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APPENDIX TO “PICARD GROUPS OF ZARISKI SURFACES”

Piotr Blass and Jeffrey Lang

In this appendix we show how to pass from *generic* to *general* and how to prove our conjecture stated in the introduction to [1] and [2] (see Theorem 7(2) below).

We use techniques of P. Samuel and Jeffrey Lang and of course the main result of [1] which was proven with the help of Deligne.

We begin by introducing some notation which is analogous to [3] and [1]. If R is a normal noetherian domain we denote by $\text{Cl } R$ its divisor class group.

k algebraically closed field of characteristic $p \geq 5$;
 $T_{i,j}$ indeterminates; we consider the polynomial ring

$$k[T_{i,j}] \quad 0 \leq i + j \leq p \text{ and two polynomials}$$

$$F(x, y) = \sum T_{i,j} X^i Y^j, \quad 0 \leq i + j \leq p \text{ and}$$

$$\Upsilon(x, y) = \sum t_{\alpha, \beta} X^\alpha Y^\beta,$$

$$0 \leq \alpha + \beta \leq p - 2 \text{ with } t_{\alpha, \beta} \text{ also indeterminates over } k.$$

We denote $\nabla = \frac{\partial^{2p-2}}{\partial x^{p-1} \partial y^{p-1}}$. We consider the system of equations:

$$(LS) \nabla(F^j \Upsilon) = 0 \quad j = 0, 1, 2, \dots, p - 2$$

$$(PLS) \nabla(F^{p-1} \Upsilon) = \Upsilon^p.$$

We consider the above equations as equalities of polynomials in X, Y . By comparing coefficients of the various monomials in X and Y we get an equivalent system

$$(LSI) \sum \underline{p}_{\alpha, \beta}^{\gamma, \delta} t_{\alpha, \beta} = 0$$

$$(PLSI) \sum Q_{\mu, \nu}^{\gamma, \delta} t_{\gamma, \delta} = t_{\mu, \nu}^p$$

$$\underline{p}, Q \in k[T_{i,j}].$$

If we specialize the indeterminates $T_{i,j}$ to have values $(c_{i,j}) \in \text{Spec}k[T_{i,j}]$, a closed point, then we denote the corresponding system $LSI(c_{i,j}) + PLSI(c_{i,j})$.

We use the following facts.

THEOREM 1 (J. Lang): *Let A be any algebraically closed field of characteristic $p > 0$. Let $G(x, y) \in A[x, y]$ be a polynomial such that $\partial G/\partial x$ and $\partial G/\partial y$ are relatively prime polynomials in $A[x, y]$. Then the surface $S: z^p = G(x, y)$ is normal and its divisor class group $Cl S$ is isomorphic to the set of polynomial solutions $t(x, y) \in A[x, y]$ of degree $\deg t \leq \deg G - 2$ of the following system of equations:*

$$\nabla(G^j t) = 0, \quad j = 0, 1, 2, \dots, p-2, \text{ and}$$

$$\nabla(G^{p-1} t) = t^p.$$

PROOF: See [3], 2.1, 2.3, 2.9.1.

THEOREM 2 (Blass-Deligne): *The only solution of $(LSI) + (PLSI)$ in $L = \overline{k}(T_{i,j})$ i.e. with $t_{\alpha, \beta} \in L$ is the identically zero solution $t_{\alpha, \beta} = 0$.*

PROOF: The set of solutions is isomorphic to $Cl \frac{L[X, Y, Z]}{(Z^p - \sum T_{i,j} X^i Y^j)}$ by above theorem, but the latter group is shown to be zero in [1]. From now on Σ means $\sum_{0 \leq i+j=p}$.

The following is simple.

LEMMA 3. *If $q = (c_{i,j}) \in \text{Spec}k[T_{i,j}]$, then the set of solutions $(t_{\alpha, \beta})$ of $LS(c_{i,j}) + PLS(c_{i,j})$ is finite.*

PROOF: Lang [3], proof of Lemma 2.8.

In what follows, Let H be the subscheme of $\text{Spec} k[T_{i,j}] \times \text{Spec} k[t_{\alpha, \beta}]$ defined by $(LS + PLS)$ or equivalently by $(LSI$ and $PLSI)$.

Consider the projection $H_{\text{red}} \rightarrow H \rightarrow \text{Spec} k[T_{i,j}]$. We denote by $\kappa^{-1}(q)$ the set (group) of closed points of H_{red} that map to q .

REMARK 4. We point out that if $q = (c_{i,j}) \in \text{Spec} k[T_{i,j}]$ is a closed point, then $\kappa^{-1}(q)$ is in one to one correspondence with the solution set of equations $LSI(c_{i,j}) + PLSI(c_{i,j})$.

PROPOSITION 5: *There exists an open and dense subset \mathcal{O}_p of $\text{Spec} k[T_{i,j}]$ such that for $q \in \mathcal{O}_p$, $\kappa^{-1}(q)$ consists of a single point with coordinates $t_{\alpha, \beta} = 0$ for all α, β .*

LEMMA 6: Let $Z \subset H_{\text{red}}$ be the subset of $\text{Spec } k[T_{i,j}] \times \text{Spec } k[t_{\alpha,\beta}]$ defined by $t_{\alpha,\beta} = 0$ (all α, β). Let C be any irreducible component of H_{red} whose image $\kappa(C)$ is dense in $\text{Spec } k[T_{i,j}]$. Then $C = Z$.

PROOF: First of all, $\dim C = \dim Z = \dim k[T_{i,j}]$ because of Lemma 3. Consider the diagram

$$\begin{array}{ccc} \mathcal{O}(C) & \leftarrow & k[T_{i,j}] \\ & \searrow m & \downarrow \\ & & \overline{k(T_{i,j})} = L. \end{array}$$

Let $[t_{\alpha,\beta}]$ be the class of $t_{\alpha,\beta}$ in $\mathcal{O}(C)$. $\mathcal{O}(C)$ has fraction field which is finite algebraic over $k[t_{i,j}]$. Hence we get an injective map m .

Suppose that for some α, β , $[t_{\alpha,\beta}] \neq 0$ in $\mathcal{O}(C)$ then $m([t_{\alpha,\beta}]) \neq 0$ and we would get a non-trivial solution of $(LS) + (PLS)$ in L which contradicts the Blass-Deligne theorem. Thus $[t_{\alpha,\beta}] = 0$ in $\mathcal{O}(C)$ for all α, β , i.e. $C \subseteq Z$ and consequently $C = Z$ since Z is irreducible.

PROOF OF PROPOSITION 5. Let $H_{\text{red}} = Z \cup C_1 \dots \cup C_s$ be a decomposition of H_{red} into irreducible components.

We have $\overline{\kappa(C_j)} \subset \text{Spec } k[T_{i,j}]$ by Lemma 6. Thus set $\mathcal{O}_{\mathbf{p}} = \text{Spec } k[T_{i,j}] - \bigcup_{j=1}^s \overline{\kappa(C_j)}$. For every $q \in \mathcal{O}(\mathbf{p})$, $\kappa^{-1}(q)$ is a single point of Z . Q.E.D.

REMARK 6. There exists an open and dense subset of $\text{Spec } k[T_{i,j}]$, for example the subset V defined in [1] (0.2), such that if $q = (c_{i,j})$ belongs to it, then

$$\kappa^{-1}(q) \leftrightarrow \text{Cl} \frac{k[x, y, z]}{(z^p - \sum c_{i,j} x^i y^j)}.$$

PROOF: For $q \in V$, $q = (c_{i,j})$, the polynomials $\partial(\sum c_{i,j} x^i y^j) / \partial x$ and $\partial(\sum c_{i,j} x^i y^j) / \partial y$ are relatively prime. Thus Remark 6 follows from Remark 4 and Theorem 1. Q.E.D.

THEOREM 6. There exists an open and dense subset $D \subset \text{Spec } k[T_{i,j}]$ such that for every closed point $q = (c_{i,j}) \in D$,

- (1) $\kappa^{-1}(q)$ consists of the single point,
- (2) $\text{Spec} \frac{k[x, y, z]}{(z^p - \sum c_{i,j} x^i y^j)}$ is a UFD,
- (3) Cl of the above ring in (2) is the zero group
- (4) The system $SLI(c_{i,j}) + PSLI(c_{i,j})$ has only the zero solution.

PROOF: Set $D = V \cap \mathcal{O}_{\mathbf{p}}$. Then (1) follows from Proposition 5 and we deduce (3) and (2) from Remark 6. Finally (4) follows because the closed

points of $\kappa^{-1}(q)$ are in one-to-one correspondence with the solutions of the system $SLI(c_{i,j}) + PLSI(c_{i,j})$. Q.E.D.

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