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THE LENZ SHIFT AND WIENER SAUSAGE IN RIEMANNIAN MANIFOLDS *

Isaac Chavel and Edgar A. Feldman

In this paper we consider a given Riemannian manifold M, of dimension $n \geqslant 3$, with associated Laplace-Beltrami operator Δ acting on functions on M. Associated, in turn, to the Laplace operator is the heat kernel $p(x, y, t)$ with attendant Brownian motion X.

When M is compact one has a unique heat kernel; when M is noncompact we consider the minimal positive heat kernel. (Cf. Chavel [1, Chapters VI–VIII], Cheeger-Yau [4], Dodziuk [5], Karp-Li [7] and Minakshisundaram [8;9], for the necessary background.) In the compact case one automatically has the *conservation of heat property*

$$\int_M p(x, y, t)\, \mathrm{d}V(y) = 1 \tag{1}$$

for all x in M and $t > 0$, where $\mathrm{d}V$ denotes the Riemannian measure on M; and we shall *assume* the validity of (1) for all x in M and $t > 0$, when M is noncompact.

For any given Brownian path $X(\tau)$ we let $W_{t,\epsilon}(X)$ denote the tubular neighborhood of $X([0, t])$, of radius ϵ, in M – *the Wiener sausage of time t and radius* ϵ – and $V_{t,\epsilon}(X)$ its volume. Let c_{n-1} denote the $(n-1)$-area of the unit sphere in Euclidean space $\mathbb{R}^n$. Our interest is in establishing the formula

$$\lim_{\epsilon \downarrow 0} \epsilon^{2-n} V_{t,\epsilon} = (n-2) c_{n-1} t \tag{2}$$

in probability $\mathrm{d}P_x$ (where $\mathrm{d}P_x$ is the probability measure on Brownian paths concentrated on those starting at x – cf. below), for all x in M.

We actually obtain a more detailed version of (2), viz., if $I_{t,\epsilon}(X)$ denotes the indicator function of $W_{t,\epsilon}(X)$, and $\mathrm{d}\Psi_{t,\epsilon}(X)$ denotes the measure on M given by

$$[\mathrm{d}\Psi_{t,\epsilon}(X)](z) = \frac{[I_{t,\epsilon}(X)](z)}{(n-2)c_{n-1}\epsilon^{n-2}} \mathrm{d}V(z),$$

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then $\mathrm{d}\Psi_{t,\epsilon}(X)$ converges weakly $*$, as $\epsilon \downarrow 0$, to the measure, supported on $X([0, t])$, given by

$$f \to \int_0^t f(X(\tau))\,\mathrm{d}\tau,$$

in probability $\mathrm{d}P_x$, for all x in M, i.e.,

$$\lim_{\epsilon \downarrow 0} \mathrm{d}\Psi_{t,\epsilon}(X) = X_*(\mathrm{d}s\,|\,[0, t]) \tag{3}$$

weakly $*$, in probability $\mathrm{d}P_x$, where $\mathrm{d}s$ is Lebesque measure on $\mathbb{R}$.

A direct consequence of this result is the extension of the Lenz-shift phenomenon for the heat kernel and associated eigenvalue problems, as discussed in Kac [6], Rauch-Taylor [13], Simon [14, Chapter VII], to the general situations enumerated above. (Also, cf. Ozawa [10], Papanicolaou-Varadhan [11].) The result is as follows:

THEOREM: *Let* $\boldsymbol{M} = M \times M \times \cdots$, *and* $\not{f} = (\not{f}_1, \not{f}_2, \ldots)$ *denote an arbitrary element of* $\boldsymbol{M}$. *Let* $\rho : M \to [0, +\infty)$ *be bounded, continuous, on* M, *with*

$$\int_M \rho\,\mathrm{d}V = 1,$$

and endow $\boldsymbol{M}$ *with the probability measure*

$$\mathrm{d}U = \rho\,\mathrm{d}V \times \rho\,\mathrm{d}V \times \ldots.$$

For every $\not{f}$ *in* $\mathbf{M}$, *let*

$$B_N(\not{f}) = \cup_{j=1}^N B(\not{f}_j;\, \epsilon_N), \quad \Omega_N(\not{f}) = M \backslash B_N(\not{f}),$$

where $B(\not{f}_j;\, \epsilon_N)$ *denotes the geodesic disk centered at* $\not{f}_j$ *having radius* ϵ_N *and let* $q_N(\ ,\ ,\ ;\not{f})$ *denote the heat kernel of* $\Omega_N(\not{f})$ *with vanishing boundary data on the common boundary of* $B_N(\not{f})$ *and* $\Omega_N(\not{f})$. *Then*

$$\lim_{N \uparrow +\infty} N\epsilon_N^{n-2} = \alpha \quad \text{in } (0, +\infty)$$

implies that the heat semi-group associated to $q_N(\ ,\ ,\ ;\not{f})$ *converges strongly, in probability* $\mathrm{d}U$, *to*

$$\mathrm{e}^{t\{\Delta-(n-2)c_{n-1}\alpha\rho\}}$$

Also, in the compact case, when $\rho = 1/V(M)$, *we have* $\lambda_{j,N}(\not{f}) \to \lambda_j + (n-2)c_{n-1}\alpha/V(M)$, *as* $N \uparrow +\infty$, *in probability* $\mathrm{d}U$, *where* $\{\lambda_j\}_{j=1}^{\infty}$ *is the* *spectrum of the eigenvalue problem on M, and $\{\lambda_{j,N}(\not{f})\}_{j=1}^{\infty}$ is the spectrum for the Dirichlet eigenvalue problem on $\Omega_N(\not{f})$.*

We leave the details of the theorem to the references cited above. Here we shall devote our attention to (2) and (3).

The previous derivations of (2) in $\mathbb{R}^n$ use the Riemannian symmetry of $\mathbb{R}^n$ to study

$$\lim_{t\uparrow+\infty} V_{t,\epsilon}/t$$

via a result of Spitzer [15], and then obtain (2) using the scaling of Brownian motion, with respect to the radial variable ϵ, to change the "time" asymptotic result to a "radial" asymptotic result. In the general situation one does not necessarily have the Riemannian symmetry of M; one certainly does not have rescaling; and, most importantly, one easily sees that the asymptotic character of $V_{t,\epsilon}$, with respect to large t, reflects the specific geometry of M, in contrast to the universal character of (2).

We now give the basic idea of (2). The space of Brownian paths $\boldsymbol{W}$ under consideration in the compact case is the collection of all continuous maps of $[0, +\infty)$ into M; and in the noncompact case the collection of all continuous maps of $[0, \infty)$ into $M^* \equiv: M\cup\{\infty\}$ (the 1-point compactification of M) with the property that if $X(t_0) = \infty$ for some $t_0 > 0$, then $X(t) = \infty$ for all $t \geqslant t_0$. To each x in M is associated the probability measure $\mathrm{d}P_x$ on $\boldsymbol{W}$, concentrated on those paths starting at x, with the property that for any Borel set B in M, and $t > 0$, we have

$$P_x(X(t) \in B) = \int_B p(x, y, t)\,\mathrm{d}V(y).$$

The conservation property (1) states that $X(t)$ is in M, almost surely $\mathrm{d}P_x$, for all x in M and $t > 0$. We let E_x denote the expectation associated to $\mathrm{d}P_x$, i.e., for any measurable f on $\boldsymbol{W}$, in $L^1(\mathrm{d}P_x)$,

$$E_x(f) = \int_{\boldsymbol{W}} f\,\mathrm{d}P_x.$$

For any y in M, and $\epsilon > 0$, let $B(y;\,\epsilon)$ denote the open metric disk of radius ϵ centered at y, and for any Brownian path $X(\tau)$ let $\boldsymbol{T}_{B(y;\epsilon)}(X)$ denote the *first hitting time of* $B(y;\,\epsilon)$ *by* X, i.e.,

$$\boldsymbol{T}_{B(y;\epsilon)}(X) = \inf\{\tau > 0\colon\ X(\tau) \in B(y;\,\epsilon)\}$$

(should $X(\tau) \notin B(y;\,\epsilon)$ for all $\tau > 0$, then $\boldsymbol{T}_{B(y;\epsilon)} \equiv: +\infty$).

The key to our approach is the formula

$$\lim_{\epsilon\downarrow 0} \epsilon^{2-n} P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t) = (n-2)\boldsymbol{c}_{n-1}\int_0^t p(x, y, \tau)\,\mathrm{d}\tau \tag{4}$$

for all distinct x, y in M, and all $t > 0$. Now

$$W_{t,\epsilon}(X) = \{ y \in M \colon \mathrm{d}(y, X([0, t])) < \epsilon \}$$
$$= \{ y \in M \colon \boldsymbol{T}_{B(y;\epsilon)}(X) \leqslant t \};$$

so

$$E_x(V_{t,\epsilon}) = \int_M P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t)\, \mathrm{d}V(y). \tag{5}$$

Ignoring convergence questions for the moment, we have

$$\begin{aligned} E_x(V_{t,\epsilon}) &= \int_M P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t)\, \mathrm{d}V(y) \\ &\sim \epsilon^{n-2}(n-2)\boldsymbol{c}_{n-1} \int_M \mathrm{d}V(y) \int_0^t p(x, y, \tau)\, \mathrm{d}\tau \\ &= \epsilon^{n-2}(n-2)\boldsymbol{c}_{n-1} \int_0^t \mathrm{d}\tau \int_M p(x, y, \tau)\, \mathrm{d}V(y) \\ &= \epsilon^{n-2}(n-2)\boldsymbol{c}_{n-1} t \end{aligned}$$

as $\epsilon \downarrow 0$, by (1). Similarly, for

$$[\mathrm{d}\psi_{t,\epsilon}(X)](z) = [I_{t,\epsilon}(X)](z)\, \mathrm{d}V(z)$$

we have

$$\begin{aligned} E_x(\psi_{t,\epsilon}(f)) &= \int_M P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t) f(y)\, \mathrm{d}V(y) \\ &\sim \epsilon^{n-2}(n-2)\boldsymbol{c}_{n-1} \int_0^t \mathrm{d}\tau \int_M p(x, y, \tau) f(y)\, \mathrm{d}V(y) \\ &= \epsilon^{n-2}(n-2)\boldsymbol{c}_{n-1} E_x\left(\int_0^t f(X(s))\, \mathrm{d}s \right) \end{aligned}$$

as $\epsilon \downarrow 0$.

A finer argument then shows the convergence of (3) in $L^2(\mathrm{d}P_x)$ (which, of course, implies convergence in probability) when M is compact. The noncompact case is then derived from this one.

We note that the asymptotic formula (4) is based on the argument in Port-Stone [12, p. 21] showing

$$P_x(\boldsymbol{T}_{\{y\}} < +\infty) = 0$$

for all distinct points x, y in $\mathbb{R}^n$. In Chavel-Feldman [2], using an argument of Rauch-Taylor [13], we proved the formula

$$\lim_{\epsilon \downarrow 0} P_x\left(T_{B_\epsilon} \leqslant t\right) = 0 \tag{6}$$

where B_ϵ denotes the tubular neighborhood, of radius ϵ, of any compact submanifold with codimension $\geqslant 2$ in M, for any of our Brownian motions. We refer the reader to our application, there, of (6) to the construction of topological perturbations of Riemannian manifolds, having negligible effect on the diffusion and the spectrum. In Chavel [1, Chapter IX], the result (6) is presented using the Port-Stone argument. Our point here is the surprising fact that the Port-Stone argument is sharp enough to prove the asymptotic formula (4). Also, it gives the estimates for the Wiener sausage of reflecting Brownian motion determined by the Neumann heat kernel, for domains in $\mathbb{R}^n$ (Chavel-Feldman [3]), a fact as yet unavailable in the literature (as far as we can tell).

The first appearance of (4), to our knowledge, is in Lemma 1 of Papanicolaou-Varadhan [11] for classical Brownian motion in $\mathbb{R}^n$. The argument there, also, appeals to the global rescaling between the space and time variables.

Finally, we note that if $n = 2$, then one has corresponding formulae, with $(n-2)c_{n-1}\epsilon^{n-2}$ replaced by $2\pi/|\ln \epsilon|$.

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1. Preliminaries

In what follows, α, T will be positive parameters; by $c(\alpha)$ we mean a constant depending only on α, larger than 1, such that $c(\alpha) \to 1$ as $\alpha \to 1$. $\mathbb{B}^n(R)$ will denote the open disk in $\mathbb{R}^n$ of radius R; and $\mathbb{B}^n = \mathbb{B}^n(1)$, having n-volume ω_n.

Recall that the heat kernel on $\mathbb{R}^n$ is given by

$$p(X, Y, t) = (4\pi t)^{-n/2}\, e^{-|X-Y|^2/4t}.$$

Also recall the classical formulae

$$\int_{\mathbb{B}^n} \frac{d\zeta}{|\omega - \zeta|^{n-2}} = \frac{n-2}{2} c_{n-1}\left\{\frac{1}{n-2} - \frac{|\omega|^2}{n}\right\}$$

if $|\omega| \leqslant 1$, and

$$\int_{\mathbb{B}^n} \frac{d\zeta}{|\omega - \zeta|^{n-2}} = (n-2)\boldsymbol{c}_{n-1}|\omega|^{2-n}/n(n-2)$$

if $|\omega| \geqslant 1$.

Given any A, T, $R > 0$, and W in $\mathbb{R}^n$, the integral

$$J = \int_0^T (4\pi\tau)^{-n/2}\, d\tau \int_{\mathbb{B}^n(R)} e^{-A|W-Z|^2/4\tau}\, dZ,$$

where dZ denotes the Lebesgue measure on $\mathbb{R}^n$, depends on A, T, R, and $|W|$. A standard substitution then yields

$$J = \frac{A^{1-n/2}}{4\pi^{n/2}} \int_{\mathbb{B}^n(R)} \frac{dZ}{|W-Z|^{n-2}} \int_{A|W-Z|^2/4T}^{\infty} e^{-\mu}\mu^{n/2-2}\, d\mu. \tag{7}$$

Thus,

$$0 \leqslant \frac{A^{1-n/2}\Gamma(n/2-1)}{4\pi^{n/2}} \int_{\mathbb{B}^n(R)} \frac{dZ}{|W-Z|^{n-2}} - J$$

$$\leqslant \text{const} \cdot R^n/T^{n/2-1},$$

i.e.,

$$0 \leqslant \frac{R^2 A^{1-n/2}}{(n-2)\boldsymbol{c}_{n-1}} \int_{\mathbb{B}^n} \frac{d\zeta}{|W/R - \zeta|^{n-2}} - J \leqslant \text{const} \cdot R^n/T^{n/2-1};$$

so for fixed A, T, $b > 0$, and $R = b\epsilon$, $|W| = \epsilon$, we have $|W|/R = b^{-1}$, which implies

$$0 \leqslant \frac{b^2\epsilon^2 A^{1-n/2}}{2}\left\{\frac{1}{n-2} - \frac{b^{-2}}{n}\right\} - J \leqslant \text{const} \cdot \frac{b^n\epsilon^n}{T^{n/2-1}} \tag{8}$$

if $b \geqslant 1$, and

$$0 \leqslant \frac{b^n\epsilon^2 A^{1-n/2}}{n(n-2)} - J \leqslant \text{const} \cdot \frac{b^n\epsilon^n}{T^{n/2-1}} \tag{9}$$

if $b \leqslant 1$.

2. The asymptotic formula for $P_x(T_{B(y;\epsilon)} \leqslant t)$

For any A in W, and f integrable $\mathrm{d}P_x$ over A, we let

$$E_x(f;\ A) = \int_A f\,\mathrm{d}P_x.$$

Also, for any y in M and $\epsilon > 0$, we let $S(y;\ \epsilon)$ denote the boundary of $B(y;\ \epsilon)$.

Given any $\alpha > 1$, $T > 0$, and $\epsilon > 0$, the strong Markov law implies that, for any x, y in M satisfying $d(x,\ y) > \alpha\epsilon$, we have

$$\int_0^{t+T} \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p(x,\ z,\ \tau)\,\mathrm{d}V(z)$$
$$\geqslant E_x\left(\int_0^T \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p\big(X(T_{B(y;\epsilon)}),\ z,\ \tau\big)\,\mathrm{d}V(z);\ T_{B(y;\epsilon)} \leqslant t\right).$$

The geometric interpretation of the inequality lies in the fact that for any Borel set $B \subseteq M$, the integral

$$\int_0^t \mathrm{d}\tau \int_B p(x,\ y,\ \tau)\,\mathrm{d}V(y)$$

is equal to the average amount of time spent, by the Brownian particle starting from x, in B during the time interval $[0,\ t]$. So the inequality is simply stating that the average time spent by the Brownian particle in the metric disk $B(y;\ \alpha\epsilon)$ during the time interval $[0,\ t+T]$ is not less than the average time (relative to $\mathrm{d}P_x\,\mathrm{d}\tau$ on all of $W \times [0,\ t+T]$) spent in $B(y;\ \alpha\epsilon)$ during the time interval $[0,\ T]$ with the time clock starting only when the Brownian particle hits $B(y;\ \epsilon)$ prior to time t.

One immediately concludes

$$\int_0^{t+T} \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p(x,\ z,\ \tau)\,\mathrm{d}V(z)$$
$$\geqslant P_x\big(T_{B(y;\epsilon)} \leqslant t\big) \inf_{w \in S(y;\epsilon)} \int_0^T \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p(w,\ z,\ \tau)\,\mathrm{d}V(z). \quad (10)$$

Similarly, one has

$$\int_0^t \mathrm{d}\tau \int_{B(y;\epsilon/\alpha)} p(x,\ z,\ \tau)\,\mathrm{d}V(z)$$
$$\leqslant P_x\big(T_{B(y;\epsilon)} \leqslant t\big) \sup_{w \in S(y;\epsilon)} \int_0^t \mathrm{d}\tau \int_{B(y;\epsilon/\alpha)} p(w,\ z,\ \tau)\,\mathrm{d}V(z). \quad (11)$$

Note that, in the inequality (10), the choice of $\alpha > 1$ and $T > 0$ are independent of each other. However, in our application below, the parameter T will be chosen only after α is chosen.

Fix any $t_0 > 0$.

Given any α in (1, 2], and a compact subset of K of M, there exists $T_1 = T_1(\alpha, K)$ in $(0, t_0)$ and $R_1 = R_1(\alpha, K)$ in (0, inf **inj**(y)) (where **inj** denotes the injectivity radius, and the infimum is taken over all y in K) such that

$$\alpha^{-1} \leqslant \frac{\mathrm{d}(z_1, z_2)}{|Z_1 - Z_2|} \leqslant \alpha, \quad \alpha^{-n} \leqslant \frac{\mathrm{d}V(z)}{\mathrm{d}Z} \leqslant \alpha^n \tag{12}$$

$$\frac{\mathrm{e}^{-\alpha^2 |Z_1 - Z_2|^2/4\tau}}{\alpha(4\pi\tau)^{n/2}} \leqslant p(z_1, z_2, \tau) \leqslant \frac{\alpha\, \mathrm{e}^{-|Z_1 - Z_2|^2/4\tau\alpha^2}}{(4\pi\tau)^{n/2}}, \tag{13}$$

for all z, z_1, z_2, in $B(y; R_1)$ (where Z, Z_1, Z_2, are the respective preimages of z, z_1, z_2, within the tangent cut locus, with respect to the exponential map of the tangent space of M, at y, onto M), y in K, and τ in $(0, T_1)$.

We henceforth assume, until further notice, that T is fixed in $(0, T_1)$.

An immediate consequence of (8), (9), (12), and (13), is the existence of ϵ_1 in $(0, R_1/2)$ such that

$$\int_0^T \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p(w, z, \tau)\, \mathrm{d}V(z)$$

$$\geqslant c(\alpha)^{-1}\left\{1 - \mathrm{const} \cdot \left(\epsilon/\sqrt{T}\right)^{n-2}\right\} \epsilon^2/n(n-2) \tag{14}$$

and

$$\int_0^T \mathrm{d}\tau \int_{B(y;\epsilon/\alpha)} p(w, z, \tau)\, \mathrm{d}V(z) \leqslant c(\alpha)\epsilon^2/n(n-2) \tag{15}$$

for all ϵ in $(0, \epsilon_1)$, w in $S(y; \epsilon)$, and y in K.

To estimate

$$\int_0^t \mathrm{d}\tau \int_{B(y;\epsilon/\alpha)} p(w, z, \tau)\, \mathrm{d}V(z)$$

from above, without requiring $t \leqslant T_1$, simply note that

$$\int_{T_1/2}^{t} \mathrm{d}\tau \int_{B(y;\epsilon/\alpha)} p(w, z, \tau)\, \mathrm{d}V(z)$$

$$\leqslant tV(B(y; \epsilon/\alpha))\{\sup p(x, z, \tau)\}$$

$$\leqslant \text{const} \cdot \epsilon^n,$$

where the sup is evaluated on $K \times K \times [T_1/2, t_0]$. So

$$\int_0^t \mathrm{d}\tau \int_{B(y;\epsilon/\alpha)} p(w, z, \tau)\, \mathrm{d}V(z)$$

$$\leqslant c(\alpha)\{1 + \text{const} \cdot \epsilon^{n-2}\}\epsilon^2/n(n-2) \tag{16}$$

for all ϵ in $(0, \epsilon_1)$, w in $S(y; \epsilon)$, and y in K.

The inequalities (10) and (11) now imply

$$P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t)$$

$$\leqslant c(\alpha)\frac{n(n-2)}{\epsilon^2}\left\{1 + \text{const} \cdot (\epsilon/\sqrt{T})^{n-2}\right\}$$

$$\times \int_0^t \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p(x, z, \tau)\, \mathrm{d}V(z) \tag{17}$$

and

$$P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t)$$

$$\geqslant c(\alpha)^{-1}\frac{n(n-2)}{\epsilon^2}\{1 - \text{const} \cdot \epsilon^{n-2}\}$$

$$\times \int_0^t \mathrm{d}\tau \int_{B(y;\epsilon/\alpha)} p(x, z, \tau)\, \mathrm{d}V(z) \tag{18}$$

for all ϵ in $(0, \epsilon_1)$.

Next, given any $R > 0$, we have the existence of ϵ_2 in $(0, \epsilon_1)$ for which

$$c(\alpha)^{-1}\epsilon^n\omega_n\int_0^{\tau_0} p(x, y, \tau)\,\mathrm{d}\tau \leqslant \int_0^{\tau_0}\mathrm{d}\tau\int_{B(y;\epsilon/\alpha)} p(x, z, \tau)\,\mathrm{d}V(z) \tag{19}$$

$$\leqslant \int_0^{\tau_0}\mathrm{d}\tau\int_{B(y;\alpha\epsilon)} p(x, z, \tau)\,\mathrm{d}V(z)$$

$$\leqslant c(\alpha)\epsilon^n\omega_n\int_0^{\tau_0} p(x, y, \tau)\,\mathrm{d}\tau \tag{20}$$

for all x, y in K satisfying $\mathrm{d}(x, y) \geqslant R$, τ_0 in $(0, t_0 + T_1]$, and ϵ in $(0, \epsilon_2)$. So (18), (19), and (17), (20) combine to imply

$$c(\alpha)^{-1}\{1 - \text{const}\cdot\epsilon^{n-2}\}(n-2)c_{n-1}\int_0^t p(x, y, \tau)\,\mathrm{d}\tau$$

$$\leqslant \epsilon^{2-n}P_x\big(T_{B(y;\epsilon)} \leqslant t\big) \tag{21}$$

$$\leqslant c(\alpha)\Big\{1 + \text{const}\cdot\big(\epsilon/\sqrt{T}\big)^{n-2}\Big\}(n-2)c_{n-1}\int_0^{t+T} p(x, y, \tau)\,\mathrm{d}\tau \tag{22}$$

for all x, y in K satisfying $d(x, y) \geqslant R$, t in $(0, t_0]$, T in $(0, T_1]$, α in $(1, 2]$, and ϵ in $(0, \epsilon_2]$, where $\epsilon_2 = \epsilon_2(\alpha, T, R)$.

One immediately concludes the validity of (4) for any distinct x, y in K. To calculate the limit

$$\epsilon^{2-n}\int_K P_x\big(T_{B(y;\epsilon)} \leqslant t\big)f(y)\,\mathrm{d}V(y)$$

as $\epsilon \downarrow 0$, it remains to bound $\epsilon^{2-n}P_x(T_{B(y;\epsilon)} \leqslant t)$ for x close to y.

We note that the upper bound of (13) may be relaxed as follows: Given R_1 for which the estimates (12) and (13) are valid, there exists a positive constant such that

$$p(z_1, z_2, \tau) \leqslant \text{const}\cdot\tau^{-n/2}\,\mathrm{e}^{-|z_1 - z_2|^2/16\tau} \tag{23}$$

for all z_1, z_2 in $B(y; R_1)$, y in K, and τ in $(0, t_0 + T_1]$.

Therefore, $4\alpha\epsilon \leqslant d(x, y) \leqslant R_1/2$ implies

$$\int_0^{t+T} \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p(x, z, \tau)\, \mathrm{d}V(z)$$

$$\leqslant \text{const} \cdot \int_0^{t+T} \tau^{-n/2}\, \mathrm{d}\tau \int_{\mathbb{B}^n(\alpha\epsilon)} \mathrm{e}^{-|X-Z|^2/16\tau}\, \mathrm{d}Z$$

(this last integral calculated in the tangent space of M at y)

$$\leqslant \text{const} \cdot \int_{\mathbb{B}^n(\alpha\epsilon)} |X-Z|^{2-n}\, \mathrm{d}Z \leqslant \text{const} \cdot \epsilon^n d^{2-n}(x, y),$$

that is,

$$\int_0^{t+T} \mathrm{d}\tau \int_{B(y;\alpha\epsilon)} p(x, z, \tau)\, \mathrm{d}V(Z) \leqslant \text{const} \cdot \epsilon^n\, d^{2-n}(x, y) \tag{24}$$

for all ϵ in $(0, R_1/16)$, t, $T > 0$, and x, y in K. Thus, applying (17) with $T = T_1$, and using (24), we have

$$P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t) \leqslant \text{const} \cdot \epsilon^{n-2}\, d^{2-n}(x, y) \tag{25}$$

for all ϵ in $(0, \epsilon_3]$, where $\epsilon_3 = \min\{\epsilon_1, R_1/16\}$, t in $(0, t_0]$, and x, y in K satisfying

$$4\alpha\epsilon \leqslant d(x, y) \leqslant R_1/2. \tag{26}$$

We conclude that

$$\int_{B(x;R)\setminus B(x;4\alpha\epsilon)} P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t)\, \mathrm{d}V(y) \leqslant \text{const} \cdot \epsilon^{n-2}R^2 \tag{27}$$

for all ϵ in $(0, \epsilon_3]$, R in $(0, R_1/2]$, t in $(0, t_0]$, α in $(1, 2]$, and x in K.

Of course, we have

$$\int_{B(x;4\alpha\epsilon)} P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t)\, \mathrm{d}V(y) \leqslant V(B(x; 4\alpha\epsilon)) \leqslant \text{const} \cdot \epsilon^n,$$

that is,

$$\int_{B(x;4\alpha\epsilon)} P_x(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t)\, \mathrm{d}V(y) \leqslant \text{const} \cdot \epsilon^n \tag{28}$$

for all ϵ in $(0, \mathrm{R}_1/16)$, $t > 0$, and x in K.

Now let f be a continuous nonnegative function on K, and set

$$F = \sup f. \tag{29}$$

Then (21), (22), (25), and the argument of (28) imply

$$c(\alpha)^{-1}\{1 - \text{const}\cdot\epsilon^{n-2}\}(n-2)\boldsymbol{c}_{n-1}$$

$$\times\int_0^t \mathrm{d}\tau \int_K p(x, y, \tau) f(y)\, \mathrm{d}V(y) - \text{const}\cdot FR^2$$

$$\leqslant \epsilon^{2-n}\int_K P_x\left(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t\right) f(y)\, \mathrm{d}V(y) \tag{30}$$

$$\leqslant c(\alpha)\left\{1 + \text{const}\cdot\left(\epsilon/\sqrt{T}\right)^{n-2}\right\}(n-2)\boldsymbol{c}_{n-1}$$

$$\times\int_0^{t+T} \mathrm{d}\tau \int_K p(x, y, \tau) f(y)\, \mathrm{d}V(y) + \text{const}\cdot F(R^2 + \epsilon^2) \tag{31}$$

for all ϵ in $(0, \epsilon_4]$, where $\epsilon_4 = \min\{\epsilon_2, \epsilon_3\}$, R in $(0, R_1/2]$, T in $(0, T_1]$, α in $(1, 2]$, x in K, and t in $(0, t_0]$.

We summarize the discussion to this point: The number $t_0 > 0$ is fixed for the whole discussion, and t varies in $(0, t_0]$. We are given α in $(1, 2]$ and the compact set K. Then α and K determine positive constants T_1, R_1, for which (12) and (13) are valid. We then obtain positive constants: H depending at most on α, K, T_1, R_1; H_T depending at most on α, K, T_1, R_1, and T in $(0, T]$; and ϵ_4 depending at most on α, K, T_1, R_1, and T in $(0, T_1]$, R in $(0, R_1/2]$, such that

$$c(\alpha)^{-1}\{1 - H\epsilon^{n-2}\}(n-2)\boldsymbol{c}_{n-1}$$

$$\times\int_0^t \mathrm{d}\tau \int_K p(x, y, \tau) f(y)\, \mathrm{d}V(y) - HFR^2$$

$$\leqslant \epsilon^{2-n}\int_K P_x\left(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t\right) f(y)\, \mathrm{d}V(y) \tag{32}$$

$$\leqslant c(\alpha)\left\{1 + H_T\epsilon^{n-2}\right\}(n-2)\boldsymbol{c}_{n-1}$$

$$\times\int_0^{t+T} \mathrm{d}\tau \int_K p(x, y, \tau) f(y)\, \mathrm{d}V(y) + HF(R^2 + \epsilon^2) \tag{33}$$

for all ϵ in $(0, \epsilon_4]$, R in $(0, R_1/2]$, T in $(0, T_1]$, α in $(1, 2]$, x in K, t in $(0, t_0]$, and continuous nonnegative functions on K.

3. The L^2-theorem in the compact case

The basic idea for this argument was gleaned from Simon [14, p. 240].

Assume we are in case (i), where M is a compact manifold. Then in the above discussion we may pick $K = M$. Recall the measure $\mathrm{d}\Psi_{t,\epsilon}(X)$ on M induced by X.

We fix our function f, and set

$$[\Phi_t(X)](f) = \int_0^t f(X(s))\,\mathrm{d}s.$$

In what follows, we suppress the X and f from our expressions.

Then we may write (32) and (33) as

$$c(\alpha)^{-1}\{1 - H\epsilon^{n-2}\}E_x(\Phi_t) - HFR^2 \leqslant E_x(\Psi_{t,\epsilon}) \tag{34}$$

$$\leqslant c(\alpha)\{1 + H_T\epsilon^{n-2}\}E_x(\Phi_{t+T}) + HF(R^2 + \epsilon^2). \tag{35}$$

One immediately has

$$\lim_{\epsilon \downarrow 0} E_x(\Psi_{t,\epsilon}) = E_x(\Phi_t). \tag{36}$$

To study $E_x(\{\Psi_{t,\epsilon} - \Phi_t\}^2)$ we first note that

$$\begin{aligned}
&E_x\left(V_{t,\epsilon}^2\right)\\
&= \iint_{M\times M} P_x\left(\left(\boldsymbol{T}_{B(y;\epsilon)} \leqslant t\right) \& \left(\boldsymbol{T}_{B(z;\epsilon)} \leqslant t\right)\right) \mathrm{d}V(y)\,\mathrm{d}V(z)\\
&\leqslant 2\iint_{M\times M} P_x\left(\boldsymbol{T}_{B(y;\epsilon)} \leqslant \boldsymbol{T}_{B(z;\epsilon)} \leqslant t\right) \mathrm{d}V(y)\,\mathrm{d}V(z)\\
&\leqslant 2\iint_{M\times M} E_x\left(P_{X(\boldsymbol{T}_{B(y;\epsilon)})}\left(\boldsymbol{T}_{B(z;\epsilon)} \leqslant t\right);\ \boldsymbol{T}_{B(y;\epsilon)} \leqslant t\right) \mathrm{d}V(y)\,\mathrm{d}V(z)\\
&\leqslant 2\epsilon^{2n-4}\Big[c(\alpha)\{1 + H_T\epsilon^{n-2}\}(n-2)\boldsymbol{c}_{n-1}(t+T)\\
&\qquad\qquad + H\{R^2 + \epsilon^2\}\Big]^2\\
&\leqslant H\{1 + H_T\epsilon^{n-2}\}^2\epsilon^{2n-4}\left[\{t+T\}^2 + R^2 + \epsilon^2\right],
\end{aligned}$$

that is,

$$E_x\left(V_{t,\epsilon}^2\right) \leqslant H\{1 + H_T\epsilon^{n-2}\}^2\epsilon^{2n-4}\left[\{t+T\}^2 + R^2 + \epsilon^2\right]. \tag{37}$$

Fix $t > 0$, a positive integer N, and ϵ in $(0, \epsilon_4]$. For each $j = 0, \ldots, N - 1$, and Brownian path X, let $W_j(X)$ denote the tubular neighborhood, of $X([tj/N, t(j+1)/N])$, having radius ϵ; let V_j denote the volume of W_j; and let $\mathrm{d}\Psi_j$, $\mathrm{d}\psi_j$ be the associated measures on M, as above. Then

$$\sum_{j=0}^{N-1} \left[\Psi_j(X)\right](f) - H(N-1)F\epsilon^2 \leqslant \left[\Psi_{t,\epsilon}(X)\right](f) \tag{38}$$

$$\leqslant \sum_{j=0}^{N-1} \left[\Psi_j(X)\right](f). \tag{39}$$

Next, let

$$\left[\Phi_j(X)\right](f) = \int_{tj/N}^{t(j+1)/N} f(X(s))\,\mathrm{d}s.$$

Then (we suppress the X and f) (38) and (39) imply that

$$0 \leqslant E_x\left(\{\Psi_{t,\epsilon} - \Phi_t\}^2\right)$$
$$\leqslant E_x\left(\sum_{j,k}\{\Psi_j\Psi_k - 2\Psi_j\Phi_k + \Phi_j\Phi_k\}\right) + H(N-1)F^2\epsilon^2,$$

which implies

$$0 \leqslant \limsup_{\epsilon \downarrow 0} E_x\left(\{\Psi_{t,\epsilon} - \Phi_t\}^2\right)$$
$$\leqslant \limsup_{\epsilon \downarrow 0} E_x\left(\sum_{j,k}\{\Psi_j\Psi_k - 2\Psi_j\Phi_k + \Phi_j\Phi_k\}\right).$$

We first estimate the sum where $j = k$. Well,

$$E_x\left(\Psi_j^2 - 2\Psi_j\Phi_j + \Phi_j^2\right)$$
$$\leqslant E_x\left(\Psi_j^2 + \Phi_j^2\right)$$
$$\leqslant F^2\left[H\{1 + H_T\epsilon^{n-2}\}^2\{(t/N + T)^2 + R^2 + \epsilon^2\} + t^2/N^2\right]$$

by (37), which implies

$$\limsup_{\epsilon \downarrow 0} E_x\left(\Psi_j^2 - 2\Psi_j\Phi_j + \Psi_j^2\right)$$
$$\leqslant F^2\left[H\{(t/N + T)^2 + R^2\} + t^2/N^2\right]$$

for all R in $(0, R_1]$ and T in $(0, T_1]$. Thus

$$\limsup_{\epsilon \downarrow 0} E_x\left(\Psi_j^2 - 2\Psi_j\Phi_j + \Psi_j^2\right) \leqslant HF^2t^2/N^2. \tag{41}$$

To estimate the sum over $j \neq k$ we first note that

$$\sum_{j \neq k} \Psi_j\Psi_k - 2\Psi_j\Phi_k + \Phi_j\Phi_k$$

$$= 2\sum_{j<k} \Psi_j\{\Psi_k - \Phi_k\} + 2\sum_{j>k}\{\Phi_j - \Psi_j\}\Phi_k. \tag{42}$$

Then, for $j < k$, we have, by the strong Markov law and (35),

$$\begin{aligned}
&E_x\left(\Psi_j\{\Psi_k - \Phi_k\}\right)\\
&\quad = E_x\left(\Psi_j E_{X(t(j+1)/N)}\left\{\Psi_{k-(j+1)} - \Phi_{k-(j+1)}\right\}\right)\\
&\quad = E_x\left(\Psi_j E_{X(t(j+1)/N)}\left\{E_{X(t(k-(j+1))/N)}[\Psi_0 - \Phi_0]\right\}\right)\\
&\quad \leqslant E_x(\Psi_j)F\left\{\left[c(\alpha)\left\{1 + H_T\epsilon^{n-2}\right\} - 1\right]t/N\right.\\
&\qquad\qquad\qquad \left.+ c(\alpha)\left\{1 + H_T\epsilon^{n-2}\right\}T + H(R^2 + \epsilon^2)\right\}\\
&\quad \leqslant F^2\left(c(\alpha)\left(1 + H_T\epsilon^{n-2}\right)(t/N + T) + H(R^2 + \epsilon^2)\right)\\
&\qquad \times\left\{\left[c(\alpha)\left(1 + H_T\epsilon^{n-2}\right) - 1\right]t/N + c(\alpha)\left\{1 + H_T\epsilon^{n-2}\right\}T\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.+ H(R^2 + \epsilon^2)\right\},
\end{aligned}$$

which implies, for $j < k$,

$$\begin{aligned}
&\limsup_{\epsilon \downarrow 0} E_x\left(\Psi_j\{\Psi_k - \Phi_k\}\right)\\
&\quad \leqslant \left[c(\alpha)\{t/N + T\} + HR^2\right]\\
&\qquad \times\left[\{c(\alpha) - 1\}t/N + c(\alpha)T + HR^2\right]F^2
\end{aligned}$$

for all T in $(0, T_1]$, and R in $(0, R_1/2]$. Therefore $j < k$ implies

$$\limsup_{\epsilon \downarrow 0} E_x\left(\Psi_j\{\Psi_k - \Phi_k\}\right) \leqslant c(\alpha)\{c(\alpha) - 1\}F^2t^2/N^2. \tag{43}$$

Similarly, the strong Markov law and (34) imply, for $j > k$,

$$\begin{aligned} &E_x\big(\Phi_k\{\Phi_j - \Psi_j\}\big) \\ &\quad \leqslant E_x(\Phi_k)F\Big\{\big[1 - c(\alpha)^{-1}(1 - H\epsilon^{n-2})\big]t/N + HR^2\Big\} \\ &\quad \leqslant F^2\Big\{\big[1 - c(\alpha)^{-1}(1 - H\epsilon^{n-2})\big]t/N + HR^2\Big\}t/N, \end{aligned}$$

which implies

$$\limsup_{\epsilon\downarrow 0} E_x\big(\Phi_k\{\Phi_j - \Psi_j\}\big) \leqslant F^2\Big[\big\{1 - c(\alpha)^{-1}\big\}t/N + HR^2\Big]t/N$$

for all R in $(0, R_1/2]$. Therefore, $j > k$ implies

$$\limsup_{\epsilon\downarrow 0} E_x\big(\Phi_k\{\Phi_j - \Psi_j\}\big) \leqslant \big\{1 - c(\alpha)^{-1}\big\}F^2t^2/N^2. \tag{44}$$

Then (40)–(44) combine to imply

$$\begin{aligned} &0 \leqslant \limsup\ E_x\big(\{\Psi_{t,\epsilon} - \Phi_t\}^2\big) \\ &\quad \leqslant HF^2t^2/N + (N-1)F^2t^2\Big[c(\alpha)\{c(\alpha) - 1\} + 1 - c(\alpha)^{-1}\Big]/N \end{aligned}$$

for all α in (1, 2] and $N = 1, 2, \ldots$ (recall: the $c(\alpha)$'s are not necessarily identical). One easily has

$$\lim_{\epsilon\downarrow 0} E_x\big(\{\Psi_{t,\epsilon} - \Phi_t\}^2\big) = 0 \tag{45}$$

4. The asymptotic law for the noncompact case

The argument we give is that of J.M. Bismut.

Let M be an arbitrary noncompact manifold with minimal positive heat kernel satisfying the conservation of heat property (1).

Given x in M, let $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$ be an exhaustion of M, with $x \in \Omega_1$, by domains Ω_j having smooth boundary and compact closure, with $\overline{\Omega_j}$ isometrically embedded into a compact Riemannian manifold M_j^* having the same dimension as M.

Recall that $\boldsymbol{W}$ denotes the Wiener space of paths in M. For each $j = 1, 2, \ldots$, let

$$\boldsymbol{W}_j = \big(X([0, t]) \subseteq \Omega_j\big),$$

i.e., W_j consists of those paths that have not left Ω_j by time t; and let $W_j{}^*$ be the Wiener space of paths in $M_j{}^*$ with probability measure $\mathrm{d}P_x^j$ associated to x. The conservation of heat property (1), and the minimality of the positive heat kernel p imply

$$\lim_{j \to +\infty} P_x(W_j) = P_x(W) = 1. \tag{46}$$

Now let f be a bounded continuous function on M.
We wish to show that

$$\lim_{\epsilon \downarrow 0} [\Psi_{t,\epsilon}(X)](f) = [\Phi_t(X)](f) \tag{47}$$

in probability $\mathrm{d}P_x$. Pick $\eta > 0$. Then

$$\begin{aligned}
&P_x(|[\Psi_{t,\epsilon}(X)](f) - [\Phi_t(X)](f)| > \eta) \\
&\quad \leqslant P_x(W - W_j) \\
&\qquad + P_x(X \in W_j,\ |[\Psi_{t,\epsilon}(X)](f) - [\Phi_t(X)](f)| > \eta) \\
&\quad = P_x(W - W_j) \\
&\qquad + P_x^j(X \in W_j,\ |[\Psi_{t,\epsilon}(X)](f) - [\Phi_t(X)](f)| > \eta) \\
&\quad \leqslant P_x(W - W_j) \\
&\qquad + P_x^j(X \in W_j^*,\ |[\Psi_{t,\epsilon}(X)](f) - [\Phi_t(X)](f)| > \eta).
\end{aligned}$$

From the result in the compact case, we have

$$\limsup_{\epsilon \downarrow 0} P_x(|[\Psi_{t,\epsilon}(X)](f) - [\Phi_t(X)](f)| > \eta) \leqslant P_x(W - W_j)$$

for all $j = 1, 2, \ldots$. But then (46) will imply

$$\limsup_{\epsilon \downarrow 0} P_x(|[\Psi_{t,\epsilon}(X)](f) - [\Phi_t(X)](f)| > \eta) = 0$$

for all $\eta > 0$; but that is the claim (47) in probability $\mathrm{d}P_x$.

We note that if one does not have the conservation of heat property (1), then the above argument will apply to the collection of paths $\{X\}$ for which $X([0, t]) \subseteq M$.

5. Remarks

1: We first note that our arguments have proved (47) in $L^2(\mathrm{d}P_x)$ for compact manifolds; and the weaker result that (47) is valid in probability $\mathrm{d}P_x$ when M is noncompact and the Brownian motion is determined by the minimal positive heat kernel satisfying (1). We now comment that it is not too hard to show the stronger result, that (47) is valid in $L^2(\mathrm{d}P_x)$ in the noncompact case if one adds the geometric assumptions on M that it is Riemannian complete with Ricci curvature bounded from below.

2: Here we note that the result (47) is extendable to variable stopping times, viz., let $\boldsymbol{T}: \boldsymbol{W} \to [0, +\infty)$ be bounded, measurable, and set

$$W_{\boldsymbol{T},\epsilon}(X) = \{ y \in M : \mathrm{d}(y, X([0, \boldsymbol{T}(X)])) < \epsilon \}.$$

Define the corresponding measures $\mathrm{d}\Psi_{\boldsymbol{T},\epsilon}(X)$, $\mathrm{d}\Phi_{\boldsymbol{T}}(X)$ on M. Then (47) can be extended to

$$\lim_{\epsilon \downarrow 0} [\Psi_{\boldsymbol{T},\epsilon}(X)](f) = [\Phi_{\boldsymbol{T}}(X)](f), \tag{48}$$

in $L^2(\mathrm{d}P_x)$ in the compact cases, and in probability $\mathrm{d}P_x$ in the noncompact case if $\boldsymbol{T}$ is bounded.

Indeed, if $\boldsymbol{T}$ is a simple function, i.e., it is a linear combination of indicator functions of measurable subsets of $\boldsymbol{W}$, then the result is certainly true. If $\boldsymbol{T}$ is bounded, then for each $N = 1, 2, \ldots,$ one sets

$$T_N = ([2^N \boldsymbol{T}] + 1)/2^N, \quad S_N = \max\big(([2^N \boldsymbol{T}] - 1)/2^N, 0\big).$$

Then S_N and T_N are simple functions, and $T_N \downarrow \boldsymbol{T}$, $S_N \uparrow \boldsymbol{T}$ uniformly on $\boldsymbol{W}$. So

$$\Psi_{S_N,\epsilon} \leqslant \Psi_{\boldsymbol{T},\epsilon} \leqslant \Psi_{T_N,\epsilon},$$

implies

$$\begin{aligned}
\Phi_{S_N} &= \lim_{\epsilon \downarrow 0} \Psi_{S_n,\epsilon} \\
&\leqslant \liminf_{\epsilon \downarrow 0} \Psi_{\boldsymbol{T},\epsilon} \\
&\leqslant \limsup_{\epsilon \downarrow 0} \Psi_{\boldsymbol{T},\epsilon} \\
&\leqslant \lim_{\epsilon \downarrow 0} \Psi_{T_N,\epsilon} = \Phi_{T_N},
\end{aligned}$$

where the limit is taken in the appropriate sense. Hence (48) follows from (47) when $\boldsymbol{T}$ is bounded.

When $\boldsymbol{T}$ is unbounded, set $T_N = \min(\boldsymbol{T}, N)$, and set $\boldsymbol{W}_N$ to be the collection of Brownian paths X for which $T_N(X) = \boldsymbol{T}(X)$. Then $T_N \uparrow \boldsymbol{T}$ and $\lim P_x(\boldsymbol{W} - \boldsymbol{W}_N) = 0$ as $N \uparrow +\infty$. This fact, used with (48) for the stopping times T_N, easily implies that (48) holds in probability $\mathrm{d}P_x$ for general $\boldsymbol{T}$.

3: The case of variable stopping times contains the special case of the first exist time of a domain in M with smooth boundary and compact closure. We therefore have the extension of the Wiener sausage law for Brownian motion with absorption at the boundary, and the Lenz-shift phenomenon for Dirichlet eigenvalues.

4: More generally, we consider M arbitrary noncompact, where $p(x, y, t)$ is the minimal positive heat kernel, and the total heat is not necessarily conserved. For the Brownian path $X(t)$, we let $\zeta(X) = \sup\{t : X(t) \in M\}$, the "first exit time from M." Given any fixed $t > 0$, set $\boldsymbol{T} = \min(t, \zeta)$. Exhaust M by domains $M_1 \subseteq \subseteq M_j \subseteq \subseteq M_{j+1} \subseteq \subseteq \ldots$, with smooth boundary and compact closures; set T_j'' to be the first exit time from M_j, and $T_j = \min(t, T_j'')$. Of course, $T_j \uparrow \boldsymbol{T}$. It is easy to see that (48) holds in probability for $\boldsymbol{T}$ follows from the similar (in fact L^2) statement for T_j. One now has the appropriate extension of (2) to nonconservative heat flows.

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