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## ABSOLUTELY EXTREMAL POINTS IN MINIMAL FLOWS

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### Abstract

A point of a minimal flow is called absolutely extremal if it is an extreme point in every affine embedding. We show that distal points are absolutely extremal and give an example of a weakly mixing minimal flow all of whose points are absolutely extremal.

### §1. Introduction

Let  $(X, T)$  be a metric flow i.e.  $X$  is compact metric and  $T$  a homeomorphism of  $X$  onto itself. A flow  $(Q, T)$  is called *affine* if  $Q$  is a convex compact subset of a locally convex linear space and  $T$  is an affine homeomorphism of  $Q$ . We say that a map  $\varphi: (X, T) \rightarrow (Q, T)$  is an *affine embedding* if  $\varphi$  is continuous, one to one, equivariant and  $\overline{co}(\varphi(X)) = Q$ . Thus for any embedding, the set  $ex(Q)$  of extreme points of  $Q$  is contained in  $\varphi(X)$ . We often identify  $X$  with  $\varphi(X)$ . If  $(X, T)$  is a minimal flow (every orbit is dense), then  $(\varphi(X), T)$  is minimal and we have  $\overline{ex}(Q) = \varphi(X)$ . Call a point  $x_0 \in X$  *absolutely extremal* if for every embedding  $\varphi: X \rightarrow Q$ ,  $\varphi(x_0)$  is an extreme point of  $Q$ .  $(X, T)$  is an absolutely extremal flow if every point of  $X$  is absolutely extremal.

Using this terminology a theorem of I. Namioka [3] asserts that a distal minimal flow is absolutely extremal. We simplify the proof of this theorem and generalize it in showing that every distal point of a minimal flow is absolutely extremal. We give examples of almost automorphic absolutely extremal flows  $(X, T)$  and  $(Y, T)$  such that in the product flow  $(X \times Y, T)$  which is minimal and almost automorphic, there are points which are not absolutely extremal. Finally using a result of del Junco and Keane we demonstrate the existence of a weakly mixing minimal absolutely extremal flow.

### §2. Distal points are absolutely extremal

For a flow  $(X, T)$  denote by  $\mathcal{P}(X)$  the space of probability measures on  $X$  equipped with the weak\* topology. The homeomorphism  $T$  of  $X$

induces an affine homeomorphism of  $\mathcal{P}(X)$  under which  $(\mathcal{P}(X), T)$  is an affine flow. For  $x \in X$ , let  $\delta_x$  be the point mass at  $x$ . If  $\varphi: X \rightarrow Q$  is an affine embedding then the barycenter map  $\beta: \mathcal{P}(X) \rightarrow Q$  is defined by

$$\beta(\lambda) = \int_X \varphi(x) d\lambda(x) \quad (\lambda \in \mathcal{P}(X)).$$

The map  $\beta$  is a continuous affine homomorphism (i.e. an equivariant map) of  $\mathcal{P}(X)$  onto  $Q$ . A point  $q \in Q$  is extremal iff  $\beta^{-1}(q) = \{\delta_x\}$  for some  $x \in X$  (with  $\varphi(x) = q$ ).

LEMMA 2.1: *Let  $(X, T)$  be a metric flow,  $\nu$  a probability measure on  $X$  and assume that in  $\mathcal{P}(X)$ ,  $\lim T^{n_i} \nu = \delta_{x_0}$  for some sequence  $\{n_i\}$  and a point  $x_0 \in X$ . Then there exist an  $F_\sigma$  subset  $A$  of  $X$  and a subsequence  $\{n'_i\}$  such that  $\nu(A) = 1$  and for every  $x \in A$ ,  $\lim T^{n'_i} x = x_0$ . In particular every two points of  $A$  are proximal.*

PROOF: Let  $n'_i \in \{n_j\}$  satisfy  $T^{n'_i} \nu(B_{1/i}(x_0)) > 1 - 2^{-(i+1)}$  ( $i = 1, 2, \dots$ ), where  $B_r(x_0)$  is the closed ball of radius  $r$  around  $x_0$ . Then

$$\nu \left( \bigcap_{i=k}^\infty T^{-n'_i} B_{1/i}(x_0) \right) > 1 - 2^{-k}$$

and thus

$$A = \bigcup_{k=1}^\infty \bigcap_{i=k}^\infty T^{-n'_i} B_{1/i}(x_0)$$

is the required set.

COROLLARY 2.2: *If  $(X, T)$  is a distal flow and  $\lim T^{n_i} \nu = \delta_{x_0}$  for some  $x_0 \in X$  and  $\nu \in \mathcal{P}(X)$ , then  $\nu = \delta_x$  for some  $x \in X$ .*

THEOREM 2.3: *Let  $(X, T)$  be a metric minimal flow,  $x_0 \in X$  a distal point then  $x_0$  is an absolutely extremal point.*

PROOF: Let  $\varphi: X \rightarrow Q$  be an affine embedding; we identify  $X$  with  $\varphi(X)$ . Let  $\beta: \mathcal{P}(X) \rightarrow Q$  be the barycenter map. If  $x_0$  is not an extreme point of  $Q$  then there exists a measure  $\nu \in \mathcal{P}(X)$  with  $\beta(\nu) = x_0$  and  $\nu \neq \delta_{x_0}$ . Taking  $1/2(\delta_{x_0} + \nu)$  instead of  $\nu$  we can assume  $\nu$  has an atom at  $x_0$ . Choose  $\bar{x} \in X$  which is extreme in  $Q$ ; then by minimality there exists a sequence  $\{n_i\}$  such that  $\lim T^{n_i} x_0 = \bar{x}$ . We can also assume that

$\lim T^{n_i} \nu = \bar{\nu}$  exists and then

$$\beta(\bar{\nu}) = \lim \beta(T^{n_i} \nu) = \lim T^{n_i} \beta(\nu) = \lim T^{n_i} x_0 = \bar{x}.$$

Thus  $\bar{\nu} = \delta_{\bar{x}}$  and  $\lim T^{n_i} \nu = \delta_{\bar{x}}$ . By lemma 2.1. there exist an  $F_\sigma$  subset  $A$  of  $X$  with  $\nu(A) = 1$  - hence  $x_0 \in A$  - and a subsequence  $\{n'_i\}$  such that  $\lim T^{n'_i} x = x_0$  for every  $x \in A$ . In particular every point of  $A$  is proximal to  $x_0$ . Since  $x_0$  is distal  $A = \{x_0\}$  and  $\nu = \delta_{x_0}$ , a contradiction. Thus  $x_0$  is extreme and the proof is completed.

### §3. Almost automorphic examples

Let  $I = [0, 1)$ ,  $\alpha$  an irrational number and  $(I, R_\alpha)$  the rotation by  $\alpha$  on  $I$ . Let  $x_0(n) = \text{sgn} \cos(2\pi n\alpha)$ ; consider  $x_0$  as an element of  $\{1, -1\}^{\mathbb{Z}} = \Omega$  and let  $T$  be the shift on  $\Omega$ . Set  $X = \bar{\mathcal{O}}(x_0)$ , the orbit closure of  $x_0$  in  $\Omega$ ; then  $(X, T)$  is an almost automorphic minimal flow. There exists a homomorphism  $\pi : (X, T) \rightarrow (I, R_\alpha)$  such that for  $\xi \in I \setminus E$ , where  $E = \{1/4 + n\alpha, 3/4 + n\alpha : n \in \mathbb{Z}\}$ ,  $\pi^{-1}(\xi) = \{\text{sgn} \cos(2\pi(n\alpha + \xi))\}$  and for  $\xi = 1/4 + k\alpha$ ,  $\pi^{-1}(\xi) = \{x^+, x^-\}$  where for  $n \neq -k$ ,  $x^+(n) = x^-(n) = \text{sgn} \cos(2\pi(n+k)\alpha)$  and  $x^+(-k) = 1$ ,  $x^-(-k) = -1$ . Similar situation exists for  $\pi^{-1}(\xi)$  where  $\xi = 3/4 + n\alpha$ . Since every point of  $\pi^{-1}(I \setminus E)$  is distal we conclude, by theorem 2.3 that these points are absolutely extremal. However it is clear that the points of the form  $x^\pm$  are also absolutely extremal. For if, say  $\beta(\nu) = x^+$  in some affine embedding then by the proof of theorem 2.3.  $\nu$  is supported by the proximal cell of  $x^+$  (i.e. the set  $\{y \in X : y \text{ is proximal to } x^+\}$ ) which in our case is the set  $\{x^+, x^-\}$ . Clearly then  $\nu = \delta_{x^+}$  and  $x$  is extremal. We have shown that every point of  $X$  is absolutely extremal. Next let  $Y$  be the orbit closure of  $y_0$  in  $\Omega$  where  $y_0(n) = \text{sgn} \cos(2\pi n\gamma)$  and  $\gamma$  is an irrational number independent of  $\alpha$ . Put  $z_0(n) = x_0(n) + \frac{1}{10}y_0(n)$  and set  $Z = \bar{\mathcal{O}}(z_0) \subset \mathbb{R}^{\mathbb{Z}}$ . Then  $(X, T)$  and  $(Y, T)$  are disjoint minimal flows and the minimal flow  $(X \times Y, T)$  is isomorphic to  $(Z, T)$ . Now  $\mathbb{R}^{\mathbb{Z}}$  with its product topology is a locally convex linear space and, denoting  $Q = \overline{\text{co}}(Z) \subset \mathbb{R}^{\mathbb{Z}}$ , we see that the inclusion map of  $Z$  into  $Q$  is an embedding of  $(Z, T)$  into the affine flow  $(Q, T)$ . Let  $x^\pm, y^\pm$  be the points in  $X$  and  $Y$  respectively, which lie over  $1/4 \in I$ . Then e.g.

$$x^+ + \frac{1}{10}y^- = \frac{10}{11}(x^+ + \frac{1}{10}y^+) + \frac{1}{11}(x^- + \frac{1}{10}y^-)$$

and we conclude that although  $x^+$  and  $y^-$  are absolutely extremal, the point  $(x^+, y^-)$  of the minimal flow  $X \times Y$  is not absolutely extremal. Thus the property of being an absolutely extremal point is not preserved under products.

Taking  $x_0(n) = f(\cos 2\pi n\alpha)$  where  $f$  is continuous on  $[-1, 1]$  except for at zero where it has say,  $[0, 1]$  as a limit set, the embedding

$X = \overline{\mathcal{O}}(x_0) \rightarrow \overline{co}(X) \subset \mathbb{R}^Z$  will yield a continuum of non absolutely extremal points of  $X$ .

If  $(X, T)$  is an almost automorphic minimal flow, there exist an almost periodic flow  $(Y, T)$  and an almost one to one homomorphism  $\pi: X \rightarrow Y$ . Clearly every pair  $(x, x') \in X \times X$  with  $\pi(x) = \pi(x')$  has the property that the only minimal set in  $\mathcal{O}(x, x')$  is the diagonal  $\Delta$ . For a general minimal flow  $(X, T)$  put

$$L = \{(x, x') \in X \times X: \Delta \text{ is the unique minimal set in } \overline{\mathcal{O}}(x, x')\}.$$

$L$  is an invariant equivalence relation but not necessarily closed. We shall use the following lemma in the next section.

LEMMA 3.1: *Let  $(X, T)$  be a minimal flow and  $\varphi: X \rightarrow Q$  an affine embedding. Suppose  $\beta(\nu) = \varphi(x)$  for some  $\nu \in \mathcal{P}(X)$  and  $x \in X$ . If  $y$  is an atom of  $\nu$  then  $(x, y) \in L$ .*

PROOF: Since  $\beta(1/2(\nu + \delta_x)) = \varphi(x)$  we can assume that  $x$  itself is an atom of  $\nu$ . Write  $\nu = a\delta_x + b\delta_y + (1 - (a + b))\theta$  where  $0 < a, b < 1$  and  $\theta \in \mathcal{P}(X)$ . If  $M \subset \overline{\mathcal{O}}(x, y)$  is a minimal set then, since  $X$  is minimal, there is a point  $(z, w) \in M$  with  $\varphi(z) \in ex(Q)$ . Let  $\{n_i\}$  be a sequence with  $\lim T^{n_i}(x, y) = (z, w)$  and we can assume that  $\lim T^{n_i}\theta = \tilde{\theta}$  and  $\lim T^{n_i}\nu = \tilde{\nu}$  exist. Then

$$\beta(\tilde{\nu}) = a\beta(\delta_z) + b\beta(\delta_w) + (1 - (a + b))\beta(\tilde{\theta}) = \varphi(z),$$

whence  $\delta_z = \delta_w = \tilde{\theta}$ . Thus  $M = \Delta$  and  $(x, y) \in L$ .

#### §4. A weakly mixing example

In this section we demonstrate the existence of a minimal absolutely extremal weakly mixing flow. Recall that a minimal flow  $(X, T)$  is P.O.D. if it is weakly mixing and for every  $x, y \in X, x \neq y$  there exists some  $n \neq 0$  with  $T^n y$  and  $x$  proximal. Every P.O.D flow is prime [1]. Let  $(X, T)$  be P.O.D. and suppose  $\varphi: X \rightarrow Q$  is an affine embedding of  $X$  and assume  $X \subset Q$ . Let  $\bar{\nu} \in \mathcal{P}(X)$  and  $x_0 \in X$  with  $\beta(\bar{\nu}) = x_0$ . If  $y$  is an atom of  $\bar{\nu}$  then by lemma 3.1.  $(x_0, y) \in L$ . However it is easy to see that in a P.O.D. flow  $L = \Delta$ , so that  $x_0 = y$  and  $\bar{\nu}$  can have an atom only at  $x_0$ . If  $\bar{\nu} \neq \delta_{x_0}$  then there exists a measure  $\nu \in \mathcal{P}(X)$  which is continuous (has no atoms) and for which  $\beta(\nu) = x_0$ .

Suppose further now that  $(X, T)$  is also strictly ergodic with an invariant measure  $\mu$  and that for every  $x \in X$  the set,

$$F_x = \{y \in X: (x, y) \text{ is not generic for } \mu \times \mu\},$$

is countable. We show that every point of  $X$  is absolutely extremal.

That such a flow exists is a result of A. del Junco and M. Keane [4]. They showed that the Chacon transformation which is P.O.D. [2] and strictly ergodic, has also the latter property. Set

$$F = \{(x, y) : (x, y) \text{ is not generic for } \mu \times \mu\},$$

then

$$\nu \times \nu(F) = \int 1_F d\nu \times \nu = \int \int 1_{F_x}(y) d\nu(y) d\nu(x) = 0,$$

since  $\nu$  is continuous and for every  $x$ ,  $F_x$  is countable.

If  $f(x, y)$  is a continuous function on  $X \times X$  then for  $(x, y) \notin F$  we have

$$\frac{1}{2N+1} \sum_{j=-N}^N f(T^j x, T^j y) \rightarrow \int f d\mu \times \mu.$$

In particular this convergence holds  $\nu \times \nu$  a.e. and integrating we get

$$\frac{1}{2N+1} \sum_{j=-N}^N \int \int f(T^j x, T^j y) d\nu(x) d\nu(y) \rightarrow \int f d\mu \times \mu.$$

Now let  $g$  be a continuous affine function on  $X$  (i.e.  $g$  is the restriction of an affine function on  $Q$ ). Then for  $f(x, y) = g(x)g(y)$  we have

$$\begin{aligned} \left( \int g d\mu \right)^2 &\leftarrow \frac{1}{2N+1} \sum_{j=-N}^N \int \int g(T^j x) g(T^j y) d\nu(x) d\nu(y) \\ &= \frac{1}{2N+1} \sum_{j=-N}^N \left( \int g(T^j x) d\nu(x) \right)^2. \end{aligned}$$

Since  $g$  is affine so is  $g \circ T^j$  and recalling our assumption that  $\beta(\nu) = x_0$  we see that  $\int g(T^j x) d\nu(x) = g(T^j x_0)$ . Thus the right hand side of the above equation equals  $1/(2N+1) \sum_{j=-N}^N (g(T^j x_0))^2$ . By strict ergodicity this tends to  $\int g^2 d\mu$  so that

$$\int g^2 d\mu = \left( \int g d\mu \right)^2.$$

Choosing  $g \neq 0$  with  $\int g d\mu = 0$  we get a contradiction. Thus  $x_0$  is extremal in  $Q$  and the proof is completed.

## PROBLEMS:

- (1) Is there a minimal flow no point of which is absolutely extremal?
- (2) Is every minimal flow with  $L = \Delta$ , absolutely extremal?
- (3) Is the homomorphic image of an absolutely extremal point an absolutely extremal point?

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*Added in proof:* This problem has been solved by D. Maon and S. Glasner; see “On absolutely extremal points”, to appear in this journal.