COMPOSITIO MATHEMATICA

JOSEPH H. SILVERMAN

Integral points on abelian surfaces are widely spaced

Compositio Mathematica, tome 61, nº 2 (1987), p. 253-266
<http://www.numdam.org/item?id=CM_1987__61_2_253_0>

Compositio Mathematica 61: 253–266 (1987)

Integral points on abelian surfaces are widely spaced

JOSEPH H. SILVERMAN
Mathematics Department, Boston University, 111 Cummington Street, Boston, MA 02215, USA

Received 2 April 1986; accepted 20 July 1986

An affine subset of an elliptic curve has only finitely many integral points, a famous result due to Siegel. Lang has conjectured that the same is true for abelian varieties of arbitrary dimension. In this paper we show that if an affine subset of an abelian surface were to have infinitely many integral points, then the (logarithmic) height of those points would grow exponentially. Since it is well known that the heights of the rational points on an abelian variety grow polynomially, our theorem shows that integral points on an abelian surface are relatively rare; or, as we shall say, that the integral points are *widely spaced*. We note that [Mumford, 1965], has similarly shown that the rational points on a curve of genus at least two are widely spaced, (a result not entirely superceded by Faltings' proof [Faltings, 1983] of the Mordell conjecture; see the discussion in [Szpiro, 1985], XI §§1,2). Our method of proof, which involves computations with local height functions, is rather different from that of Mumford; although we do use (essentially) the same lemma on almost orthogonal subsets of lattices in Euclidean space to finish our argument.

In order to state our main theorem more precisely, we set the following notation, which will remain fixed throughout this paper.

$K/\mathbb{Q}$	a number field
S	a finite set of places of K containing the infinite places
R_S	the ring of S-integers of K
A/K	an abelian variety
$U \subset A$	a (non-empty) affine open subset of A, say with affine coordinates $x_1, \dots, x_N$
$U(R_S)$	the set of S-integral points of U. (I.e. the set of $P \in U(K)$ such that $x_i(P) \in R_S$ for all $1 \leqslant i \leqslant N$. To indicate the dependence on $x_1, \dots, x_N$, we use the notation $U_x(R_S)$.)
$\hat{h}$	a (logarithmic) canonical height on A corresponding to some ample, symmetric divisor
$c, c', \dots$	positive constants which may depend on all of the above data, and which may vary from line to line.

Our main result is the following estimate for the size of the set of S-integral points in an open affine subset of an abelian surface.

Theorem 1

Assume that A is an abelian surface. (*I.e.* $\dim(A)=2$.) *Then*

$$\#\{P \in U(R_S) : \hat{h}(P) \leqslant H\} \leqslant c \log(H).$$

Since $\hat{h}$ is a positive definite quadratic form on $A(K)/(\text{torsion})$, one easily obtains an asymptotic estimate

$$\#\{P \in A(K) : \hat{h}(P) \leqslant H\} \sim cH^{r/2} \text{ as } H \to \infty,$$

where $r = \operatorname{rank} A(K)$ (see, e.g. [Lang, 1983], ch. 5, thm. 7.5). Thus Theorem 1 says that in the set of rational points on an abelian surface, the integral points are quite rare. It may be compared with the result of [Brown, 1984–85], who uses techniques from transcendence theory to prove that

$$\#\{P \in U(R_S) : \hat{h}(P) \leqslant H\} = o(H^{r/2}) \text{ as } H \to \infty.$$

(However, Brown's result is valid for abelian varieties of all dimensions.)

We also note that Theorem 1 is equivalent to the fact that if the points in $U(R_S)$ are arranged in order of increasing height, say $P_1, P_2, \ldots,$ then

$$\hat{h}(P_i) \geqslant c^i \text{ for some } c > 1 \text{ and all } i \geqslant c'.$$

(This is the way Mumford phrases his result on the Mordell conjecture [Mumford, 1965].)

§1. Reduction lemmas

Siegel's theorem for elliptic curves actually shows that a certain set of what might be called 'quasi-integral' points is a finite set (see [Silverman, 1986], IX.3.1 for a discussion.) We will prove that the 'quasi-integral' points on an abelian surface satisfy the conclusion of Theorem 1. Generally, let us say that a set $\Sigma \subset A(K)$ is *widely spaced* if

$$\#\{P \in \Sigma : \hat{h}(P) \leqslant H\} \leqslant c \log(H).$$

We also set the further notation:

- h_D absolute logarithmic Weil height corresponding to the divisor D (cf. [Lang, 1983], ch. 4, ch. 10).
- $\lambda_D(\cdot, v)$ local height function corresponding to a divisor D and place v (cf. [Lang, 1983] ch. 10, where they are called Weil functions). More generally, $\lambda_X(\cdot, v)$ is the local height function corresponding to the subscheme X and place v (see [Silverman, to appear] for the definition and standard properties of λ_X. Intuitively,

$$\lambda_X(P, v) = -\log(\text{"}v\text{-adic distance from } P \text{ to } X\text{"}).$$

Thus $\lambda_X(P, v)$ is large if and only if P is v-adically close to X.)

$\epsilon > 0$ a constant (the c's may also depend on ϵ).

We will prove the following theorem, from which Theorem 1 can be easily deduced (see Lemma 3).

Theorem 2

Assume that A is an abelian surface, and let $D \in \mathrm{Div}(A)$ be an ample, effective divisor. Then the set

$$\left\{P \in A(K) : P \notin \mathrm{Supp}(D) \quad \textit{and} \sum_{v \in S} \lambda_D(P, v) \geqslant \epsilon h_D(P)\right\}$$

is widely spaced.

Remark

The set in Theorem 2 consists of the points with 'D-defect' at least ϵ, in the sense of Vojta's Nevanlinna-type conjectures (cf. [Vojta, 1986], §3). More precisely, the defect $\delta(D)$ is defined to be the largest $\epsilon \geqslant 0$ for which the set in Theorem 2 is Zariski dense in A. Then Vojta's generalization of Lang's conjecture asserts that $\delta(D) = 0$ ([Vojta, 1986], conj. 3.3), which is (essentially) equivalent to the fact that for every $\epsilon > 0$, the set in Theorem 2 is finite. Thus Theorem 2 provides some slight additional evidence for Vojta's conjectures.

Most of our proof of Theorem 2 will be valid for abelian varieties of arbitrary dimension. We will thus postpone the assumption that $\dim(A) = 2$ for as long as possible. As will be seen, the problem becomes that of dealing with a certain exceptional set, which we can presently only handle in dimension 2. (And even there, we will have recourse to Faltings' proof of the Mordell conjecture!) We start with a number of reduction steps, for which we set the following additional notation.

- $\mathrm{O}(1)$ a bounded function (i.e. with the property that $|\mathrm{O}(1)| \leqslant c$.)
- $\mathrm{O}_v(1)$ an M_k-bounded function (i.e. a collection of bounded functions with the property that for all but finitely many $v \in M_K$, $\mathrm{O}_v(1)$ is identically 0; cf. [Lang, 1983], ch. 10).

Further, for each effective divisor $D \in \mathrm{Div}(A)$, define subsets of $A(K)$ as follows.

$$\Sigma_1(D, \epsilon) = \left\{P \in A(K) : P \notin \mathrm{Supp}(D) \text{ and } \sum_{v \in S} \lambda_D(P, v) \geqslant \epsilon h_D(P)\right\}$$

$$\Sigma_2(D, \epsilon) = \left\{P \in A(K) : P \notin \mathrm{Supp}(D) \text{ and } \sum_{v \in S} \lambda_D(P, v) \geqslant \epsilon \hat{h}(P)\right\}.$$

We start by showing that Theorem 2 implies Theorem 1.

Lemma 3

Suppose that $\Sigma_1(D, \epsilon)$ is widely spaced for every ample, effective divisor $D \in \mathrm{Div}(A)$ and every $\epsilon > 0$. Then $U(R_S)$ is widely spaced.

Proof

First, suppose that $y_1, \ldots, y_M$ is another set of affine coordinates for U. Then each $y_i \in \Gamma(U, \mathcal{O}_U\} = K[x_1, \ldots, x_N]$. Let S' be a set of places of K containing S, and such that $y_i \in R_{S'}[x_1, \ldots, x_N]$ for all $1 \leqslant y \leqslant M$. Then it is clear that $U_x(R_S) \subset U_y(R_{S'})$. Since the set S is arbitrary, this shows that it suffices to prove Lemma 3 for any one choice of affine coordinates for U.

Next, let $D_i = (x_i)_\infty$ be the polar divisor of x_i, and let $D = \sup\{D_i\}$ (i.e. $D \geqslant D_i$ for all i; and if $D' \geqslant D_i$ for all i, then $D' \geqslant D$). Then the support of D is clearly $A - U$ (i.e., the complement of U in A). It follows from ([Mumford, 1974], p. 62) that D is ample.

Further, we note that for any $n \geqslant 1$, a basis for the global sections to the line bundle $\mathcal{O}_A(nD)$ is a set of affine coordinates for U. Hence replacing D by nD, it suffices to prove that $U_x(R_S)$ is widely spaced under the assumption that there is a very ample effective divisor D such that $\{x_0, x_1, \ldots, x_N\}$ is a basis for $\Gamma(A, \mathcal{O}_A(D))$. We may further assume that $x_0 = 1$.

Let

$$f = [x_0, \ldots, x_N] : A \to \mathbb{P}^N$$

be the embedding of A corresponding to our choice of basis for $\Gamma(A, \mathcal{O}_A(D))$; and let $H \in \mathrm{Div}(\mathbb{P}^N)$ be the hyperplane $X_0 = 0$, so $f^*H = D$. Further, for each i, write

$$(x_i) = E_i - D \text{ with } E_i \in \mathrm{Div}(A).$$

Since D is very ample, and the x_i's form a basis for $\Gamma(A, \mathcal{O}_A(D))$, it follows that $\cap\,\mathrm{Supp}(E_i) = \emptyset$. (I.e. The E_i's have disjoint support.) Hence from ([Lang, 1983], ch. 10, cor. 3.3), we conclude that

$$\min_{0 \leqslant i \leqslant N} \{\lambda_{E_i}(\cdot, v)\} = \mathrm{O}_v(1).$$

Further, Weil's decomposition theorem ([Lang, 1983], ch. 10, thm. 3.7) says that

$$v(x_i(P)) = \lambda_{E_i}(P, v) - \lambda_D(P, v) + \mathrm{O}_v(1).$$

Now let $P \in U(R_S)$. Then

$$\begin{aligned} h_D(P) &= h_H(f(P)) + O(1) && \text{since } D = f^*H \\ &= \sum_{\text{all } v} \max_{0 \leqslant i \leqslant N} \{-v(x_i(P))\} + O(1) && \text{def. of } h_H \\ &= \sum_{v \in S} \max_{0 \leqslant i \leqslant N} \{-v(x_i(P))\} + O(1) && \text{since } P \in U(R_S) \\ &= \sum_{v \in S} \max_{0 \leqslant i \leqslant N} \{\lambda_D(P, v) - \lambda_{E_i}(P, v) + O_v(1)\} + O(1) \\ &= \sum_{v \in S} \lambda_D(P, v) - \sum_{v \in S} \min_{0 \leqslant i \leqslant N} \{\lambda_{E_i}(P, v)\} + O(1) \\ &= \sum_{v \in S} \lambda_D(P, v) + O(1). \end{aligned}$$

Hence if the O(1) in this last time is bounded by c, then it follows that we have an inclusion

$$U(R_S) \subset \Sigma_1(D, \tfrac{1}{2}) \cup \{P \in A(K) : h_D(P) \leqslant 2c\}.$$

Since D is (very) ample, the latter set is finite. Hence if $\Sigma_1(D, \frac{1}{2})$ is widely spaced, then so is $U(R_S)$. □

From Lemma 3, we are reduced to studying sets of the form $\Sigma_1(D, \epsilon)$ for ample, effective divisors D. For reasons which will become apparent later, it turns out to be technically easier to deal only with irreducible divisors, so our next reduction step is to show that this is sufficient. (However, note that in order to do this, we must look at $\Sigma_2(D, \epsilon)$. It is certainly possible for $\Sigma_1(D, \epsilon)$ to be non-widely spaced for an irreducible divisor D.)

Lemma 4

If $\Sigma_2(D, \epsilon)$ is widely spaced for every irreducible effective divisor D and every $\epsilon > 0$, then $\Sigma_1(D, \epsilon)$ is widely spaced for every ample effective divisor D and every $\epsilon > 0$.

Proof

Let D be an ample effective divisor, and write D as a sum of (not necessarily distinct) irreducible effective divisors

$$D = \sum_{i=1}^{N} D_i.$$

Then for all $P \in \Sigma_1(D, \epsilon)$,

$$\epsilon h_D(P) \leqslant \sum_{v \in S} \lambda_D(P, v) = \sum_{i=1}^{N} \Big(\sum_{v \in S} \lambda_{D_i}(P, v) \Big) + O(1).$$

Thus

$$\frac{\epsilon}{N} h_D(P) \leqslant \max_{1 \leqslant i \leqslant N} \Big\{ \sum_{v \in S} \lambda_{D_i}(P, v) \Big\} + O(1).$$

Next, since D is ample, ([Lang, 1983], ch. 4, prop. 5.4) implies that

$$c\hat{h}(P) + O(1) \leqslant h_D(P).$$

Combining these last two inequalities, we see that

$$\Sigma_1(D, \epsilon) \subset \bigcup_{i=1}^{N} \Sigma_2\Big(D_i, \frac{c\epsilon}{2N}\Big) \cup \Big\{ P \in A(K) : \hat{h}(P) \leqslant \frac{2Nc'}{c\epsilon} \Big\}.$$

Since the latter set is finite, and by assumption all of the $\Sigma_2(D_i, \epsilon)$'s are widely spaced, it follows that $\Sigma_1(D, \epsilon)$ is also widely spaced. □

For the next reduction step, we consider functions

$$\tau : S \to [0, 1] \text{ satisfying } \sum_{v \in S} \tau_v = 1.$$

For such a function, define

$$\Sigma_3(D, \epsilon, \tau) = \{ P \in A(K) : P \notin \operatorname{Supp}(D) \text{ and}$$
$$\lambda_D(P, v) \geqslant \epsilon \tau_v \hat{h}(P) \text{ for all } v \in S \}.$$

Lemma 5

$\Sigma_2(D, \epsilon)$ *is widely spaced for every* $\epsilon > 0$ *if and only if* $\Sigma_3(D, \epsilon, \tau)$ *is widely spaced for every* $\epsilon > 0$ *and every function* τ *as above.*

Proof

One direction is trivial, and the other is the usual *reduction to simultaneous approximation* (see, for example, [Lang, 1983], ch. 7, §2). One actually obtains an inclusion of the form

$$\{P \in \Sigma_2(D, \epsilon) : h_D(P) \leqslant H\} \subset \cup \left\{P \in \Sigma_3\left(D, \frac{\epsilon}{2}, \tau\right) : h_D(P) \leqslant H\right\},$$

where the union is over a certain finite collection of τ's which depends only on the number of elements in S. □

We will also have need of the following elementary counting lemma, which says that almost orthogonal subsets of a lattice are widely spaced. This is similar to the lemma used by [Mumford, 1965].

Lemma 6

Let Λ be a finite dimensional lattice, q a positive definite quadratic form on Λ, n an integer, and a, $b > 0$ constants. Suppose that $\Sigma \subset \Lambda$ is a subset with the property that for any n distinct points $P_1, \ldots, P_n \in \Sigma$,

$$\max_{1 \leqslant i, j \leqslant n} \{q(P_i - P_j)\} \geqslant a \min_{1 \leqslant i \leqslant n} \{q(P_i)\} - b.$$

Then there is a constant $c > 0$, depending on Λ, q, n, a, and b, such that

$$\#\{P \in \Sigma : q(P) \leqslant x\} \leqslant c \log(x) \quad \textit{for all } x \geqslant 2.$$

Proof

This is a standard counting argument. Intuitively, the condition on Σ says that for any n points of Σ having approximately the same length, at least two of them make a fairly wide angle. □

Finally, it turns out to be easier to deal only with divisors which do not equal any of their translates. The following lemma often lets one reduce to this case.

Lemma 7

For each effective divisor $D \in \mathrm{Div}(A)$, let

$$\Phi = \Phi(D) = \{T \in A : D_T = D\}.$$

(Here D_T is the translation of D by T. Note that Φ is the product of an abelian subvariety and a finite subgroup of A.) Further, let $\varphi : A \to A/\Phi$ be the natural projection. Then for every $\epsilon > 0$ there exists an $\epsilon' > 0$ such that

$$\varphi(\Sigma_2(D, \epsilon)) \subset \Sigma_2(\varphi D, \epsilon') \cup (\textit{finite set}).$$

Proof

From the definition of Φ, we have $D = \varphi^*(\varphi D)$. Hence for all $P \in \Sigma_2(D, \epsilon)$,

$$\epsilon \hat{h}(P) \leqslant \sum_{v \in S} \lambda_D(P, v) = \sum_{v \in S} \lambda_{\varphi^*(\varphi D)}(P, v)$$

$$= \sum_{v \in S} \lambda_{\varphi D}(\varphi P, v) + O_v(1).$$

Further, since $\hat{h}$ is defined using an ample divisor on A, $\hat{h} \circ \varphi$ will dominate any height on A/Φ. In other words, for any choice of $\hat{h}$ on A/Φ, there is an inequality $\hat{h} \geqslant c\hat{h} \circ \varphi + O(1)$. (Note the two $\hat{h}$'s are different.) It follows that for all $P \in \Sigma_2(D, \epsilon)$,

$$c\epsilon \hat{h}(\varphi P) \leqslant \sum_{v \in S} \lambda_{\varphi D}(\varphi P, v) + c';$$

and so

$$\varphi(\Sigma_2(D, \epsilon)) \subset \Sigma_2(\varphi D, \tfrac{1}{2}c\epsilon) \cup \left\{ P \in \frac{A}{\Phi}(K) : \hat{h}(P) \leqslant \frac{2c'}{c\epsilon} \right\}. \qquad \square$$

§2. An orthogonality result

We are now ready to prove our main orthogonality result, which says that the points of $\Sigma_3(D, \epsilon, \tau)$ are almost orthogonal (in the sense of Lemma 6), provided that each n-tuple of points in $\Sigma_3(D, \epsilon, \tau)$ satisfies a certain Zariski open condition. The idea of the proof is as follows. Assume for simplicity that $S = \{v\}$ contains only one element. Then points of $\Sigma_2(D, \epsilon)$ can be thought of as points which are v-adically close to D. Given two such points P and Q, we want to show that $\hat{h}(P - Q)$ is fairly large. Now $P - Q$ need not be v-adically close to D, but it is certainly close to the set of differences

$$D - D = \{x - y : x, y \in \mathrm{Supp}(D)\}.$$

(Of course, generally $D - D$ equals all of A; but if we work instead on a product A^n, then this difficulty disappears.) Using this idea, we get the desired estimate for $\hat{h}(P - Q)$ *provided that* $P - Q$ does not actually belong to the set $D - D$. It is this last proviso which leads to an exceptional set of points which must be treated separately.

Theorem 8

Let $D \in \mathrm{Div}(A)$ *be an effective divisor. For each integer* $n \geqslant 1$, *let*

$$W_n = \{(P_1, \dots, P_n) \in A^n : \text{there is a } T \in A \text{ such that } P_i - T \in \mathrm{Supp}(D)$$
$$\text{for all } 1 \leqslant i \leqslant n\}.$$

(I.e. W_n *is the set of all translates of* $\mathrm{Supp}(D)^n \subset A^n$ *by an element of the diagonal of* A^n.*) Suppose that* $P_1, \dots, P_n \in \Sigma_3(D, \epsilon, \tau)$ *have the property that* $(P_1, \dots, P_n) \notin W_n$. *Then*

$$\max_{1 \leqslant i,j \leqslant n} \{\hat{h}(P_i - P_j)\} \geqslant c \min_{1 \leqslant i \leqslant n} \{\hat{h}(P_i)\} - c'.$$

Remark

Note that

$$\dim(W_n) \leqslant n \dim(D) + \dim(A) = \dim(A^n) + \dim(A) - n.$$

Thus if we choose $n > \dim(A)$, then W_n will be a proper Zariski-closed subset of A^n; so in general one would expect that a random set of n points would not lie on W_n.

Proof

Define a differencing map

$$\sigma : A^n \to A^{n^2}$$
$$(P_1, \dots, P_n) \to (\dots, P_i - P_j, \dots)_{1 \leqslant i,j \leqslant n}.$$

Notice that another description of W_n is then given by

$$W_n = \sigma^{-1}(\sigma(\mathrm{Supp}(D)^n)).$$

Now let $P_1, \dots, P_n \in A(K)$ be points with $\boldsymbol{P} = (P_1, \dots, P_n) \notin W_n$. We do a formal calculation using local height functions.

First, note that D^n (considered as a subscheme of A^n) is the intersection of $\pi_i^*(D)$ for $1 \leqslant i \leqslant n$, where $\pi_i : A^n \to A$ is the i^{th} projection. Hence from ([Silverman, to appear], thm. 2.1b),

$$\min_{1 \leqslant i \leqslant n} \{\lambda_D(P_i, v)\} = \lambda_{D^n}(\boldsymbol{P}, v) + O_v(1).$$

Next, we clearly have an inclusion $D^n \subset \sigma^*(\sigma(D^n))$, so ([Silverman, to appear], thm. 2.1d,h) implies

$$\lambda_{D^n}(\boldsymbol{P}, v) \leqslant \lambda_{\sigma^*(\sigma(D^n))}(\boldsymbol{P}, v) + \mathrm{O}_v(1) = \lambda_{\sigma(D^n)}(\sigma\boldsymbol{P}, v) + \mathrm{O}_v(1).$$

Next,we choose a finite set of divisors $E_1, \dots, E_r \in \mathrm{Div}(A^{n^2})$ such that $\cap E_i = \sigma(D^n)$ ([Silverman, to appear], lemma 2.2). Then from ([Silverman, to appear], thm. 2.1b)

$$\lambda_{\sigma(D^n)}(\sigma\boldsymbol{P}, v) = \min_{1 \leqslant i \leqslant n} \{\lambda_{E_i}(\sigma\boldsymbol{P}, v)\} + \mathrm{O}_v(1)$$

$$\leqslant \min_{1 \leqslant i \leqslant n} \{h_{E_i}(\sigma\boldsymbol{P})\} + \mathrm{O}(1).$$

In this last line, the functions h_{E_i} are height functions with respect to divisors. In other words, they are Weil height functions, which are defined at all points of A^{n^2}. We emphasize that the validity of this last inequality depends on the fact that $\sigma\boldsymbol{P} \notin \mathrm{Supp}(\cap E_i)$ (i.e. $\boldsymbol{P} \notin W_n$). In general, one has an inequality $h_E \geqslant \lambda_E(\cdot, v) + \mathrm{O}_v(1)$ which is valid for all points not in the support of the divisor E. (In essence, this inequality is nothing more than a fancy version of the product formula.)

Now let $\pi_{ij}: A^{n^2} \to A$ be the various projections; and let $\Delta \in \mathrm{Div}(A)$ be the ample divisor used to define $\hat{h}$. Then $\sum \pi_{ij}^*(\Delta)$ is ample on A^{n^2}. Therefore ([Lang, 1983], ch. 4, prop. 5.4),

$$h_{E_i} \leqslant c h_{\sum \pi_{ij}^* \Delta} + \mathrm{O}(1).$$

Finally, we note that

$$h_{\sum \pi_{ij}^*\Delta}(\sigma\boldsymbol{P}) = \sum_{1 \leqslant i,j \leqslant n} h_\Delta(\pi_{ij}(\sigma\boldsymbol{P})) + \mathrm{O}(1) = \sum_{1 \leqslant i,j \leqslant n} h_\Delta(P_i - P_j) + \mathrm{O}(1)$$

$$\leqslant n^2 \max_{1 \leqslant i,j \leqslant n} \{h_\Delta(P_i - P_j)\} + \mathrm{O}(1)$$

$$= n^2 \max_{1 \leqslant i,j \leqslant n} \{\hat{h}(P_i - P_j)\} + \mathrm{O}(1).$$

Combining the above inequalities, we have now proven that for every $\boldsymbol{P} = (P_1, \dots, P_n) \in A^n(K)$ with $\boldsymbol{P} \notin W_n$, there is an inequality

$$\min_{1 \leqslant i \leqslant n} \{\lambda_D(P_i, v)\} \leqslant c \max_{1 \leqslant i,j \leqslant n} \{\hat{h}(P_i - P_j)\} + \mathrm{O}(1).$$

If we now add the additional hypothesis that $P_1, \dots, P_n \in \Sigma_3(D, \epsilon, \tau)$, and sum over all $v \in S$, then we obtain the lower bound

$$\sum_{v \in S} \min_{1 \leqslant i \leqslant n} \{\lambda_D(P_i, v)\} \geqslant \sum_{v \in S} \min_{1 \leqslant i \leqslant n} \{\epsilon \tau_v \hat{h}(P_i)\}$$

$$= \epsilon \min_{1 \leqslant i \leqslant n} \{\hat{h}(P_i)\} \text{ since } \sum_{v \in S} \tau_v = 1.$$

Since S is finite, combining these last two inequalities gives the desired result,

$$\max_{1 \leqslant i,j \leqslant n} \{\hat{h}(P_i - P_j)\} \geqslant c \min_{1 \leqslant i \leqslant n} \{\hat{h}(P_i)\} + \mathrm{O}(1). \quad \square$$

§3. Abelian surfaces

We now add the assumption that $\dim(A) = 2$, and proceed with the proof of Theorem 2. Combining Theorem 7 with Lemma 6 (and the other reduction lemmas), we see that it suffices to show that for sufficiently large n, any set of *n distinct* points $P_1, \ldots, P_n \in \Sigma_2(D, \epsilon)$ satisfy $(P_1, \ldots, P_n) \notin W_n$. Further, from Lemma 4, we may assume that D is an irreducible divisor. Since A is an abelian surface, this means that D is an irreducible curve contained in A. As in Lemma 7, for each effective divisor $D \in \mathrm{Div}(A)$ we define

$$\Phi = \Phi(D) = \{T \in A : D_T = D\}.$$

We start with the case that $\Phi(D)$ is infinite.

Proposition 9

Let $D \in \mathrm{Div}(A)$ be an effective divisor on an abelian surface. Suppose that $\Phi(D)$ is infinite. Then $\Sigma_2(D, \epsilon)$ is finite.

Proof

From Lemma 7, we have

$$\varphi(\Sigma_2(D, \epsilon)) \subset \Sigma_2(\varphi D, \epsilon') \cup (\text{finite set}).$$

Now the fact that $\Phi = \Phi(D)$ is infinite means that it is the product of a finite group with an elliptic curve. (Remember we are assuming that $\dim(A) = 2$.) Hence the quotient A/Φ is an elliptic curve. Further, $\varphi D \in \mathrm{Div}(A/\Phi)$ is an effective divisor, hence ample. It follows from the refined version of Siegel's theorem (cf. [Silverman, 1986], IX.3.1) that $\Sigma_2(\varphi D, \epsilon')$ is finite, and so the above inclusion shows that $\varphi(\Sigma_2(D, \epsilon))$ is also finite.

Next, suppose that we fix a point $\pi \in A/\Phi$, and look at the set of $P \in \Sigma_2(D, \epsilon)$ such that $\varphi(P) = \pi$. Then using the definition of $\Sigma_2(D, \epsilon)$ and the fact that $\varphi^*(\varphi D) = D$, we see that

$$\epsilon \hat{h}(P) \leqslant \sum_{v \in S} \lambda_D(P, v) = \sum_{v \in S} \lambda_{\varphi^*(\varphi D)}(P, v) = \sum_{v \in S} \lambda_{\varphi D}(\pi, v) + \mathrm{O}(1).$$

Now for a given π, the right-hand-side is fixed, so there are only finitely many choices for P. This completes the proof that $\Sigma_2(D, \epsilon)$ is finite. □

Combining Lemma 7 and Proposition 9, we are essentially reduced to the case of divisors satisfying $\Phi(D) = \{0\}$ (i.e. D is not equal to any of its translates). We now prove a purely geometric lemma which describes how the differencing map affects these divisors. As in the proof of Theorem 8, we define the differencing map on A by

$$\sigma : A^n \to A^{n^2}$$

$$(P_1, \ldots, P_n) \to (\ldots, P_i - P_j, \ldots)_{1 \leqslant i, j \leqslant n}.$$

We will also use the following notation: for any set Σ, let

$$\Sigma^{\langle n \rangle} = \{(x_1, \ldots, x_n) \in \Sigma : x_1, \ldots, x_n \text{ are distinct}\}.$$

Lemma 10

Let $D \in \mathrm{Div}(A)$ be an effective divisor on an abelian surface with the property that $\Phi(D) = \{0\}$. Then for all sufficiently large n, the map

$$\sigma : \mathrm{Supp}(D)^{\langle n \rangle} \to A^{n^2}$$

is injective. (In other words, distinct n-tuples of distinct points in the support of D have distinct sets of differences.)

Proof

For any $T \in A$, the divisor D_T is algebraically equivalent to D, so the intersection product $D.D_T$ is independent of T. Hence, if we choose $n > D^2$, then we will have

$$n > \#(\mathrm{Supp}(D) \cap \mathrm{Supp}(D_T)) \quad \text{for all } T \notin \Phi(D).$$

Now suppose that $\boldsymbol{P}, \boldsymbol{Q} \in \mathrm{Supp}(D)^{\langle n \rangle}$ and $\sigma(\boldsymbol{P}) = \sigma(\boldsymbol{Q})$. Since σ is a homomorphism whose kernel is the diagonal of A^n, it follows that there is a $T \in A$ such that $P_i = Q_i + T$ for every $1 \leqslant i \leqslant n$. Therefore

$$P_i \in \mathrm{Supp}(D) \cap \mathrm{Supp}(D_T) \quad \text{for all } 1 \leqslant i \leqslant n.$$

But by definition of $\mathrm{Supp}(D)^{\langle n \rangle}$, the P_i's are distinct. It follows from our choice of n above that $D = D_T$, so $T \in \Phi(D)$. But by hypothesis, $\Phi(D) = \{0\}$, from which we conclude that $T = 0$ and $\boldsymbol{P} = \boldsymbol{Q}$. This proves that σ is injective on $\mathrm{Supp}(D)^{\langle n \rangle}$. □

The next proposition provides the last fact needed to prove Theorem 2. In it we use Faltings' theorem (Mordell conjecture) to prove that for trivial $\Phi(D)$ and sufficiently large n, the exceptional set in Theorem 8 is empty.

Proposition 11

Let $D \in \mathrm{Div}(A)$ be an irreducible effective divisor on an abelian surface, and assume that $\Phi(D) = \{0\}$. Then for all sufficiently large n, the set

$$A(K)^{\langle n \rangle} \cap W_n = \emptyset.$$

(I.e. There are no points $(P_1, \dots, P_n) \in W_n$ with the property that the P_i's are distinct and defined over K. For the definition of W_n, see Theorem 8.)

Proof

Since D is irreducible, we may treat it as an irreducible curve lying on the abelian surface A. Now abelian varieties do not contain rational curves, so $\mathrm{genus}(D) \neq 0$. Similarly, if $\mathrm{genus}(D) = 1$, then D would be a translate of an elliptic curve, and so $\Phi(D)$ would be infinite. We conclude that $\mathrm{genus}(D) \geqslant 2$. (Note that if D is singular, the genus of D means the genus of the normalization of D.) From Faltings' theorem [Faltings, 1983], we know that $D(K)$ is finite. Choose n to be any integer greater than $\#D(K)$ such that the map in Lemma 10 is injective.

Now suppose that $\boldsymbol{P} \in A(K)^{\langle n \rangle} \cap W_n$. From the definition of W_n, we have $W_n = \sigma^{-1}(\sigma(D^n))$, so $\sigma(\boldsymbol{P}) \in \sigma(D^n)$. Further, since the P_i's comprising $\boldsymbol{P}$ are distinct, we see that $\sigma(\boldsymbol{P}) \in \sigma(D^{\langle n \rangle})$. Now we use Lemma 10, which says that σ is injective on $D^{\langle n \rangle}$. In other words, the map

$$\sigma : D^{\langle n \rangle} \to A^{n^2}$$

is an injective morphism defined over K. It follows that any K-rational point in the image (such as $\sigma(\boldsymbol{P})$) comes from a K-rational point in $D^{\langle n\rangle}$. Let $\boldsymbol{Q}$ be the inverse image of $\boldsymbol{P}$. Then $Q_1,\dots,Q_n \in D(K)$, and the Q_i's are distinct. This contradicts the choice of n. Therefore $A(K)^{\langle n\rangle} \cap W_n = \emptyset$. □

Proof of Theorems 1 and 2

From Lemma 3, we see that Theorem 2 implies Theorem 1. It thus suffices to prove Theorem 2. Next, using Lemma 4, it suffices to prove that $\Sigma_2(D, \epsilon)$ is widely spaced for every irreducible effective divisor D and every $\epsilon > 0$.

Next consider the set $\Phi(D)$. If $\Phi(D)$ is infinite, then from Proposition 9, the set $\Sigma_2(D, \epsilon)$ is actually finite. On the other hand, if $\Phi(D)$ is finite, then the map $\varphi: A \to A/\Phi(D)$ is finite. So using Proposition 7, we see that it suffices to show that $\Sigma_2(\varphi D, \epsilon')$ is widely spaced. In this way, we reduce to the case that $\Phi(D) = \{0\}$.

Now let n be an integer large enough so that the conditions of Proposition 11 hold, namely

$$A(K)^{\langle n\rangle} \cap W_n = \emptyset.$$

It follows in particular that

$$\Sigma_3(D, \epsilon, \tau)^{\langle n\rangle} \cap W_n = \emptyset.$$

Therefore Theorem 8 holds for *every* n-tuple of distinct points

$$(P_1,\dots,P_n) \in \Sigma_3(D, \epsilon, \tau)^{\langle n\rangle}.$$

Now Lemma 6 implies that $\Sigma_3(D, \epsilon, \tau)$ is widely spaced, and then Lemma 5 allows the same conclusion to be made for $\Sigma_2(D, \epsilon)$. □

References

Brown, M.L.: Schémas en groupes et lemme de zéros. *Problèmes Diophantiens, Publ. Math. de l'Univ. P.&M. Curie* No. 73, ed. by D. Bertrand and M. Waldschmidt (1984–85).

Faltings, G: Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Inv. Math.* 73 (1983) 349–366.

Lang, S.: *Fundamentals of Diophantine Geometry*. Springer-Verlag (1983).

Mumford, D.: A remark on Mordell's conjecture. *Am. J. of Math.* 87 (1965) 1007–1016.

Mumford, D.: *Abelian Varieties*, Oxford University Press, 1974.

Silverman, J.: *The Arithmetic of Elliptic Curves, Graduate Texts in Math.* 106, Springer-Verlag (1986).

Silverman, J.: Arithmetic distance functions and height functions in Diophantine geometry. To appear.

Szpiro, L.: Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell, *Astérisque* 127, Soc. Math. de France (1985).

Vojta, P.: A higher dimensional Mordell conjecture. In: *Arithmetic Geometry*, ed. by G. Cornell and J. Silverman. Springer-Verlag (1986) 334–346.