COMPOSITIO MATHEMATICA

ARNO VAN DEN ESSEN

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Compositio Mathematica, tome 56, nº 2 (1985), p. 259-269
<http://www.numdam.org/item?id=CM_1985__56_2_259_0>

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THE COKERNEL OF THE OPERATOR $\partial/\partial x_n$ ACTING ON A $\mathscr{D}_n$-MODULE, II

Arno van den Essen

In [5] we showed that the cokernel of the operator $\partial/\partial x_n$ of a holonomic $\mathscr{D}_n$-module M is a holonomic $\mathscr{D}_{n-1}$-module if M is so-called x_n-regular (see definition 1.1 below).

In this paper we generalize this result to arbitrary x_n-regular $\mathscr{D}_n$-modules, which are not necessary holonomic. In fact we show that for the category of x_n-regular $\mathscr{D}_n$-modules $d(\overline{M}) \leqslant d(M) - 1$, where $\overline{M} := M/\partial_n M$ and $\partial_n := \partial/\partial x_n$.

In [7] and [8] Kashiwara showed that $\mathscr{E}xt^i_{\mathscr{D}}(\mathscr{M}, \mathscr{N})$ is a $\mathbb{C}$-constructible sheaf if $\mathscr{M}$ and $\mathscr{N}$ are sheaves of holonomic $\mathscr{D}$-modules. So in particular its stalks are finite dimensional $\mathbb{C}$-vectorspaces. His proofs rests heavily on some purely analytic results. In section 2 we give an algebraic proof of this finiteness by using the cokernel results (Corollary 1.7). In fact, since our proof is algebraic we can also treat the case of formal power series at the same time (see Theorem 2.1).

In the remainder of this paper we use the following notations:
$\mathbb{N}^*$ = the set of natural numbers, $\mathbb{N} := \mathbb{N}^* \cup \{0\}$.
$n \in \mathbb{N}^*$.
k = a field of characteristic zero.
$\mathscr{O} = \mathscr{O}_n$ the ring of formal (resp. convergent) power series in $x_1, \dots, x_n$ over k (resp. $\mathbb{C}$).
$\mathscr{D} = \mathscr{D}_n = \mathscr{O}[\partial_1, \dots, \partial_n]$ is the ring of differential operators over $\mathscr{O}$, where $\partial_i := \partial/\partial x_i$. In particular we put $\partial := \partial_n$.

On $\mathscr{D}_n$ we have the increasing filtration $\{\mathscr{D}_n(v)\}_v$ where $\mathscr{D}_n(v)$ is the set of differential operators of order $\leqslant v$. Then $\mathrm{gr}(\mathscr{D}) \simeq \mathscr{O}[\zeta_1, \dots, \zeta_n]$, the ring of polynomials in $\zeta_1, \dots, \zeta_n$ with coefficients in $\mathscr{O}$. We identify these two rings. If $P \in \mathscr{D}$, $\sigma(P) \in \mathscr{O}[\zeta_1, \dots, \zeta_n]$ denotes the principal symbol of P and if $\mathscr{L}$ is a left ideal in $\mathscr{D}$ then $\sigma(\mathscr{L})$ means the ideal in $\mathscr{O}[\zeta_1, \dots, \zeta_n]$ generated by the elements $\sigma(P)$, where P runs through $\mathscr{L}$.

Let M be a left $\mathscr{D}$-module (all $\mathscr{D}$-modules will be left $\mathscr{D}$-modules).

$M_* :=$ the kernel of $\partial_n : M \to M$.

$\overline{M} :=$ the cokernel of $\partial_n : M \to M$ i.e. $\overline{M} := M/\partial_n M$.

$d(M) :=$ the dimension of M.

If R is a commutative ring dim $R :=$ krull dim R.

Let τ be a k (resp. $\mathbb{C}$)-derivation of $\mathcal{O}$ and $m \in M$. Then

$$E_\tau(m) := \sum_{i=0}^{\infty} \mathcal{O}\tau^i m.$$

An element $g \in \mathcal{O}$ is x_n-regular if $g(0,\ldots,0, x_n) \neq 0$.

If A is an arbitrary ring $\boldsymbol{M}(A)$ denotes the category of left A-modules of finite type. Finally, $K^{\cdot}(\partial_1,\ldots,\partial_n \mid M)$ denotes the Koszul complex of the commuting operators $\partial_1,\ldots,\partial_n$ on M (see [1], Chap. 2, 4.13). The i^{th} cohomology group of this complex is denoted by $H^iK(\partial_1,\ldots,\partial_n \mid M)$. Finally, if $\mathfrak{m}$ is an ideal in some ring A then $r(\mathfrak{m})$ is the radical ideal of $\mathfrak{m}$, i.e. the set of all $a \in A$ such that $a^P \in \mathfrak{m}$.

§1. An estimate for the dimension of $\overline{M}$

DEFINITION 1.1: A $\mathcal{D}$-module is called x_n-regular if there exists an $f \in \mathcal{O}$, x_n-regular such that $E_{f\partial}(m) \in \boldsymbol{M}(\mathcal{O})$, all $m \in M$. The category of x_n-regular $\mathcal{D}$-modules which satisfy the condition above for the element f is denoted $nR(f)$.

REMARK 1.2: Let M be a cyclic $\mathcal{D}$-module i.e. $M = \mathcal{D}m$ for some $m \in M$. Since for every derivation τ of $\mathcal{O}$ and every $P \in \mathcal{D}$ $E_\tau(m) \in \boldsymbol{M}(\mathcal{O})$ implies $E_\tau(Pm) \in \boldsymbol{M}(\mathcal{O})$ (cf. [2], Ch. II, prop. 1.3.2)) M belongs to $nR(f)$ iff $E_{f\partial}(m) \in \boldsymbol{M}(\mathcal{O})$. In this case there exists $r \in \mathbb{N}$ such that

$$(f\partial)^r m \in \sum_{j=0}^{r-1} \mathcal{O}(f\partial)^j m.$$

Consequently $(f\zeta_n)^r \in \sigma(\mathcal{L})$, where $\mathcal{L} = \mathrm{Ann}_{\mathcal{D}} m$. So $f\zeta_n \in J(M) := r(\sigma(\mathcal{L}))$. So we proved for $M = \mathcal{D}m$:
(1.3) If $M \in nR(f)$, then $f\zeta_n \in J(M)$.

Observe that the converse of (1.3) does not hold: take $M := \mathcal{D}_2/(\partial_2^2 - \partial_1)$. Then it is easy to see that $M \notin 2R(f)$, for every $f \in \mathcal{O}$ which is x_2-regular, however $J(M) = (\zeta_n)$.

LEMMA 1.4: *Let* $0 \to M_1 \to M \to M_2 \to 0$ *be an exact sequence of left* $\mathcal{D}$*-modules. Then* $M \in nR(f)$ *iff* M_1 *and* $M_2 \in nR(f)$.

PROOF: Obvious. cf [2], Ch. II, Lemma 1.8.

REMARK 1.5: Each holonomic $\mathcal{D}$-module is x_n-regular on some suitable coordinates $x_1,\ldots,x_n$. This can be seen as follows. In [4] we showed that if M is a holonomic $\mathcal{D}$-module, then $M[g^{-1}] \in \boldsymbol{M}(\mathcal{O}[g^{-1}])$, for some $g \neq 0$, $g \in \mathcal{O}$. Making a suitable coordinate transformation we can achieve that g is x_n-regular. But this implies that M is x_n-regular (cf. [5]).

The main result of this section is

THEOREM 1.6: *If $M \in \boldsymbol{M}(\mathscr{D})$ is x_n-regular and $M \neq 0$, then*
(1) $\overline{M} \in \boldsymbol{M}(\mathscr{D}_{n-1})$.
(2) $d(\overline{M}) \leqslant d(M) - 1$.

COROLLARY 1.7: *If M is holonomic and x_n-regular, then $\overline{M}$ is holonomic.*

PROOF: Let $\overline{M} \neq 0$. Then Th. 1.6 gives $\overline{M} \in \boldsymbol{M}(\mathscr{D}_{n-1})$, whence $d(\overline{M}) \geqslant n - 1$ (cf. [1], Ch. 2, Th. 7.1 and Ch. 3, prop. 1.8). Also by Th. 1.6 $d(\overline{M}) \leqslant n - 1$, implying $d(\overline{M}) = n - 1$.

COROLLARY 1.8: *If M is holonomic, then there exist coordinates $x_1, \dots, x_n$ such that $\overline{M}$ is holonomic.*

PROOF: Apply remark 1.5 and Cor. 1.7.

PROOF OF TH. 1.6 STARTED: (reduction to the case of a cyclic $\mathscr{D}$-module).

Let $0 \to M_1 \to M \to M_2 \to 0$ be exact. Then $\overline{M_1} \to \overline{M} \to \overline{M_2}$ is an exact sequence of $\mathscr{D}_{n-1}$-modules. It is easy to verify that the following holds:

If $\overline{M_1}, \overline{M_2} \in \boldsymbol{M}(\mathscr{D}_{n-1})$, then $\overline{M} \in \boldsymbol{M}(\mathscr{D}_{n-1})$ and if $d(\overline{M_i}) \leqslant d(M_i) - 1$, all $1 \leqslant i \leqslant 2$, then $d(\overline{M}) \leqslant d(M) - 1$.

Consequently, since $M \in \boldsymbol{M}(\mathscr{D})$ an induction on the number of generators of M shows that it suffices to treat the case of a cyclic $\mathscr{D}$-module i.e. $M = \mathscr{D}m$ for some $m \in M$.

Before we continue the proof of Th. 1.6 we recall Cor. 3 of [5]:

LEMMA 1.9: *Let $M = \mathscr{D}m$ be x_n-regular. Then*
(1) *$\overline{\Gamma}$ is a good filtration on $\overline{M}$, where $\overline{\Gamma}_v := \Gamma_v + \partial M/\partial M$ and $\Gamma_v = \mathscr{D}(v)m$, all $v \geqslant 0$.*
(2) $\overline{M} \in \boldsymbol{M}(\mathscr{D}_{n-1})$.

PROOF OF TH. 1.6 (FINISHED): Notations as in lemma 1.9, $\Gamma := \{\Gamma_v\}_v$, $\overline{\Gamma} := \{\overline{\Gamma}_v\}_v$. Put $\mathscr{L} := \operatorname{Ann}_{\mathscr{D}} m$. Then

$$\operatorname{gr}_\Gamma(M) = \operatorname{gr}(\mathscr{D})m \simeq \operatorname{gr}(\mathscr{D})/\sigma(\mathscr{L}).$$

Furthermore, putting $\psi(x + \Gamma_{v-1}) = x + \Gamma_{v-1} + \partial M$, all $x \in \Gamma_v$, all $v \geqslant 0$ we get a $\operatorname{gr}(\mathscr{D}_{n-1})$-linear map from $\operatorname{gr}_\Gamma(M)$ onto $\operatorname{gr}_{\overline{\Gamma}}(\overline{M})$ and one easily verifies that $\zeta_n \operatorname{gr}_\Gamma(M) \subset \operatorname{Ker} \psi$.

So we get a surjective $\operatorname{gr}(\mathscr{D}_{n-1})$-linear map from $\operatorname{gr}_\Gamma(M)/\zeta_n \operatorname{gr}_\Gamma(M)$ onto $\operatorname{gr}_{\overline{\Gamma}}(\overline{M})$. This implies

$$I_0 := \operatorname{gr}(\mathscr{D}_{n-1}) \cap (\sigma(\mathscr{L}) + (\zeta_n)) \subset \bar{I} := \operatorname{Ann}_{\operatorname{gr}(\mathscr{D}_{n-1})} \operatorname{gr}_{\overline{\Gamma}}(\overline{M}).$$

Put

$$R := \mathcal{O}_{n-1}[\zeta_1, \ldots, \zeta_{n-1}],\ S := \mathcal{O}_n[\zeta_1, \ldots, \zeta_n]$$

and

$$J := r(\sigma(\mathscr{L})).$$

By lemma 1.9 we get $d(\overline{M}) = \dim R/\bar{I}$. Since $I_0 \subset \bar{I}$ we get

$$d(\overline{M}) = \dim R/\bar{I} \leqslant \dim R/I_0 = \dim R/R \cap (J + (\zeta_n)).$$

Since by Gabber's theorem J is an involutive ζ-homogeneous ideal in S theorem 1.6. follows from (1.3) and

LEMMA 1.10: *Let f be an x_n-regular element of $\mathcal{O}$ and let J be an involutive ζ-homogeneous radical ideal in S satisfying $f\zeta_n \in J$ and $J \neq S$. Then*

$$\dim R/R \cap (J + (\zeta_n)) = \dim S/J - 1.$$

PROOF: (1) Let $J = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_t$, $t \in \mathbb{N}$ be the decomposition of J in minimal prime components. Then

$$r(J + (\zeta_n)) = \bigcap_{i=1}^{t} r(\mathfrak{p}_i + (\zeta_n)).$$

So

$$d := \dim R/R \cap (J + (\zeta_n)) = \dim R/R \cap r(J + (\zeta_n))$$

$$= \dim R/ \bigcap_{i=1}^{t} r(\mathfrak{p}_i + (\zeta_n)) \cap R$$

$$= \max \dim R/R \cap (\mathfrak{p}_i + (\zeta_n)).$$

We shall prove

$$\dim R/R \cap (\mathfrak{p}_i + (\zeta_n)) = \dim S/\mathfrak{p}_i - 1, \quad \text{all } i \tag{1.11}$$

whence

$$d = \max \dim S/\mathfrak{p}_i - 1 = \dim S/J - 1$$

as desired. So it remains to prove (1.11).

(2) Let $1 \leqslant i \leqslant t$ and put $\mathfrak{p} := \mathfrak{p}_i$. Since $f\zeta_n \in J$ we distinguish two cases

(i) $f \in \mathfrak{p}$.

(ii) $f \notin \mathfrak{p}$. Then $\zeta_n \in \mathfrak{p}$ (since $f\zeta_n \in \mathfrak{p}$).

Before we consider these two cases we need

LEMMA 1.12: $\mathfrak{p}$ *is involutive.*

PROOF: Since $\cap_{j \neq i} \mathfrak{p}_j \not\subset \mathfrak{p}$, there eixsts $c \in \cap_{j \neq i} \mathfrak{p}_j$ with $c \notin \mathfrak{p}$. Let a, $b \in \mathfrak{p}$. We must show $\{a, b\} \in \mathfrak{p}$ (where $\{\ ,\ \}$ denotes the Poisson-bracket on S). Obviously ac, $bc \in J$. So $\{ac, bc\} \in J$, since J is involutive. The Poisson-bracket is a bi-derivation on S, so we get

$$a\{c, b\}c + a\{c, c\}b + c\{a, b\}c + c\{a, c\}b \in \mathfrak{p}.$$

Since a, $b \in \mathfrak{p}$ it follows that $c^2\{a, b\} \in \mathfrak{p}$. Finally $c \notin \mathfrak{p}$ implies $\{a, b\} \in \mathfrak{p}$.

Case (i): Since f is x_n-regular we have $\mathcal{O}/f\mathcal{O} \in \boldsymbol{M}(\mathcal{O}_{n-1})$ implying $B := S/\mathfrak{p} + (\zeta_n) \in \boldsymbol{M}(R)$. Put $A := R/R \cap (\mathfrak{p} + (\zeta_n))$. Then $A \to B$ is a finite and hence integral extension of noetherian rings and [9], Th. 20, p. 81 implies that dim $A =$ dim B.

So it remains to prove that dim $S/\mathfrak{p} + (\zeta_n) =$ dim $S/\mathfrak{p} - 1$. First observe that $\mathfrak{p}$ is ζ-homogeneous (since J is so). Hence $\mathfrak{p} + (\zeta_n) \neq S$, for otherwise $1 \in \mathfrak{p} + (\zeta_n)$ implying $1 \in \mathfrak{p}$, a contradiction. Finally we show $\zeta_n \notin \mathfrak{p}$, which then gives dim $B =$ dim $S/\mathfrak{p} - 1$.

Let $\zeta_n \in \mathfrak{p}$. By lemma 1.12 $\partial(f) = \{\zeta_n, f\} \in \mathfrak{p}$ (since $f \in \mathfrak{p}$). Similarly $\partial^2(f) = \{\zeta_n, \partial(f)\} \in \mathfrak{p}$. Repeating this argument we find $\partial^d(f) \in \mathfrak{p}$ where d is the x_n-order of $f(0, \ldots, x_n)$. Hence $1 \in \mathfrak{p}$, a contradiction. So $\zeta_n \notin \mathfrak{p}$.

Case (ii) So $f \notin \mathfrak{p}$ and $\zeta_n \in \mathfrak{p}$. Put

$$\mathfrak{p}_0 := \{a(x', \zeta', 0, 0) \mid a \in \mathfrak{p}\},\ x' := (x_1, \ldots, x_{n-1}),$$
$$\zeta' := (\zeta_1, \ldots, \zeta_{n-1}).$$

Then $\mathfrak{p}_0$ is an ideal in R and $\mathfrak{p}_0 + (\zeta_n) + (x_n) = \mathfrak{p} + (\zeta_n) + (x_n)$. So since $\zeta_n \in \mathfrak{p}$ we have

$$\mathfrak{p}_0 + (\zeta_n) + (x_n) = \mathfrak{p} + (x_n). \tag{1.14}$$

We claim that $\mathfrak{p} + (x_n) \neq S$. This can be seen as follows. Since J is ζ-homogeneous, $\mathfrak{p}$ is ζ-homogeneous. Suppose $1 \in \mathfrak{p} + (x_n)$. Then $1 + ax_n \in \mathfrak{p}$, for some $a \in S$. Consequently $1 + a_0 x_n \in \mathfrak{p}$, where $a = a_0 + a_1 + \ldots + a_d$ is the development of a in ζ-homogeneous parts. In particular $a_0 \in \mathcal{O}$. Hence $1 + a_0 x_n \in \mathfrak{p}$ implies $1 \in \mathfrak{p}$, a contradiction. So

$$\mathfrak{p} + (x_n) \neq S. \tag{1.15}$$

Consequenctly we will derive

$$\bigcap_{q=1}^{\infty} \mathfrak{p} + (x_n^q) = \mathfrak{p}. \tag{1.16}$$

For apply Krull's theorem to the Noetherian integral domain $A: = S/\mathfrak{p}$ and the ideal $\mathfrak{m}: = x_n A$. By (1.15) $\mathfrak{m} \neq A$, so we have $\cap \mathfrak{m}^q = (0)$, which implies (1.16). Now we need.

LEMMA 1.17: $\mathfrak{p} \cap R = \mathfrak{p}_0$.
Assume this lemma, then we get

$$R/(\mathfrak{p} + (\zeta_n)) \cap R = R/\mathfrak{p}_0 \tilde{\to} S/\mathfrak{p}_0 + (x_n) + (\zeta_n)$$

$$= S/\mathfrak{p} + (x_n) \quad (\text{by } (1.14))$$

Hence

$$\dim R/\mathfrak{p} + (\zeta_n)) \cap R = \dim S/\mathfrak{p} + (x_n). \tag{1.18}$$

By lemma 1.12 $\mathfrak{p}$ is involutive. Consequently $x_n \notin \mathfrak{p}$ (if $x_n \in \mathfrak{p}$, then $1 = \{\zeta_n, x_n\} \in \mathfrak{p}$, a contradiction). Since by (1.15) $\mathfrak{p} + (x_n) \neq S$ we derive

$$\dim S/\mathfrak{p} + (x_n) = \dim S/\mathfrak{p} - 1. \tag{1.19}$$

So by (1.18) and (1.19) we get (1.11) as desired.

PROOF OF LEMMA 1.17: Obviously $\mathfrak{p} \cap R \subset \mathfrak{p}_0$. Now we show $\mathfrak{p}_0 \subset \mathfrak{p} \cap R$. Since $\zeta_n \in \mathfrak{p}$ it suffices to prove

If $a: = \Sigma a_i x_n^i \in \mathfrak{p}$, with $a_i \in R$, all $i \in \mathbb{N}$, then $a_0 \in \mathfrak{p}$.

So let $a \in \mathfrak{p}$. Then $a_0 \in \mathfrak{p} + x_n S'$, where $S': = \mathcal{O}[\zeta_1, \ldots, \zeta_{n-1}]$. By induction on q we will prove

$$a_0 \in \mathfrak{p} + x_n^q S', \quad \text{all } q \in \mathbb{N}^*. \tag{1.21}$$

Consequently, since obviously $S' \subset S$ (1.16) gives $a_0 \in \mathfrak{p}$. So it remains to prove (1.21). The case $q = 1$ is clear. Now assume

$$a_0 = g + x_n^q b, \quad g \in \mathfrak{p}, \quad b \in S'. \tag{1.22}$$

Write $b = \Sigma b_i x_n^i$, $b_i \in R$. Then

$$0 = \{\zeta_n, a_n\} = \{\zeta_n, g\} + q x_n^{q-1} b + x_n^q \{\zeta_n, b\}.$$

Since $\zeta_n,\ g \in \mathfrak{p}$, $\{\zeta_n,\ g\} \in \mathfrak{p}$, whence

$$x_n^{q-1} b_0 \in \mathfrak{p} + x_n^q S'.$$

Consequently $x_n^q b_0 \in \mathfrak{p} + x_n^{q+1} S'$. Substitute this in (1.22) and we obtain $a_0 \in \mathfrak{p} + x_n^{q+1} S'$ which proves (1.21), and this completes the proof of Lemma 1.17.

§2. Finite dimensionality of some Ext-groups

The main result of this section is

THEOREM 2.1: *Let M and N be holonomic left $\mathcal{D}$-modules. Then* $\mathrm{Ext}^i_{\mathcal{D}}(M, N)$ *are finite dimensional k-vectorspaces, all i.* By methods due to Kashiwara one can reduce the proof of this theorem to the case where $M = \mathcal{O}$ (see the proof of theorem 4.8 in [8]).

This reduction uses the fact that $\mathrm{Tor}_i^{\mathcal{O}}(M, N)$ is holonomic if M and N are holonomic (cf [1], Ch. 3, Th. 4.3). This result is an easy consequence of the following companion of Cor. 1.7: if M is a holonomic $\mathcal{D}_n$-module, then $M/x_n M$ is a holonomic $\mathcal{D}_{n-1}$-module (cf [1], Ch. 3, Th. 4.2). The proof of this last result essentially uses the existence of b-functions (for $f^s u$).

PROOF OF THEOREM 2.1: As remarked before it suffices to prove: if M is a holonomic left $\mathcal{D}$-module, then $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{O}, M)$ is a finite dimensional k-vectorspace. Since by [1], Chap. 6, Prop. 2.5.1

$$\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{O}, M) \tilde{\rightarrow} H^i_{DR}(M), \quad \text{all } 0 \leqslant i \leqslant n$$

where $H^i_{DR}(M)$ is the i^{th} cohomology group of the DeRham complex of M, theorem 2.1 follows from

PROPOSITION 2.2: *Let M be a holonomic left $\mathcal{D}$-module. Then $H^i_{DR}(M)$ is a finite dimensional k-vectorspace, all i.*

The proof of Prop. 2.2 is based on Cor. 1.8 and théorème (iii) of [3] which states that the ∂_n-kernel M_* of a holonomic $\mathcal{D}_n$-module is a holonomic $\mathcal{D}_{n-1}$-module.

If $n = 1$, then $H^0(M) = M_*$ and $H^1(M) = \overline{M}$. So for $n = 1$, prop. 2.2 is clear.

PROOF OF PROP. 2.2: By induction on n. By Cor. 1.8 we can choose coordinates $x_1, \ldots, x_n$ for $\mathcal{O}$ such that $\overline{M}$ is a holonomic left $\mathcal{D}_{n-1}$-module. Also by [3] théorème (iii) M_* is a holonomic $\mathcal{D}_{n-1}$-module. By [1], Chap. 2, Prop. 4.13 we have an exact sequence of k-vectorspaces

$$H^i K(M_*, \partial_1, \ldots, \partial_{n-1}) \rightarrow H^i(M) \rightarrow H^{i-1} K(\overline{M}, \partial_1, \ldots, \partial_{n-1}).$$

Hence our proposition follows immediately from this sequence by applying the induction hypothesis to M_* and $\overline{M}$.

§3. Miscellaneous results

Let M be a holonomic $\mathscr{D}$-module. As observed before, we showed in [3] that M_* is a holonomic $\mathscr{D}_{n-1}$-module. So we did not assume any x_n-regularity condition on M. However, if according to Remark 1.5 we have coordinates $x_1, \ldots, x_n$ such that $M[g^{-1}] \in \boldsymbol{M}(\mathscr{O}[g^{-1}])$, where $g \in \mathscr{O}$ is x_n-regular, then we can prove that M_* is either zero or a free $\mathscr{O}_{n-1}$-module of finite rank. The precise result is

Proposition 3.1: *Let M be a $\mathscr{D}$-module such that $M[g^{-1}] \in \boldsymbol{M}(\mathscr{O}[g^{-1}])$ for some x_n-regular element $g \in \mathscr{O}$ of order d. Then M_* is either zero or $M_* \simeq \mathscr{O}_{n-1}^r$, as $\mathscr{D}_{n-1}$-modules, for some $r \in \mathbb{N}^*$.*

Observe that we do not need to assume that M is a $\mathscr{D}$-module of finite type in Prop. 3.1. This is partially explained by

Lemma 3.2: *Let M be a $\mathscr{D}$-module such that $M[g^{-1}] = 0$, for some x_n-regular $g \in \mathscr{O}$ of order d. Then $M_* = 0$.*

To prove Prop. 3.1 and Lemma 3.2 we use the following crucial result of [3].

Lemma 3.3: *Let M be a $\mathscr{D}$-module and let $m_1, \ldots, m_s \in M_*$. If $a_1 m_1 + \ldots + a_s m_s = 0$, then $a_{1j} m_1 + \ldots + a_{sj} m_s = 0$, all $j \in \mathbb{N}$ (if $a \in \mathscr{O}$, then $a = \Sigma a_j x_n^j$, $a_j \in \mathscr{O}_{n-1}$).*

Corollary 3.4: *If a_1 is x_n-regular (of order d), then $m_1 \in \mathscr{O}_{n-1} m_2 + \ldots + \mathscr{O}_{n-1} m_s$ (read $m_1 = 0$ if $s = 1$).*

Proof: Apply Lemma 3.3 to $j = d$.

Proof of Lemma 3.2: Let $m \in M_*$. Then $g^q m = 0$, some $q \in \mathbb{N}^*$. Now apply Cor. 3.4 with $s = 1$ and $a_1 = g^q$.

Proof of Prop. 3.1: Put

$$M(T\colon g) = \{ m \in M \mid g^q m = 0, \text{ some } q \in \mathbb{N}^* \}.$$

Then $M(T\colon g)$ is a $\mathscr{D}$-module satisfying $M(T\colon g)[g^{-1}] = 0$, whence $M(T\colon g)_* = 0$, by Lemma 3.2. Consider the exact sequence

$$0 \to M(T\colon g) \to M \to \tilde{M} \to 0, \quad \text{where } \tilde{M} := M/M(T\colon g). \tag{*}$$

Since $\tilde{M}$ has no g-torsion $\tilde{M}$ is a submodule of $\tilde{M}[g^{-1}] \cong M[g^{-1}]$. This last module is holonomic since it belongs to $\boldsymbol{M}(\mathcal{O}[g^{-1}])$(cf [2], Ch. 3, Prop. 3.4). Consequently the submodule $\tilde{M}$ is also holonomic so in particular we have $\tilde{M} \in M(\mathscr{D})$. From (*) we deduce

$$0 \to M(T: g)_* \to M_* \to \tilde{M}_*.$$

Using $M(T: g)_* = 0$ we find the exact sequence

$$0 \to M_* \to \tilde{M}_* (\text{of } \mathscr{D}_{n-1}\text{-modules}).$$

Since $\tilde{M} \in M(\mathscr{D})$ we have $\tilde{M}_* \in \boldsymbol{M}(\mathscr{D}_{n-1})$ (by théorème i) of [3]). Whence $M_* \in \boldsymbol{M}(\mathscr{D}_{n-1})$. So it suffices to prove

$$E_{\partial_i}(\mathcal{O}_{n-1} m) \in \boldsymbol{M}(\mathcal{O}_{n-1}), \quad \text{all} \quad m \in M_*, \text{all } 1 \leqslant i \leqslant n-1, \qquad (**)$$

for then we get $M_* \in \boldsymbol{M}(\mathcal{O}_{n-1})$ and hence our proposition follows from 7.1, Chap. 5 of [1].

Proof of (**). Let $m \in M_* \subset M$. Since $M[g^{-1}] \in \boldsymbol{M}(\mathcal{O}[g^{-1}])$ there exists $h \in \mathbb{N}$ such that

$$g^h \partial_i^r m \in \sum_{j=0}^{r-1} \mathcal{O} \partial_i^j m, \quad \text{all } 1 \leqslant i \leqslant n.$$

Since $\partial_i^j m \in M_*$, all i, j Cor. 3.4 gives

$$\partial_i^r m \in \sum_{j=0}^{r-1} \mathcal{O}_{n-1} \partial_i^j m, \quad \text{all } i.$$

But this implies (**) which completes the proof of Prop. 3.1.

In the next proposition we will give an explicit description of $H'(M)$ in terms of the zero[th] DeRham group of a holonomic $\mathscr{D}_{n-1}$-module.

Let M be a holonomic $\mathscr{D}_n$-module. Then arguing as in remark 1.5 there exist coordinates $x_1, \ldots, x_n$ of $\mathcal{O}$ and an x_n-regular element $g \in \mathcal{O}$ such that $M[g^{-1}] \in \boldsymbol{M}(\mathcal{O}[g^{-1}])$. Since this implies that M is x_n-regular Cor. 1.7 gives that $M/\partial_n M$ is a holonomic $\mathscr{D}_{n-1}$-module. Now we can repeat this argument to the holonomic $\mathscr{D}_{n-1}$-module $M/\partial_n M$. So there exists coordinates $y_1, \ldots, y_{n-1}$ of $\mathcal{O}_{n-1}$ and an y_{n-1}-regular element $\Delta_{n-1} \in \mathcal{O}_{n-1}$ such that $(M/\partial_n M)[\Delta_{n-1}^{-1}] \in \boldsymbol{M}(\mathcal{O}_{n-1}[\Delta_{n-1}^{-1}])$. Observe that this new coordinates do not change x_n. We finally arrive at

Proposition 3.5: *Let M be a holonomic $\mathscr{D}_n$-module. Then there exist coordinates $x_1, \ldots, x_n$ of $\mathcal{O}$ and elements $\Delta_n, \Delta_{n-1}, \ldots, \Delta_1 \in \mathcal{O}$, where Δ_h is an x_h-regular element of $\mathcal{O}_h$ such that*

$$M/\partial_n M + \ldots + \partial_{h+1} M[\Delta_h^{-1}] \in \boldsymbol{M}(\mathcal{O}_h[\Delta_h^{-1}]), \quad \text{all } 1 \leqslant h \leqslant n. (3.6)$$

(if $h = n$, read $M[\Delta_n^{-1}] \in \boldsymbol{M}(\mathcal{O}[\Delta_n^{-1}])$.

In the next proposition we consider the situation of prop. 3.5 and give an explicit description of $H^i(M)$. The precise result is

PROPOSITION 3.7: *Let* x_i, Δ_i *be as om* Prop. 3.5. *Let* $n \geqslant 2$. *Then*

$$H^i(M) \simeq H^0K(\partial_1,\dots,\partial_{n-i} \mid M/\partial_n M + \dots + \partial_{n-i+1}M),$$

all $1 \leqslant i \leqslant n-1$.

PROOF: By [1], Chap. 2, prop. 4.13 we have an exact sequence

$$H^iK(\partial_1,\dots,\partial_{n-1} \mid M_*) \to H^i(M) \to H^{i-1}K(\partial_1,\dots,\partial_{n-1} \mid \overline{M})$$

$$\to H^{i+1}K(\partial_1,\dots,\partial_{n-1} \mid M_*).$$

By Prop. 3.1 M_* is either zero or isomorphic to $\mathcal{O}^r_{n-1}$. In both cases $H^iK(\partial_1,\dots,\partial_{n-1} \mid M_*) = 0$ if $i \geqslant 1$. So we conclude

$$H^i(M) \simeq H^{i-1}K(\partial_1,\dots,\partial_{n-1} \mid \overline{M}) \text{ if } 1 \leqslant i \leqslant n-1. \tag{3.8}$$

So if $n = 2$ we are done. Now let $n \geqslant 2$. If $i = 1$ we are done by (3.8). So let $i \geqslant 2$. We use induction on n. The hypothesis on M immediately gives the same hypothesis on the $\mathcal{D}_{n-1}$-module $\overline{M}$. So the induction gives

$$H^{i-1}K(\partial_1,\dots,\partial_{n-1} \mid \overline{M})$$

$$\simeq H^0K\big(\partial_1,\dots,\partial_{(n-1)-(i-1)} \mid \overline{M}/\partial_{n-1}\overline{M} + \dots + \partial_{(n-1)-(i-1)+1}\overline{M}\big)$$

$$\simeq H^0K(\partial_1,\dots,\partial_{n-i} \mid M/\partial_n M + \dots + \partial_{n-i+1}M).$$

Combining this with (3.8), our proposition follows.

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(Oblatum 4-XI-1983 & 30-VII-1984)

Institute for Mathematics
Catholic University
Toernooiveld
6525 ED Nijmegen
The Netherlands