdot \theta_3$ to which sends $(w, v \circ v)$ to $f(x + w) + \frac{1}{2}(f(x + v) + f(x - v))$. The derivative of this last map at $0 \in E' \oplus d \circ d$ is seen to be equal to $(\mathrm{d}f_x|E', -\frac{1}{2}d^2 f_x|d \circ d)$: $E' \oplus d \circ d \to F$, which by condition 4.1 iv) is a generic morphism. This shows that $\mathrm{d}\bar{S}_3(f)|\delta'(f, 3)$ is a generic morphism.

The following corollary should have some interest by itself.

4.4. COROLLARY: *Let $f$: $V \to W$ be proper and assume it satisfies condition 4.1 (iv). Then $f$ is excellent in a neighborhood of $\bar{\Sigma}^{1,1}(f)$.*

PROOF: We can work locally, with a map $f$: $U \to F$ as in the proof of 4.3. As shown there, the fact that $\mathrm{d}\bar{S}_3|\delta'(f, 3)$ is generic follows from 4.3 (iv). Therefore the section $\tilde{S}_3(f)$: $U \times \tilde{N}_2(f) \to \mathrm{HOM}(\gamma_3, F)$ constructed in the proof of 2.1 is transversal to the zero section on $\tilde{\delta}'(f, 3)$ and therefore also on a neighborhood of $\tilde{\delta}'(f, 3)$. The corollary follows easily.

PROOF OF 4.2: The fact that $f$ is proper implies easily that the same holds for $\pi|\tilde{M}_3$. Let's now work locally with $f$: $U \to F$ as in the proof of 4.3. Define:

$$M_3' = \{(x_1, [x_2, x_3])|x_i \quad \text{all different}, \quad f(x_1) = f(x_2) = f(x_3)\},$$

$$M_3'' = \{(x_1, [x_2, x_3]) \in M_3'|x_1 \quad \text{or} \quad x_2 \quad \text{or} \quad x_3 \in \Sigma(f)\}$$

and

$$M_4' = \{(x_1, [x_2, x_3]) \in M_3' \mid \#(f^{-1}f(x_1)) \geqslant 4\}.$$

It follows from condition 4.1 i) that $M_4'$ and $M_3''$ are of codimension at least one in $M_3'$. It follows from 4.3 that $S_3(f)$ satisfies the conditions of 2.1; therefore $\tilde{S}_3(f)$: $U \times \tilde{N}_2(f) \to \mathrm{HOM}(\gamma_3, F)$ is transversal to the zero section, as well as $\tilde{S}_3(f)|\tilde{\Delta}'(f, 3)$. Moreover, $\tilde{S}_3(f)|\tilde{\delta}'(f, 3)$ is also transversal to the zero section. Now $\tilde{S}_3(f)^{-1}(0) = M_3' \cup \tilde{M}_{1,0} \cup \tilde{M}_{0,1} \cup \tilde{\Sigma}^{1,1}$ and therefore $\tilde{M}_3^0 = M_3' - (M_4' \cup M_3'')$ is dense in $\tilde{S}_3(f)^{-1}(0)$. Recalling that $\tilde{M}_3$ has been defined as the closure of $\tilde{M}_3^0$, it follows that $\tilde{M}_3 = \tilde{S}_3(f)^{-1}(0)$. All remaining assertions are easily checked.

## §5. The cohomology classes dual to $\overline{M}_3(f)$, $\overline{M}_{0,1}(f)$ and $\overline{M}_{1,0}(f)$

Here we show how to find the expressions for $m_3$, $m_{0,1}$ and $m_{1,0}$. We shall work in the complex case (but see Remark 5.3).

Let $m_2 = D(\overline{M}_2(f), V)$ and $n_2 = D(f(\overline{M}_2(f)), W)$. We will write $c_i$ for $c_i(N_f)$. According to [11], th. 2.6 or [4], $m_2 = f^*f_!(1) - c_r$.

5.1. Proposition: *Let $f$: $V^n \to W^{n+r}$ be an excellent map between complex manifolds. We have*:

$$m_3 = f^*(n_2) - m_2 c_r + \sum_{h=1,\dots,r} 2^{h-1} c_{r+h} c_{r-h}$$

$$m_{1,0} = f^*f_!(1) c_{r+1} - c_r c_{r+1} - \sum_{h=1,\dots,r+1} 2^{h-1} c_{r+h} c_{r+1-h}$$

$$m_{0,1} = f^*f_!(c_{r+1}) - c_r c_{r+1} - \sum_{h=1,\dots,r+1} 2^{h-1} c_{r+h} c_{r+1-h}$$

*We complete the picture by recalling that* (*see* [11] *or* [8] *or* [9]):

$$D(\overline{\Sigma}^{1,1}(f), V) = c_{r+1}^2 + \sum_{h=1,\dots,r+1} 2^{h-1} c_{r+h+1} c_{r-h+1}$$

To make it possible to have a global version of the section $S_3(f)$ we will need a $C^\infty$ spray $e_W$: $TW \to W$. Recall that by definition (see [5], chap. 4, §3.4) for every $y \in W$ there is an open neighborhood $U_y$ of zero in $TW_y$ such that $e_y = e_W|U_y$ is a $C^\infty$ diffeomorphism onto an open neighborhood of $y$ in $W$; in addition, we assume that $U_y$ is convex and that $\cup_y U_y$ is open in $TW$. We let $\Omega \subset V \times \tilde{N}_2(f)$ be a closed neighborhood of $\sigma(M_3) \cup \Delta'(f, 3)$ such that if $(x_1, [x_2, x_3])$ or $(x_1, x_2, d)$ are in $\Omega$, then $f(x_2)$ and $f(x_3)$ are in $e_{f(x_1)}(U_{f(x_1)})$. We choose $\Omega$ to be a $C^\infty$

manifold with boundary $\partial\Omega$ and let $\tilde{\Omega} = \sigma^{-1}(\Omega)$. Let $TW'$ be the pull-back of $TW$ on $\Omega$ and define the section $S_3(f)\colon \Omega \to TW'$ by:

$$S_3(f)_{(x_1,x_2,x_3)} = -\tfrac{1}{2}\Big(e^{-1}_{f(x_1)}(f(x_2)) + e^{-1}_{f(x_1)}(f(x_3))\Big)$$

and

$$S_3(f)_{(x_1,x_2,d)} = -e^{-1}_{f(x_1)}(f(x_2)).$$

This $S_3(f)$ coincides locally with the one defined for prop. 4.3.

Let $P_1\colon \Omega \to V$ and $P_2\colon \Omega \to \tilde{N}_2(f)$ be the restrictions to $\Omega$ of the projections of $V \times \tilde{N}_2(f)$ on the first and second factor respectively. Let $\bar{f}\colon \tilde{N}_2(f) \to W$ be the map induced by $f$ (see [7], prop. 2.5; $\bar{f}$ is a desingularisation of $f(\overline{M}_2)$) and $f'\colon V \to W$ be a $C^\infty$ map approximating $f$ which is transversal to $\bar{f}$. Let $M_3'$ be the pull-back of $\bar{f}$ by $f'$; we have a diagram:

$$(*)\qquad \begin{tikzcd}
M_3' \arrow[r, "P_2|M_3'"] \arrow[d, "P_1|M_3'"] & N_2(f) \arrow[d, "\bar{f}"] \\
V \arrow[r, "f'"] & W
\end{tikzcd}$$

where $M_3'$ plays the role of $Z'$ in prop. 2.1. We choose $f'$ near enough to $f$ so that there is a homotopy $f_t$, $0 \leqslant t \leqslant 1$, $f_0 = f$, $f_1 = f'$ *such that*:

$$\big(f_t \times \bar{f}\big)(\partial\Omega) \cap \Delta_W(2) = \varnothing,\ 0 \leqslant t \leqslant 1.$$

We recall on a diagram various maps and spaces that will be needed for the proof of 5.1:

$$\begin{tikzcd}
\gamma_3 \arrow[r] & V \times N_2(f) \arrow[r, "\sigma"] & V \times \tilde{N}_2(f) & & \\
\tilde{M}_3 \subset & \tilde{\Omega} \arrow[u, phantom, sloped, "\subset", "i"'] \arrow[r, "\sigma|\tilde{\Omega}"] & \Omega \arrow[u, phantom, sloped, "\subset"] \arrow[r, "P_2"] \arrow[dr, "p_1"] & \tilde{N}_2(f) \arrow[dr, "\bar{f}"] & \\
P(N(\Delta'(f,3),\Omega)) = & \tilde{\Delta}'(f,3) \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\bar{p}"] & \Delta'(f,3) \arrow[u, phantom, sloped, "\subset"] \arrow[dd, "12g"] & V \arrow[r, "f'"] & W \\
\tilde{M}_{0,1} & \tilde{M}_{1,0} & & & \\
& \gamma_2 \arrow[r] & \tilde{M}_2 \arrow[uur, "p_1"'] & &
\end{tikzcd}$$

where $i$ is the natural inclusion, $\bar{p} = \sigma \cdot i$ and $p_1$ and $p_2$ are deduced from the projection of $V \times V$ on the first and second factor respectively. We will write $\pi$ for $\pi|\Omega$ and set $t_2 = e(\gamma_2)$, $t_3 = e(\gamma_3)$. The diagram becomes commutative if $f'$ is replaced by $f$.

5.2. LEMMA: *We have*:

(i) $$(p_1)_!\big(c_s\big(p_1^*(N_f) + \gamma_2 - \gamma_2 \circ \gamma_2\big)\big) = c_s(N_f) \cdot m_2 - \sum_{h=1,\dots,s} 2^{h-1} \cdot c_{r+h} \cdot c_{s-h}$$

(ii) $$(p_1)_!\big(c_s\big(p_2^*(N_f) + \gamma_2 - \gamma_2 \circ \gamma_2\big)\big) = f^*f_!(c_s) - c_r \cdot c_s - \sum_{h=1,\dots,s} 2^{h-1} \cdot c_{r+h} \cdot c_{s-h}$$

PROOF: Let $c(\ ) = 1 + c_1 + ..$ denote the total Chern class. We have:

$$c(\gamma_2 - \gamma_2 \circ \gamma_2) = \frac{1 + t_2}{1 - 2 \cdot t_2} = 1 + \sum_{h \geqslant 1} (-1)^h \cdot 2^{h-1} \cdot t_2^h.$$

Therefore:

$$c_s\big(p_j^*(N_f) + \gamma_2 - \gamma_2 \circ \gamma_2\big) = c_s\big(p_j^*(N_f)\big) + \sum_{h \geqslant 1} (-1)^h \cdot 2^{h-1} \cdot t_2^h \cdot c_{s-h}\big(p_j^*(N_f)\big)$$

for $j = 1, 2$. The desired formula follows now from ([7], Prop. 4.6; due to a misprint, in (ii) $c_r$ has been replaced by $c_1$).

PROOF OF 5.1: It follows from 4.2 that $\pi_!(D(\tilde{M}_3, \tilde{\Omega})) = m_3$, $\pi_!(D(\tilde{M}_{1,0}, \tilde{\Omega})) = m_{1,0}$ and $\pi_!(D(\tilde{M}_{0,1}, \tilde{\Omega})) = m_{0,1}$, where the dual classes are taken in $H^*(\tilde{\Omega}, \partial\tilde{\Omega})$. In what follows we will write $TW$ for various pull-backs of $TW$.

(i) $m_3$: since by Prop. 4.3 $S_3(f)$: $\Omega \to TW$ satisfies the hypothesis of Prop. 2.1 we have:

$$D(\tilde{M}_3, \tilde{\Omega}) = \sigma^*\big(D(M_3', \Omega)\big) - i_!\left(\sum_{h=0,\dots,n+r-1} (-1)^h \cdot t_3^h \cdot c_{n+r-1}(TW)\right).$$

(ia) $\pi_!(\sigma^*(D(M_3', \Omega)) = (P_1)_!\sigma_!(\sigma^*(D(M_3', \Omega))) = (P_1)_!(D(M_3', \Omega))$. Since in the Pullback diagram (*) $f'$ and $\bar{f}$ are transversal, we have $(P_2|M_3')^* \cdot (P_1|M_3')_! = f'^* \cdot \bar{f}_!$ and therefore $(P_1)_!(D(M_3', \Omega)) = (P_1|M_3')_!(1) = f'^*(\bar{f}_!(1)) = f^*(n_2)$.

(ib) $\pi_! \cdot i_!(\sum_h (-1)^h \cdot t_3^h \cdot c_{n+r-1}(TW))) = (p_1)_! \cdot \bar{g}_! \cdot \bar{p}_!$ (the same). Let $N = (g^{-1})^*(N(\Delta'(f, 3), \Omega))$; according to 3.2 (i) we have an isomorphism of virtual bundles: $N \simeq p_1^*(TV) + \gamma_2 \circ \gamma_2 - \gamma_2$. Using that $p_!((-1)^h \cdot t_2^h) = c_{h-n+1}(-N)$ (see e.g. [10], II-4 i), the previous expression equals:

$$(p_1)_!\left(\sum_{h=0,\ldots,n+r} c_{h-n+1}\left(-p_1^*(TV) - \gamma_2 \circ \gamma_2 + \gamma_2\right) \cdot c_{n+r-1-h}(TW)\right)$$

$$= (p_1)_!\left(c_r\left(p_1^*(N_f) + \gamma_2 - \gamma_2 \circ \gamma_2\right)\right).$$

Applying Lemma 5.2 $(i)$ and $(ia)$ we obtain the desired formula.

(ii) $m_{1,0}$: since $\tilde{M}_{1,0} = (\tilde{S}_3(f)|\tilde{\Delta}'(f, 3))^{-1}(0)$ we have: $D(\tilde{M}_{1,0}, \tilde{\Delta}'(f, 3)) = i^*(e(\gamma_3^* \otimes TW)$. Proceeding as in *(ib)* above, this equals $(p_1)_!(c_{r+1}(p_1^*(N_f) + \gamma_2 - \gamma_2 \circ \gamma_2))$. Applying 5.2 *(i)* gives the desired formula.

(iii) $m_{0,1}$: since $\tilde{M}_{0,1} = \tau(\tilde{M}_{1,0})$, it follows from *(ii)* above that $m_{0,1} = (p_1)_!(c_{r+1}(p_2^*(N_f) + \gamma_2 - \gamma_2 \circ \gamma_2))$. We now apply 5.2 *(ii)*.

5.3. Remark: The calculations for $m_{1,0}$ and $m_{0,1}$ remain valid in the real case. They yield: $m_{1,0} = f^*f_!(1) \cdot c_{r+1}$ and $m_{0,1} = f^*f_!(c_{r+1}) = 0$. We already saw why $m_{0,1}$ is zero. In fact, we even have $f_!(c_{r+1}) = 0$. Indeed, $c_{r+1} = D(\overline{\Sigma}^1(f), V)$ and hence $f_!(c_{r+1}) = D(f(\overline{\Sigma}^1), W)$; but $f(\overline{\Sigma}^1)$ is the boundary of $f(\overline{M}_2)$, considered as a chain (see also [2], Th. 5.1).

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