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A. HAEFLIGER

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## DEFORMATIONS OF TRANSVERSELY HOLOMORPHIC FLOWS ON SPHERES AND DEFORMATIONS OF HOPF MANIFOLDS

A. Haefliger

### Abstract

In this note we consider a transversely holomorphic foliation  $\mathcal{F}$  of dimension one on  $S^{2n-1}$  obtained by intersecting the orbits of a holomorphic flow on  $\mathbb{C}^n$  having zero as a contracting fixed point. It is shown that any deformation of  $\mathcal{F}$  (in the class of transversely holomorphic foliations) is still obtained by intersecting  $S^{2n-1}$  with the orbits of a deformation of the holomorphic flow.

We use an analogue of the theorem of Kodaira-Spencer on the existence of a versal deformation for transversely holomorphic foliation (see [6] or [7]) and the classification of germs of holomorphic contracting vector fields (Poincaré-Dulac theorem) as explained in the book of Arnold [1]. This book was the main inspiration for this paper.

In an appendix which can be read independently, we show that parallel considerations leads to a complete classification of Hopf manifolds. This completes results of C. Borcea [2].

### 1. Statement of the main theorem

1.1.  $\lambda$ -RESONANT VECTOR FIELDS. Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a sequence of complex numbers with strictly negative real part. Following Arnold ([1], p. 178), an additive \*)  $\lambda$ -resonant monomial vector field in  $\mathbb{C}^n$  is a vector field of the form  $az^m \partial / \partial z_s$ , where  $m = (m_1, \dots, m_n)$  is a multiindex of non negative integers  $m_i$  such that

$$\lambda_s = (m, \lambda).$$

Here  $(m, \lambda) = \sum m_i \lambda_i$ ,  $z^m = z^{m_1} \dots z^{m_n}$  and  $a \in \mathbb{C}$ . This condition implies that the  $m_i$  are not all zero.

DEFINITION:  $\mathcal{g}_\lambda$  denotes the vector space of  $\lambda$ -resonant vector fields, i.e. vector fields which are sum of  $\lambda$ -resonant monomial vector fields. It is a finite dimensional vector space. It can also be characterized as the subspace of holomorphic vector fields on  $\mathbb{C}^n$  commuting with the diago-

\* In the appendix, we shall define multiplicative  $\mu$ -resonant vector fields.

nal vector field  $\sum \lambda_s \partial / \partial z_s$ . Therefore  $\mathcal{g}_\lambda$  is a Lie subalgebra of the Lie algebra of holomorphic vector fields on  $\mathbb{C}^n$ .

For generic  $\lambda$ ,  $\dim \mathcal{g}_\lambda = n$ . For  $n = 2$ ,  $\dim \mathcal{g}_\lambda$  can be 2, 3 or 4. For  $n \geq 3$ ,  $\dim \mathcal{g}_\lambda$  is not bounded.

1.2. THE THEOREM OF POINCARÉ-DULAC. Let  $\xi = \sum a_m^s z^m \partial / \partial z_s$  be a holomorphic vector field on a neighbourhood of 0 in  $\mathbb{C}^n$ , vanishing at 0 and such that the eigenvalues  $\lambda_1, \dots, \lambda_n$  of the matrix  $(a_i^s)$  of the linear part of  $\xi$  have strictly negative real parts (in other words the flow generated by  $\xi$  is contracting). We can order the  $\lambda_i$ 's so that  $\text{Re } \lambda_1 \leq \dots \leq \text{Re } \lambda_n < 0$ .

The theorem of Poincaré-Dulac (Cf. Arnold [1], p. 183) asserts that one can find new coordinates such that  $\xi$  is under the form of a  $\lambda$ -resonant vector field, with linear part under Jordan form, so

$$\xi = \sum \lambda_s z_s \partial / \partial z_s + \sum_{(s,m)} a_m^s z^m \partial / \partial z_s \tag{1.2.1}$$

where the sum is over the sequences  $(s, m)$  such that

$$\lambda_s = \sum m_i \lambda_i, \quad m_i = 0 \quad \text{for } i \geq s.$$

Remark that after a diagonal change of coordinates, one can assume that the coefficients  $a_m^s$  are as small as we want. Indeed if  $h$  is the diagonal linear map with entries  $\mu_1, \dots, \mu_n$  in the diagonal, then

$$(h_* \xi)(z) = \sum \lambda_s z_s \partial / \partial z_s + \sum_{(s,m)} \frac{\mu_s}{\mu^m} a_m^s z^m \partial / \partial z_s.$$

So we can choose  $\mu_k = \epsilon^{-k}$ , where  $\epsilon$  is very small.

In other words, under the action of the group of linear automorphisms of  $\mathbb{C}^n$ , the orbit of the diagonal vector field  $\sum \lambda_s z_s \partial / \partial z_s$  is in the adherence of the orbit of  $\xi$ .

The vector field  $\xi$  generates a global holomorphic flow  $z(t)$  of the form

$$z_s(t) = e^{\lambda_s t} \left( z_s(0) + \sum_{(s,m),r} b_{m,r}^s t^r z^m(0) \right) \tag{1.2.2}$$

(the coefficients  $b_{m,r}^s$  are determined step by step, starting with  $s = n$ ).

The orbits of this flow in  $\mathbb{C}^n - \{0\}$  are complex curves which are the leaves of a holomorphic foliation  $\mathcal{F}_\xi$ . Note that  $\xi$  vanishes only at 0. Those leaves are transversal to the unit sphere  $S^{2n-1}$  in  $\mathbb{C}^n$ , if the coefficients  $a_m^s$  are small enough. It follows that their intersection with  $S^{2n-1}$  are curves which are the leaves of a transversely holomorphic foliation  $\mathcal{F}_\xi^0$  on  $S^{2n-1}$  induced by  $\mathcal{F}_\xi$ .

**THEOREM:** *Let  $S$  be a small enough neighbourhood of 0 in a vector subspace of  $\mathcal{g}_\lambda$  (cf. 1.1) complementary to the vector subspace generated by  $[\xi, \mathcal{g}_\lambda]$  and  $\xi$ .*

*The family  $\mathcal{F}_{\xi+s}^0$  of transversely holomorphic foliations on  $S^{2n-1}$  obtained by intersecting the orbits of the flows generated by  $\xi + s$  is a versal deformation of  $\mathcal{F}_\xi^0$  parametrized by  $s \in S$  (in the sense of [7]).*

Note that if  $\xi$  is the diagonal vector field  $\sum \lambda_s z_s \partial / \partial z_s$ , then the dimension of  $S$  is  $\dim \mathcal{g}_\lambda - 1$ . In any case,  $\dim S \geq n - 1$ .

Let  $\theta_{F_\xi^0}^{\text{tr}}$  be the sheaf of germs of transversely holomorphic vector fields for  $\mathcal{F}_\xi^0$  (cf. Lemma 3.2), we obtain that

$$\dim H^0(S^{2n-1}, \theta_{F_\xi^0}^{\text{tr}}) = \dim H^1(S^{2n-1}, \theta_{F_\xi^0}^{\text{tr}}) = \dim S$$

and for  $i > 1$

$$H^i(S^{2n-1}, \theta_{F_\xi^0}^{\text{tr}}) = 0.$$

**REMARK:** An open neighbourhood of 0 in a vector subspace of  $\mathcal{g}_\lambda$  complementary to  $[\xi, \mathcal{g}^\lambda]$  parametrizes a versal deformation of the holomorphic vector field  $\xi$  (see Arnold [1], p. 302, and N. Brouchlińskaia [2] where the existence of a versal deformation is proved).

**2. Infinitesimal deformation defined by a deformation of a vector field**

2.1. Let  $\xi$  be an everywhere non zero holomorphic vector field on a complex manifold  $X$  and let  $\mathcal{F}$  be the holomorphic foliation whose leaves are the orbits of the flow generated by  $\xi$ .

We consider the following sheaves:

- $\theta$  = sheaf of germs of holomorphic vector fields on  $X$ ,
- $\theta_{\mathcal{F}}$  = subsheaf of  $\theta$  of vectors fields preserving  $\mathcal{F}$ ,
- $\theta^\xi$  = subsheaf of  $\theta$  of vector fields commuting with  $\xi$ ,
- $\theta_{\mathcal{F}}^{\text{tr}}$  = sheaf of germs of transversely holomorphic vector fields for  $\mathcal{F}$  (quotient of  $\theta_{\mathcal{F}}$  by vector fields tangent to the leaves of  $\mathcal{F}$ )
- $\sigma$  = sheaf of germs of holomorphic functions on  $X$
- $\sigma_{\mathcal{F}}^{\text{tr}}$  = subsheaf of germs of holomorphic functions locally constant on the leaves of  $\mathcal{F}$ .

We have the following exact sequences of sheaves

$$0 \rightarrow \sigma_{\mathcal{F}}^{\text{tr}} \rightarrow \sigma \xrightarrow{L_\xi} \sigma \rightarrow 0 \tag{2.1.1}$$

$$0 \rightarrow \theta^\xi \rightarrow \theta \xrightarrow{L_\xi} \theta \rightarrow 0 \tag{2.1.2}$$

$$0 \rightarrow \sigma_{\mathcal{F}}^{\text{tr}} \rightarrow \theta^\xi \rightarrow \theta_{\mathcal{F}}^{\text{tr}} \rightarrow 0 \tag{2.1.3}$$

where  $L_\xi: \sigma \rightarrow \sigma$  is the derivative in the direction of  $\xi$ ,  $L_\xi: \theta \rightarrow \theta$  the Poisson bracket with  $\xi$  and the inclusion  $\sigma_{\mathcal{F}}^u \rightarrow \theta^\xi$  the multiplication by  $\xi$ . The multiplication by  $\xi$  send the sequence (2.1.1) in the sequence (2.1.2).

To check exactness, choose local coordinates  $(z, w_1, \dots, w_{n-1})$  such that  $\xi$  is given by  $\partial/\partial z$ .

Let  $S$  be a germ of analytic space with a base point 0; its Zariski tangent space of 0 is denoted by  $T_0S$ . Let  $\xi_s$  be a holomorphic family of everywhere non zero vector fields on  $X$  parametrized by  $S$  and such that  $\xi_0 = \xi$ . Denote by  $\mathcal{F}_s$  the corresponding family of holomorphic foliations on  $X$ .

2.2. PROPOSITION: *The Kodaira-Spencer map*

$$\rho: T_0S \rightarrow H^1(X, \theta_{\mathcal{F}})$$

measuring the infinitesimal deformations of the family  $\mathcal{F}_s$  is given by

$$\rho(\partial/\partial s) = -i \cdot \delta(\partial\xi_s/\partial s|_{s=0})$$

where  $\delta: H^0(X, \theta) \rightarrow H^1(X, \theta^\xi)$  is the connecting homomorphism associated to (2.1.2) and  $i: H^1(X, \theta^\xi) \rightarrow H^1(X, \theta_{\mathcal{F}})$  is induced by the inclusion of  $\theta^\xi$  in  $\theta_{\mathcal{F}}$ .

PROOF: Let  $\{U_i\}_{i \in I}$  be an open covering of  $X$  such that there are families of holomorphic charts  $\varphi_i^s: U_i \rightarrow \mathbb{C} \times \mathbb{C}^{n-1}$  so that  $\varphi_{i*}(\xi_s) = \partial/\partial z$ , where  $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$ .

Let  $g_{ij}^s$  be the change of charts defined by  $\varphi_i^s = g_{ij}^s \varphi_j^s$  on  $U_i \cap U_j$ .

The element  $\rho(\partial/\partial s)$  of  $H^1(X, \theta_{\mathcal{F}})$  corresponding to the infinitesimal deformation  $\partial/\partial s$  is represented by the 1-cocycle associating to  $U_i \cap U_j$  the vector field

$$\theta_{ij} = d/ds \left( (\varphi_i^0)^{-1} \varphi_i^s (\varphi_j^s)^{-1} \varphi_j^0 \right)_{s=0} = \left( (\varphi_i^0)^{-1} \right)_* \left( \frac{d}{ds} g_{ij}^s \Big|_{s=0} \right).$$

Let  $\eta_i$  be the holomorphic vector field on  $U_i$  defined by  $\eta_i = d/ds \left( (\varphi_i^s)^{-1} \varphi_i^0 \right)_{s=0}$ . On  $U_i \cap U_j$ , we have  $\theta_{ij} = \eta_j - \eta_i$ .

Moreover  $L_\xi \eta_i = -d/ds \xi_s|_{s=0}$  because  $\xi_s = ((\varphi_i^s)^{-1} \varphi_i^0)_* \xi$  hence  $d/ds \xi_s|_{s=0} = [\eta_i, \xi]$ .

This shows that  $\rho(\partial/\partial s) = -i \circ \delta(d\xi_s/ds|_{s=0})$ .

2.3. COROLLARY: *If we consider  $\mathcal{F}_s$  as a family of transversely holomorphic foliations, then the Kodaira-Spencer map*

$$\rho: T_0S \rightarrow H^1(X, \theta_{\mathcal{F}}^u)$$

is the composition of  $\delta$  and of the map  $p: H^1(X, \theta^\xi) \rightarrow H^1(X, \theta_{\mathcal{F}}^u)$  induced by the projection  $\theta^\xi \rightarrow \theta_{\mathcal{F}}^u$ .

### 3. Proof of the theorem

3.1. We denote by  $\mathcal{F}$  the holomorphic foliation on  $W = \mathbb{C}^n - \{0\}$  whose leaves are the orbits of the flow generated by  $\xi$  and by  $\mathcal{F}^0$  the transversely holomorphic foliation on  $S^{2n-1}$  induced from  $\mathcal{F}$  by the inclusion  $i: S^{2n-1} \rightarrow \mathbb{C}^n$ . As before we denote by  $\theta_{\mathcal{F}}^{\text{tr}}$  (resp.  $\theta_{\mathcal{F}^0}^{\text{tr}}$ ) the sheaf of germs of transversely holomorphic vector fields for  $\mathcal{F}$  (resp.  $\mathcal{F}^0$ ); clearly  $\theta_{\mathcal{F}^0}^{\text{tr}} = i^* \theta_{\mathcal{F}}^{\text{tr}}$ .

By the analogue of the Kodaira-Spencer theorem proved in [6] or in [7] for transversely holomorphic foliations, it will be sufficient to prove that the Kodaira-Spencer map  $\rho: T_0 S \rightarrow H^1(S^{2n-1}, \theta_{\mathcal{F}^0}^{\text{tr}})$  is an isomorphism.

For the real parameter  $t$ , the orbits (1.2.2) of the flow generated by  $\xi$  are tangent to the leaves of  $\mathcal{F}$ , transversal to the sphere  $S^{2n-1}$ , tend to 0 for  $t \rightarrow +\infty$  and to infinity in norm for  $t \rightarrow -\infty$ . So there is a projection  $\pi: W \rightarrow S^{2n-1}$  mapping the point  $z$  on the point of intersection with  $S^{2n-1}$  of the orbit passing through  $z$ . The pull back by  $\pi$  of  $\mathcal{F}^0$  is the transversely holomorphic foliation associated to  $\mathcal{F}$ . Hence  $\sigma_{\mathcal{F}}^{\text{tr}} = \pi^* \sigma_{\mathcal{F}^0}^{\text{tr}}$  and

$$H^i(S^{2n-1}, \sigma_{\mathcal{F}^0}^{\text{tr}}) \approx H^i(W, \sigma_{\mathcal{F}}^{\text{tr}}) \tag{3.1.1}$$

because  $W$  retracts by deformation on  $S^{2n-1}$  along the orbits. For the same reasons and with the notations of 2, we have

$$H^i(S^{2n-1}, i^* \theta^{\xi}) \approx H^i(W, \theta^{\xi}). \tag{3.1.2}$$

For  $n > 1$ , any holomorphic vector field on  $W$  extends to a holomorphic vector field on  $\mathbb{C}^n$ , so any element of  $H^0(W, \theta)$  is represented by a convergent series  $\sum a_m^s z^m \partial / \partial z_s$  (here  $\theta$  denotes the sheaf of germs of holomorphic vector fields on  $W$ ; it is isomorphic to  $\sigma^n$ , where  $\sigma$  is the sheaf of germs of holomorphic functions on  $W$ ).

Consider the cohomology long exact sequences associated to the short exact sequences (2.1.1) and (2.1.2):

$$0 \rightarrow H^0(W, \sigma_{\mathcal{F}}^{\text{tr}}) \rightarrow H^0(W, \sigma) \xrightarrow{L_{\xi}} H^0(W, \sigma) \rightarrow H^1(W, \sigma_{\mathcal{F}}^{\text{tr}}) \rightarrow \dots \tag{3.1.3}$$

$$0 \rightarrow H^0(W, \theta_{\mathcal{F}}^{\xi}) \rightarrow H^0(W, \theta) \xrightarrow{L_{\xi}} H^0(W, \theta) \rightarrow H^1(W, \theta_{\mathcal{F}}^{\xi}) \rightarrow \dots \tag{3.1.4}$$

Let  $\mathcal{g}_{\lambda}^{\perp}$  be the vector subspace of  $H^0(W, \theta)$  of holomorphic vector fields which are sum of monomial vector fields  $az^m \partial / \partial z_s$ , which are not  $\lambda$ -resonant. It is clear that  $L_{\xi}$  maps  $\mathcal{g}_{\lambda}$  in  $\mathcal{g}_{\lambda}$  and  $\mathcal{g}_{\lambda}^{\perp}$  in  $\mathcal{g}_{\lambda}^{\perp}$ , because the

bracket of a  $\lambda$ -resonant monomial vector field with a monomial vector field which is not  $\lambda$ -resonant is not  $\lambda$ -resonant.

3.2. LEMMA:

- (a)  $H^i(W, \sigma) = H^i(W, \theta) = 0$  for  $i \neq 0, n - 1$ .  $H^{n-1}(W, \sigma)$  (resp.  $H^{n-1}(W, \theta)$ ) is isomorphic to the vector space of convergent series  $\sum a_m z^m$  (resp.  $\sum a_m^s z^m \partial / \partial z_s$ ) on  $(\mathbb{C} - \{0\})^n$  where the sum is over the sequences  $m$  such that all  $m_i < 0$ .
- (b) For  $i > 0$ , the maps  $L_\xi: H^i(W, \sigma) \rightarrow H^i(W, \sigma)$  and  $L_\xi: H^i(W, \theta) \rightarrow H^i(W, \theta)$  are isomorphisms.
- (c) The kernel and cokernel of  $L_\xi: H^0(W, \sigma) \rightarrow H^0(W, \sigma)$  are generated by the constant function 1.  $L_\xi: \mathfrak{g}_\lambda^\perp \rightarrow \mathfrak{g}_\lambda^\perp$  is an isomorphism.

PROOF OF (A): Consider the covering  $\mathcal{U} = \{U_i\}$  of  $W$  by the Stein open sets  $U_i = \{z \in W: z_i \neq 0\}$ . By the theorem of Leray,  $H^k(W, \sigma)$  is isomorphic to the Čech cohomology  $H^k(\mathcal{U}, \sigma)$  computed using alternate cochains. Hence  $H^k(W, \sigma) = 0$  for  $k \geq n$ . In dimension  $n - 1$ , cochains are cocycles and are holomorphic functions on  $\cap U_i = (\mathbb{C} - \{0\})^n$ ; their Laurent expansion are of the form  $\sum a_m z^m$ , where  $m = (m_1, \dots, m_n)$ ,  $m_i \in \mathbb{Z}$ . Modulo the coboundaries, each element of  $H^{n-1}(W, \sigma)$  has a unique representative with all  $m_i < 0$ .

To prove that  $H^k(W, \sigma) = 0$  for  $0 < k < n - 1$ , one can consider  $W$  as the union of  $\{z \in W: z_1, \dots, z_{n-1} \text{ not all zero}\} = (\mathbb{C}^{n-1} - \{0\}) \times \mathbb{C}$  and  $\{z \in W: z_n \neq 0\} = (\mathbb{C} - \{0\}) \times \mathbb{C}^{n-1}$ , and write the Mayer-Vietoris cohomology exact sequence associated to this covering. Part a) of the lemma for  $\sigma$  follows by induction on  $n$ , using Künneth formula.

As  $\theta = \sigma^n$ , the similar result for  $H^i(W, \theta)$  follows.

PROOF OF (B): If  $\xi$  is diagonal, namely if  $\xi = \sum \lambda_s z_s \partial / \partial z_s$ , then

$$L_\xi(z^m) = (m, \lambda) z^m \quad \text{and}$$

$$L(z^m \partial / \partial z_s) = [(m, \lambda) - \lambda_s] z^m \partial / \partial z_s.$$

If all the  $m_i$  are strictly negative,  $(m, \lambda)$  and  $(m, \lambda) - \lambda_s$  have strictly positive real part, hence are non zero. It follows that the endomorphism  $L_\xi$  of  $H^{n-1}(W, \sigma)$  and  $H^{n-1}(W, \theta)$  are injective. Surjectivity is also easy because  $|(m, \lambda)|^{-1}$  and  $|(m, \lambda) - \lambda_s|^{-1}$  are smaller than 1 for  $|m|$  big enough.

In general we can assume that  $\xi$  is under the form 1.2.1. With respect to the lexicographic order,

$$L_\xi(z^m) = (m, \lambda) z^m + \text{bigger monomials}, \quad \text{and}$$

$$L_\xi(z^m \partial / \partial z_s) = [(m, \lambda) - \lambda_s] z^m + \text{bigger monomial vector fields},$$

if we decide that  $z^m \partial / \partial z_s < z^n \partial / \partial z_t$  for  $s > t$ . It follows that  $L$  is injective. One should be able to prove directly that  $L$  is also surjective.

We give another argument using the upper semi-continuity of the dimension of the space of solutions of a differential elliptic operator on a compact manifold depending smoothly on a parameter (cf. Kodaira-Spencer III, [8] Th. 4, p. 48). When  $\xi$  is diagonal, we have checked that  $L_\xi$  is an isomorphism. Hence from the exact sequence 3.1.3, we have  $H^i(W, \sigma_{\mathcal{F}}^{\text{tr}}) = H^i(S^{2n-1}, \sigma_{\mathcal{F}_0}^{\text{tr}}) = 0$  for  $i \geq 2$ . As this group is isomorphic to the space of solutions of an elliptic differential operator (cf. Kalka-Duchamp [5]), for all  $\xi$  close enough to a diagonal vector field (this is always the case by 1.2), we still have  $H^i(S^{2n-1}, \sigma_{\mathcal{F}_0}^{\text{tr}}) = 0$  for  $i \geq 2$ . Hence  $L_\xi: H^i(W, \sigma) \rightarrow H^i(W, \sigma)$  is an isomorphism for  $i > 0$ . The similar argument works for  $\sigma$  replaced by  $\theta$ .

PROOF OF (C): The elements of  $H^0(W, \sigma)$  are convergent series  $\sum a_m z^m$ , with all  $m_i \geq 0$ . For  $\xi$  diagonal, it is easy to check that the kernel and cokernel of  $L_\xi$  are generated by 1. So from the exact sequence (3.1.3) and (3.1.1), we have  $\dim H^0(S^{2n-1}, \sigma_{\mathcal{F}}^{\text{tr}}) = \dim H^1(S^{2n-1}, \sigma_{\mathcal{F}}^{\text{tr}}) = 1$ . Hence  $\sum (-1)^i \dim H^i(S^{2n-1}, \sigma_{\mathcal{F}_0}^{\text{tr}}) = 0$ . This number is the index of an elliptic complex (cf. Kalka-Duchamp [5]), so it is constant under deformation. Hence when  $\xi$  is not diagonal, we still have  $\dim H^0(S^{2n-1}, \sigma_{\mathcal{F}_0}^{\text{tr}}) = \dim H^1(S^{2n-1}, \sigma_{\mathcal{F}_0}^{\text{tr}}) \leq 1$  (we also use semi-continuity as above). But the kernel of  $L_\xi$  contains 1, hence  $\dim H^1(S^{2n-1}, \sigma_{\mathcal{F}_0}^{\text{tr}}) = \dim H^1(W, \sigma_{\mathcal{F}}^{\text{tr}}) = 1$ . Therefore  $L_\xi$  surjects on the space of holomorphic functions vanishing at zero.

Similarly  $L_\xi$  restricted to  $\mathcal{G}_\lambda^\perp$  is injective and, when  $\xi$  is diagonal, it is an isomorphism on  $\mathcal{G}_\lambda^\perp$ . As above, we see that  $\dim H^0(W, \theta^\xi) = \dim H^1(W, \theta)$ , for all  $\xi$ . But  $H^0(W, \theta^\xi) = \text{Ker } L_\xi = \text{Ker}(L_\xi | \mathcal{G}_\lambda) = \mathcal{G}_\lambda / L_\xi(\mathcal{G}_\lambda)$ , because  $\mathcal{G}_\lambda$  is finite dimensional, and  $H^1(W, \theta^\xi) = \text{Coker } L_\xi$ . Hence  $L_\xi$  maps  $\mathcal{G}_\lambda^\perp$  surjectively on itself.

3.3. END OF THE PROOF OF THE THEOREM. Consider the commutative diagram

$$\begin{array}{cccccccc}
 \rightarrow & H^0(W, \sigma) & \rightarrow & H^0(W, \sigma) & \xrightarrow{\delta} & H^1(W, \sigma_{\mathcal{F}}^{\text{tr}}) & \rightarrow & H^1(W, \sigma) & \rightarrow & H^1(W, \sigma) & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 \rightarrow & H^0(W, \theta) & \rightarrow & H^0(W, \theta) & \xrightarrow{\delta} & H^1(W, \theta^\xi) & \rightarrow & H^1(W, \theta) & \rightarrow & H^1(W, \theta) & \rightarrow \\
 & & & & \searrow & & & \downarrow p & & & \\
 & & & & & H^1(W, \theta_{\mathcal{F}}^{\text{tr}}) & & \downarrow & & & \\
 & & & & & & & H^2(W, \theta_{\mathcal{F}}^{\text{tr}}) & & & 
 \end{array}$$

where the first row is mapped in the second one by the multiplication by



$\xi$ , and the vertical column is the cohomology exact sequence associated to (2.1.3).

By 2.3, we have to check that the restriction of  $p \circ \delta$  to the subspace  $T_0S$  of  $H^0(W, \theta)$  is an isomorphism on  $H^1(W, \theta_{\mathcal{F}}^{\text{tr}})$ .

By the lemma, the map  $p$  as well as both maps  $\delta$  are surjective. Also the restriction of  $\delta$  to the vector subspace of  $H^0(W, \theta)$  generated by  $T_0S$  and  $\xi$  is an isomorphism on  $H^1(W, \theta^{\xi})$ . But  $\delta\xi$  generates the kernel of  $p$ ; hence  $p \circ \delta|_{T_0S}$  is an isomorphism.

### Appendix: Versal deformation of Hopf manifolds

A.1.  $\mu$ -RESONANT MAPS. Let  $\mu = (\mu_1, \dots, \mu_n)$  be a sequence of non zero complex numbers such that  $|\mu_i| < 1$ . A (multiplicative)  $\mu$ -resonant monomial (cf. Arnold [1], p. 185) is a polynomial map of  $\mathbb{C}^n$  in  $\mathbb{C}^n$  of the form  $z \rightarrow az^m e_s$  such that

$$\mu_s = \mu^m.$$

Here  $m = (m_1, \dots, m_n)$  is a sequence of positive integers,  $\mu^m = \mu_1^{m_1} \dots \mu_n^{m_n}$ ,  $a \in \mathbb{C}$  and  $e_1, \dots, e_n$  is the canonical basis of  $\mathbb{C}^n$ . We shall assume that

$$0 < |\mu_1| \leq |\mu_2| \leq \dots \leq |\mu_n| < 1.$$

A  $\mu$ -resonant polynomial map  $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a sum of  $\mu$ -resonant monomials. Equivalently  $f$  is a polynomial map of  $\mathbb{C}^n$  in  $\mathbb{C}^n$  which commutes with the diagonal linear map  $d_\mu: (z_1, \dots, z_n) \rightarrow (\mu_1 z_1, \dots, \mu_n z_n)$ . Note that  $f(0) = 0$ .

The set of  $\mu$ -resonant polynomial maps is a subalgebra  $R_\mu$  of the algebra of polynomial maps of  $\mathbb{C}^n$  in  $\mathbb{C}^n$ . It is finite dimensional. The elements of degree one in  $R_\mu$  are represented by matrices with possibly non zero blocs along the diagonal with size equal to the number of times a  $\mu_i$  is repeated.

Let  $G_\mu$  be the group of invertible elements in  $R_\mu$ . An element  $f$  in  $R_\mu$  is invertible iff its linear part (or equivalently its derivative  $f'(0)$  at 0) is invertible. Indeed, after conjugation with a linear automorphism in  $R_\mu$ , one can assume that  $f$  is under lower triangular Jordan form. The  $i^{\text{th}}$ -coordinate is the sum of a non zero multiple of  $z_i$  and a polynomial containing only  $z_1, \dots, z_{i-1}$ .

$G_\mu$  is a connected complex Lie group, open in  $R_\mu$ . The kernel of the projection of  $G_\mu$  on the group  $G_\mu^1$  of linear automorphisms in  $R_\mu$  is nilpotent.

The Lie algebra  $\mathcal{G}_\mu$  of  $G_\mu$  is the space of (multiplicatively)  $\mu$ -resonant vector fields on  $\mathbb{C}^n$ , namely those vector fields which are linear combination of vector fields of the form  $z^m \partial / \partial z_s$ , where  $\mu_s = \mu^m$ . The  $\mu$ -resonant

vector fields can also be characterized as vector fields invariant by  $d_\mu$ . As a vector space,  $\mathcal{G}_\mu$  is isomorphic to  $R_\mu$ .

**A.2. DEFINITION OF HOPF MANIFOLDS.** In this paragraph, we give several equivalent definitions for a Hopf manifold. We begin with the one which is apparently the most general.

A Hopf manifold of dimension  $n > 1$  is a complex manifold  $W_f$  which is the quotient of  $W = \mathbb{C}^n - \{0\}$  by an infinite cyclic group acting holomorphically, and properly discontinuously on  $W$ . It is proved by C. Borcea [2] that this group is generated by an element  $f$  such that  $\sup_{|z| < a} |f^m(z)| \rightarrow 0$  when  $m \rightarrow \infty$  for any  $a$ , and that  $W_f$  is compact.

$f$  extends to an automorphism of  $\mathbb{C}^n$ . We claim that the eigenvalues of the differential  $f'(0)$  of  $f$  at 0 are of absolute value smaller than one. Indeed let  $v$  be an eigenvector corresponding to an eigenvalue  $\mu$  of  $f'(0)$ , and let  $p$  be a linear projection of  $\mathbb{C}^n$  on the one-dimensional subspace  $V$  generated by  $v$ . For  $m$  big enough, the restriction of  $f^m$  to  $V$  composed with  $p$  is a holomorphic map mapping the unit disk in  $V$  in a disk of smaller radius. The derivative at 0 of this map is  $\mu^m$ , and by Schwarz lemma,  $|\mu|$  must be smaller than one.

So we could have defined a Hopf manifold of dimension  $n$  as a compact complex manifold  $W_f$  which is the quotient of  $W$  by a properly discontinuous group generated by an automorphism  $f$  of  $\mathbb{C}^n$  fixing 0 and such that the eigenvalues  $\mu_1, \dots, \mu_n$  of  $f'(0)$  are inside the unit circle. According to the Poincaré-Dulac theorem (Cf. Arnold [1], p. 187), there is a holomorphic isomorphism  $h$  of a neighbourhood of 0 on a neighbourhood of 0 such that  $hfh^{-1}$  is the restriction of a polynomial map  $\tilde{f}$  of  $\mathbb{C}^n$  which is  $\mu$ -resonant. We have seen in A.1 that  $\tilde{f}$  is bijective and as each orbit of  $f$  and  $\tilde{f}$  meets an arbitrarily small neighbourhood of 0 in  $\mathbb{C}^n$ , the map  $\tilde{h}$  extends to a global automorphism  $h$  of  $\mathbb{C}^n$  such that  $\tilde{f} = \tilde{h}f\tilde{h}^{-1}$ .

Eventually we can equivalently define a Hopf manifold as the quotient  $W_f$  of  $W = \mathbb{C}^n - \{0\}$  by a polynomial automorphism  $f$  of  $\mathbb{C}^n$  whose derivative  $f'(0)$  at 0 has eigenvalues  $\mu = (\mu_1, \dots, \mu_n)$  inside the unit circle, and such that  $f$  is  $\mu$ -resonant.

**A.3. THEOREM:** *Let  $f$  be as above. A versal deformation of the Hopf manifold  $W_f$  is obtained as follows. Let  $S$  be a small complex submanifold in  $G_\mu$  passing through  $f$  and whose tangent space at  $f$  is complementary to the tangent space of the orbit of  $f$  under the action of  $G_\mu$  by conjugation on itself. Then the family  $W_s$ , where  $s \in S$ , is a versal deformation of  $W_f$  parametrized by  $S$ .*

*Let  $\theta_f$  be the sheaf of germs of holomorphic vector fields on  $W_f$ . Then  $H^i(W_f, \theta_f) = 0$  for  $i > 1$  and  $\dim H^0(W_f, \theta_f) = \dim H^1(W_f, \theta_f)$  is the dimension of the centralizer of  $f$  in  $G_\mu$  (which is also the dimension of the kernel of the endomorphism  $1 - f_*$  of  $\mathcal{G}_\mu$ ).*

For instance if  $f$  is diagonal, then  $\dim H^1(W_f, \theta_f) = \dim \mathcal{g}_\mu$ . In general  $n \leq \dim H^1(W_f, \theta_f) \leq \dim \mathcal{g}_\mu$ .

For  $n \geq 3$ , the dimension of  $\mathcal{g}_\mu$  is not bounded. For instance, if  $n = 3$  and  $\mu_1, \mu_2, \mu_3$  are real numbers such that  $\mu_2 = \mu_3$  and  $\mu_1 = \mu_2^2$ , then  $\dim \mathcal{g}_\mu = p + 6$ .

**A.4. PROOF OF THE THEOREM:** Consider the exact sequence (cf. [4], [2], [9]) analogous to (3.1.4):

$$\begin{aligned} 0 \rightarrow H^0(W_f, \theta_f) \rightarrow H^0(W, \theta) \xrightarrow{1-f_*} H^0(W, \theta) \rightarrow H^1(W_f, \theta_f) \\ \rightarrow H^1(W, \theta) \xrightarrow{1-f_*} H^1(W, \theta) \end{aligned} \tag{A.4.1}$$

where  $\theta$  is the sheaf of germs of holomorphic vector fields on  $W$ .

The main facts are:

(a) for  $i > 0$ ,  $1 - f_* : H^i(W, \theta) \rightarrow H^i(W, \theta)$  is an isomorphism.

(b) Denote by  $\mathcal{g}_\mu^\perp$  the subspace of  $H^0(W, \theta)$  of those vector field of the form  $\sum a_m^s z^m \partial / \partial z_s$ , where the monomials  $a_m^s z^m \partial / \partial z_s$  are not  $\mu$ -resonant. Then  $1 - f_*$  maps  $\mathcal{g}_\mu$  in  $\mathcal{g}_\mu$  and is an isomorphism of  $\mathcal{g}_\mu^\perp$  on  $\mathcal{g}_\mu^\perp$ .

(c) Let  $f_s$  be a holomorphic family of automorphisms of  $\mathbb{C}^n$  depending on a parameter  $t$  in a small neighbourhood of 0 in  $\mathbb{C}$ , such that  $f = f_0$  and  $f_t \in G_\mu$ . For the corresponding family  $W_{f_t}$  of Hopf manifolds, the infinitesimal deformation  $\rho(\partial / \partial t) \in H(W_f, \theta_f)$  is the image by  $\delta$  of the vector field  $d/dt(f_t f_0^{-1})|_{t=0}$  on  $W$ .

Those facts imply the theorem. By (a) and (b), the restriction of  $\delta$  to the subspace  $\mathcal{g}_\mu$  of  $H^0(W, \theta)$  is a surjection on  $H^1(W_f, \theta_f)$ . The differential  $f^{-1}_*$  of the right translation by  $f^{-1}$  in  $G_\mu$  maps isomorphically the tangent space at  $f$  to the orbit of  $f$  on the subspace  $(1 - f_*)\mathcal{g}_\mu$  of  $\mathcal{g}_\mu$ , and maps isomorphically  $T_f S$  on a complement in  $\mathcal{g}_\mu$  of  $(1 - f_*)\mathcal{g}_\mu$ . By c), the Kodaira-Spencer map  $\rho : T_f S \rightarrow H^1(W_f, \theta_f)$  is the composition of  $f^{-1}_*$  with  $\delta$ , hence is an isomorphism by the exactness of (A.4.1).

Also (a), (b) and the exactness of (A.4.1) imply that  $H^i(W_f, \theta_f) = 0$  for  $i > 1$  and that  $H^0(W_f, \theta_f) = H^1(W_f, \theta_f)$ . The Lie algebra of the centralizer of  $f$  is the kernel of the map  $1 - f_* : \mathcal{g}_\mu \rightarrow \mathcal{g}_\mu$ , so is canonically isomorphic to  $H^0(W_f, \theta_f)$ . Hence the connected component of the centralizer of  $f$  in  $G_\mu$  (which acts naturally on  $W_f$ ) is the connected component of the group of analytic automorphisms of  $W_f$ .

(c) is proved in Douady [4] and the proof of a) and b) is parallel to the proof of lemma 3.2 and is partly contained in [2] or [9]. We have seen in lemma 3.2 that  $H^i(W, \theta) = 0$  for  $i \neq 0, n - 1$  and that  $H^{n-1}(W, \theta)$  is isomorphic to the convergent series in  $(\mathbb{C} - \{0\})^n$  of the form  $\sum a_m^s z^m \partial / \partial z_s$ , with  $m_i < 0$ . It is easy to check directly that  $(1 - f_*)|_{\mathcal{g}_\mu^\perp}$  is injective, as

well as  $1 - f_*: H^i(W, \theta) \rightarrow H^i(W, \theta)$  for  $i > 0$ . Indeed we can assume that  $0 < |\mu_1| \leq \dots \leq |\mu_n| < 1$  and that the linear part of  $f$  is under upper triangular form. Then for the order defined in 3.2 (proof of b)), we have

$$(1 - f_*)z^m \partial / \partial z_s = (1 - \mu_s / \mu^m) z^m \partial / \partial z_s + \text{bigger terms.}$$

Moreover  $(1 - \mu_s / \mu^m) \neq 0$  if all  $m_i$  are negative or if  $z^m \partial / \partial z_s \in \mathcal{G}_\mu^\perp$ . The surjectivity of  $(1 - f_*): H^i(W, \theta) \rightarrow H^i(W, \theta)$  for  $i > 0$  is also obvious if  $f$  is diagonal.

From the preceding discussion and the exactness of A.3.1, it follows that  $H^i(W_f, \theta_f) = 0$  for  $i > 1$  in case  $f$  is diagonal. By upper semi-continuity, this is still true for  $f$  close enough to a diagonal map; this is always the case up to conjugation (cf. A.2). As in the proof of lemma 3.2, c), we see that  $H^0(W_f, \theta_f) = H^1(W_f, \theta_f)$ , because  $\Sigma(-1)^i H^i(W_f, \theta_f) = 0$  for  $f$  diagonal, hence also for  $f$  close to diagonal. The restriction of  $1 - f_*$  to  $\mathcal{G}_\mu$  has kernel and cokernel in  $\mathcal{G}_\mu$  of the same dimension. We have seen that the kernel of  $1 - f_*$  restricted to  $\mathcal{G}_\mu^\perp$  is zero, so its cokernel in  $\mathcal{G}_\mu^\perp$  must also be zero. This completes the proof of b).

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Université de Genève  
 Faculté des Sciences  
 Section de Mathématiques  
 2–4, rue du Lièvre  
 Case Postale 124  
 1211 Genève 24  
 Switzerland