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ON TORSION IN THE SLOPE SPECTRAL SEQUENCE

Richard Crew *

Introduction

The purpose of this paper is to prove, and give some simple applications of, a formula relating the Hodge numbers of a variety X , smooth and proper over a perfect field k , and certain numerical invariants that can be extracted from the slope spectral sequence of X . These invariants are of two kinds; the first, $m^{i,j}(X)$, only depend on the Newton polygon of the crystalline cohomology $H_{\text{crys}}^{i+j}(X)$, while the second, $T^{i,j}(X)$, describe the torsion in the E_1 terms of the slope spectral sequence of X . We will make extensive use of the Illusie-Raynaud structure theory [4] of this spectral sequence. After proving this formula (Theorem 4 below) we give some simple applications. These all concern a *surface* X ; we give a criterion, in terms of the Hodge and crystalline cohomology of X , for the slope spectra sequence to degenerate, and prove a semicontinuity theorem for $T^{0,2}$ which generalizes a result of Nygaard [5]. Applications of theorem 4 to higher dimensional varieties figure in recent work of Ekedahl [2], who gives, notably, a general criterion for degeneration of the slope spectra sequence in terms of the Hodge and crystalline cohomology.

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Notation

All unexplained notation and terminology will be as in Illusie-Raynaud [4]. In what follows k is a perfect field, W its ring of Witt vectors, and K the fraction field of W . If M is a graded W -module, then we will take $\text{length}_W(M)$ to be the function which to the integer i assigns the number $\text{length}_W(M^i)$.

We will never make explicit use of hypercohomology, so that an expression such as $H^i(\Omega_X)$ denotes the cohomology of a graded sheaf, not a hypercohomology group.

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Let X be a smooth proper scheme over a perfect field k of characteristic $p > 0$. The *slope spectra sequence*

$$E_1^{p,q} = H^q(X, W\Omega_X^p) \Rightarrow H_{\text{cris}}^{p+q}(X/W) \quad (0.1)$$

is defined and studied in [3], [4]. We recall from [4] that the rows $H^q(X, W\Omega_X^p)$ in the matrix of E_1 terms can be viewed as graded modules over a graded ring R^* , the *Raynaud algebra* (c.f. [4] for the definition). One central result is a finiteness theorem for the R^* -modules $H^q(X, W\Omega_X^p)$, namely the assertion that they are *coherent* R^* -modules; we recall that

1. DEFINITION: A graded R^* -module M^* is *coherent* if it possesses a finite filtration with quotients of the following types:

- Type I_a: A W -module of finite length concentrated in a single degree
- Type I_b: A finite free W -module concentrated in a single degree. Via the action of F, V , such a module is an F -crystal with slopes contained in the interval $[0,1)$.
- Type II_i: The module U_i defined in [4] I 2.14.3. It is concentrated in two consecutive degrees and is killed by p .

(In [4], the notation of a coherent R^* -module is actually defined by an equivalent condition.)

Recall that the right R^* -module R_n^* is defined by $R_n^* = (V^n, dV^n) \setminus R^*$. If M^* is coherent, then any graded component of $\text{Tor}_i^R(R_n^*, M^*)$ has finite W -length. We denote by $\ell(M^*)(j)$ the \mathbb{Z} -valued function on \mathbb{Z} defined by

$$\ell(M^*)(j) = \sum_i (-1)^i \text{length}_W \text{tor}_i^R(R_n^*, M^*)^j \quad (1.1)$$

It is clear that ℓ is additive for exact sequences of coherent R^* -modules. Its value on the various modules of Types I_a, I_b, II_i is as follows:

2. LEMMA:

- (i) If M^* is of Type I_a, then $\ell(M^*) = 0$
- (ii) If M^* is of Type I_b, concentrated in degree 0, then

$$\begin{aligned} \ell(0) &= \text{length}_W(M/VM) \\ \ell(1) &= -\text{length}_W(M/FM) \end{aligned} \quad (M = M^0)$$

and $\ell(M^*)(j) = 0$ for $j \neq 0, 1$.

(iii) If M^* is of Type II_i , concentrated in degrees 0, 1, then

$$\ell(0) = \ell(2) = 1, \quad l(1) = 2$$

$$\ell(j) = 0 \quad \text{for } j \neq 0, 1, 2.$$

PROOF: This is a consequence of the explicit calculation of the Tors made in [4]. For M^* of Type I_a or I_b concentrated in degree 0, we have

$$\text{Tor}_0(R_1^*, M^*)^0 = M/VM \quad \text{Tor}_1(R_1^*, M^*)^0 = {}_V M$$

$$\text{Tor}_1(R_1^*, M^*)^1 = M/FM \quad \text{Tor}_2(R_1^*, M^*)^1 = {}_F M$$

Then (ii) follows from M being p -torsion free (hence F - and V -torsion free) and (i) follows from the exactness of

$$0 \rightarrow {}_V M \rightarrow M \xrightarrow{V} M \rightarrow M/VM \rightarrow 0$$

$$0 \rightarrow {}_F M \rightarrow M \xrightarrow{F} M \rightarrow M/FM \rightarrow 0.$$

The proof of (iii) is similar.

We want to define the numerical invariants alluded to in the introduction. If M^* is coherent, we recall that $M^i \otimes \mathbb{Q}$ is an F -isocrystal with slopes in $[0, 1)$ for each i , and set

$$m_\lambda^i(M^*) = \text{the multiplicity of the slope } \lambda \text{ in } M^i \otimes \mathbb{Q}$$

Now choose a filtration for M^* as in Definition 1, and let

$$T^i(M^*) = \text{the number of times any module of the form } U_j(-i)$$

appears as a quotient in the given filtration.

In this situation we have

3. LEMMA: *The number $T^i(M^*)$ is independent of the filtration chosen for M^* , and we have*

$$\begin{aligned} \ell(M^*)(i) = & \sum_{\lambda \in [0, 1)} (1 - \lambda) m_\lambda^i(M^*) - \sum_{\lambda \in [0, 1)} \lambda m_\lambda^{i-1}(M^*) \\ & + T^i(M^*) + 2T^{i-1}(M^*) + T^{i-2}(M^*) \end{aligned}$$

PROOF: Given formula 3.1, the independence is obvious. It is enough, given the additivity of l , to check 3.1 when M^* is of Type I_a , I_b or II_i ,

and for these it is a consequence of lemma 2, once we show that for M^* of Type I_b in degree 0,

$$\sum_{\lambda \in [0, 1)} (1 - \lambda) m_{\lambda}^0(M^*) = \text{length}_W(M/VM) \quad M = M^0$$

$$\sum_{\lambda \in [0, 1)} \lambda m_{\lambda}^0(M^*) = \text{length}_W(M/FM) \quad M = M^0.$$

In fact, the right hand sides of the above equations are isogeny invariants, as are the left hand sides; it is therefore enough to check them for M of "standard type," i.e. of the form $D/(F^r - V^s)$, where D is the Dieudonne ring. For these latter modules the above equalities are clear.

Turning again to X smooth and proper over k , we can now define

$$\begin{aligned} m^{i,j}(X) = & \sum_{\lambda \in [i-1, i)} (\lambda - i + 1) \dim_K H_{\text{cris}}^{i-j}(X/W)_{\lambda} \\ & + \sum_{\lambda \in [i, i+1)} (i + 1 - \lambda) \dim_K H_{\text{cris}}^{i+j}(X/W)_{\lambda} \end{aligned} \quad (3.2)$$

where $H_{\text{cris}}^{i+j}(X/W)_{\lambda}$ is the part of $H_{\text{cris}}^{i+j}(X/W) \otimes \mathbb{Q}$ with slope λ , and

$$T^{i,j}(X) = T^i(H^j(X, W\Omega_X^i)) \quad (3.3)$$

The $m^{i,j}(X)$ could be called the Hodge-Newton numbers of X , for it is easily checked that they have the following interpretation: for each n , we form the polygon $\text{HN}(n)$ whose break-points are at the points $(0, 0)$ and

$$\left(\sum_{0 \leq \ell \leq i} m^{\ell, n-\ell}, \sum_{0 \leq \ell \leq i} \ell m^{\ell, n-\ell} \right)$$

for $0 \leq i \leq n$. Then $\text{HN}(n)$ is the uppermost convex polygon with integer slopes and integral breakpoints lying below the Newton polygon of $H_{\text{cris}}^n(X/W)$. To interpret the $m^{i,j}(X)$ in terms of the slopes spectral sequence 0.1 of X , we recall that $H^j(X, W\Omega_X^i) \otimes \mathbb{Q}$ is canonically isomorphic to the part of $H_{\text{cris}}^{i+j}(X/W) \otimes \mathbb{Q}$ where the geometric Frobenius acts with slopes in the interval $[i, i+1)$ (cf [3] II3.2), and that, via the isomorphism, the corresponding action on $H^j(X, W\Omega_X^i) \otimes \mathbb{Q}$ is $p^i F$. This means that the formula for $m^{i,j}(X)$ can be rewritten

$$\begin{aligned} m^{i,m}(X) = & \sum_{\lambda \in [0, 1)} \lambda \dim_K H^{j-1}(W\Omega^{i-1})_{\lambda} \otimes K \\ & + \sum_{\lambda \in [0, 1)} (1 - \lambda) \dim_K H^j(W\Omega^i)_{\lambda} \otimes K \end{aligned} \quad (3.4)$$

Now since $R\Gamma(X, W\Omega_X^\bullet)$ is an element of $D^b(R')$ whose homology is coherent we may apply ℓ : the result, by 3.1, 3.3, and 3.4, is

$$\begin{aligned} \ell(R\Gamma(X, W\Omega_X^\bullet))(i) &= \sum_j (-1)^j \ell(H^j(X, W\Omega_X^\bullet))(i) \\ &= \sum_j (-1)^j m^{i,j} + \sum_j (-1)^j T^{i,j} \\ &\quad + 2 \sum_j (-1)^j T^{i-1,j} + \sum_j (-1)^j T^{i-2,j} \end{aligned}$$

Let us write simply

$$\begin{aligned} m^i(X) &= \sum_j m^{i,j}(X) (-1)^j \\ T^i(X) &= \sum_j (-1)^j T^{i,j}(X) \end{aligned}$$

Then we have

4. THEOREM: *If X/k is a proper smooth variety over a perfect field, then for all i ,*

$$m^i(X) + T^i(X) + 2T^{i-1}(X) + T^{i-2}(X) + \chi(\Omega_X^i)$$

PROOF: By II. Theorem 1.2 of [4] we have

$$R_1 \otimes^L W\Omega_X^\bullet \simeq R_1 \otimes W\Omega_X^\bullet \simeq \Omega_X^\bullet$$

whence

$$R_1 \otimes^L R\Gamma(W\Omega_X^\bullet) \simeq R\Gamma(\Omega_X^\bullet)$$

This gives a spectral sequence

$$E_{p,q}^2 = \text{Tor}_p(R_1, H^q(W\Omega_X^\bullet)) \Rightarrow H^{q-p}(\Omega^\bullet)$$

which implies

$$\begin{aligned} \sum_{p,q} (-1)^{p+q} \text{length}_W \text{Tor}_p(R_1, H^q(W\Omega_X^\bullet)) \\ = \sum_q (-1)^q \text{length}_W H^q(\Omega_X^\bullet) \end{aligned} \tag{4.2}$$

The left hand side is just

$$\sum_q (-1)^q \ell(H^q(W\Omega_X)),$$

so that 4.1 follows from 4.2 and 3.5.

REMARK: If we introduce, following Ekedahl [2], the ‘‘Hodge-Witt’’ numbers

$$h_W^{p,q} \stackrel{\text{def}}{=} m^{p,q} + T^{p,q} - 2T^{p-1,q+1} + T^{p-2,q+2}$$

then the formula 4.1 takes the form

$$\sum_q (-1)^q h_W^{p,q} = \sum_q (-1)^q h^{p,q}$$

used by Ekedahl.

In order to illustrate 4.1 we shall consider the case of a *surface* X/k ; then in 4.1 the only relation of interest is the one given by $i = 0$, the other being linearly dependent on this one. After some rearrangement 4.1 gives, for $i = 0$ and $\dim X = 2$,

$$m^{0,2} + T^{0,2} + \delta = h^{0,2} \quad (4.3)$$

where δ is the ‘‘defect of smoothness’’

$$\begin{aligned} \delta &= h^{0,1} - m^{0,1} \\ &= \dim \text{Pic } X / \text{Pic}^{\text{red}} X \end{aligned} \quad (4.4)$$

To get the second line of 4.4, we recall that $H^1(W\mathcal{O}_X)$ is the covariant Dieudonne module of $\text{Pic}^{\text{red}} X$, so that $m^{0,1} = \dim_k H^1(W\mathcal{O}) / VH^1(W\mathcal{O}) = \dim \text{Pic}^{\text{red}} X$. We should also recall that for a surface X , $T^{0,2}$ is the only one of the $T^{i,j}$ that can possibly be nonzero (since the only differential in 0.1 that can be nonzero is $d_1^{0,2}$). Since degeneration of the slope spectral sequence at E_1 is equivalent to the vanishing of *all* the $T^{i,j}$, we obtain

5. COROLLARY: *If X/k is a surface, then the slope spectral sequence*

$$E_1^{p,q} = H^q(X, W\Omega_X^p) = H_{\text{cris}}^{p+q}(X/W)$$

degenerates at E_1 if and only if

$$m^{0,2} + \delta = h^{0,2}$$

6. COROLLARY: *If X/k is a surface and $m^{0,2} + \delta = h^{0,2}$, then all 1-forms on X are closed.*

PROOF: The hypothesis implies that the slope spectral sequence degenerates at E_1 , and it is known (e.g. [3]) that this implies that the 1-forms are closed.

The next theorem and its corollary were inspired by a result of Nygaard ([5], 3.1 and 3.2):

7. THEOREM: *If S is a k -scheme and X/S is proper and smooth of relative dimension 2, then the function on geometric points $\bar{s} \rightarrow S$ of S*

$$\bar{s} \rightarrow T^{0,2}(X_{\bar{s}})$$

is upper semicontinuous on S .

PROOF: For $i = 0$, 4.1 reads

$$m^{0,2}(X_{\bar{s}}) - m^{0,1}(X_{\bar{s}}) + T^{0,2}(X_{\bar{s}}) = h^{0,2}(X_{\bar{s}}) - h^{0,1}(X_{\bar{s}})$$

Now it is well known that $h^{0,2}(X_{\bar{s}}) - h^{0,1}(X_{\bar{s}}) = p_a(X_{\bar{s}})$ and $m^{0,1}(X_{\bar{s}}) = \frac{1}{2}b_1(X_{\bar{s}})$ are constant on S . In order, then, to show that $T^{0,2}(X_{\bar{s}})$ is upper semicontinuous in \bar{s} , it is enough to show that $m^{0,2}(X_{\bar{s}})$ is lower semicontinuous in \bar{s} , it is enough to show that $m^{0,2}(X_{\bar{s}})$ is lower semicontinuous. Now $m^{0,2}$ is just the length of the slope zero segment of the polygon $\text{HN}(2)$ associated to $H_{\text{cris}}^2(X_{\bar{s}})$. By [1], Theorem 2.6, we know that the Newton polygon of $H_{\text{cris}}^2(X_{\bar{s}})$ and hence the polygon $\text{HN}(2)$, rises under specialization; and this implies that $m^{0,2}$ is lower semicontinuous.

8. COROLLARY: *With X/S as before, suppose in addition that S is connected. If there is a geometric point $\bar{s} \rightarrow S$ such that $m^{0,2}(X_{\bar{s}}) + \delta(X_{\bar{s}}) = h^{0,2}(X_{\bar{s}})$, then the differential $d: f_*\Omega_{X/S}^1 \rightarrow f_*\Omega_{X/S}^2$ is zero.*

PROOF: Since $f_*\Omega_{X/S}^2$ is locally free, the subset of S on which $d = 0$ is closed. To show that it contains the generic point of S , we need only remark that $T^0(X_{\bar{s}}) = 0$ by 4.5, and that the condition $T^0 = 0$ is open, by Theorem 7. The result follows then from Corollary 6.

We conclude with a discussion of algebraic surfaces X satisfying $q = -p_a$, where $q = \dim \text{Alb } X$.

9. PROPOSITION: ([6], Prop. 4) *Let X/k be a smooth, complete surface. Then $q(X) = -p_a(X)$ if and only if $H^2(W\mathcal{O}_X)$ is V -torsion.*

PROOF: Since $H^2(W\Omega_X)$ is coherent, one has that $H^2(W\mathcal{O})$ is V -torsion if and only if $T^{0,2} = m^{0,2} = 0$. By 4.3 and 4.4, this last condition is equiv-

alent to $q(X) = -p_a(X)$, since $q(X) = m^{0,1}$.

In particular, $H^2(W\mathcal{O}_X)$ is of finite length for such a surface.

10. COROLLARY: *If X is a smooth proper surface with $q(X) = -p_a(X)$, then X is Hodge-Witt and $H_{cris}^2(X/W)$ is purely of slope 1. In particular, all global 1-forms on X are closed.*

PROOF: This follows from 9,5, and 6.

From Proposition 9 and its corollary, one may go on to compute the Hodge numbers of X in terms of the structure of the group $\text{Pic } X/\text{Pic}^{\text{red}} X$ and $\dim \text{Alb}(X)$. We refer the reader to [6] for the result and the details.

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