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Vyjayanthi Chari
Andrew Pressley

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VYJAYANTHI CHARI[1,*†] & ANDREW PRESSLEY[2]
[1]*School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA*
(*Permanent address: School of Mathematics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Bombay 400 005, India) [2]Department of Mathematics, King's College,
Strand, London WC2R 2LS, UK*

Introduction

The representation theory of the Virasoro algebra plays an important role in many areas of Mathematics and Physics. As examples, we may cite the theory of affine Lie algebras, statistical mechanics and two-dimensional conformal quantum field theory. In all these areas, the unitary representations are particularly significant. In this paper we shall give a complete classification of the unitary representations of the Virasoro algebra which have finite multiplicities under the rotation subalgebra. The result proves a special case of a conjecture of Victor Kac. We also obtain similar results for the Ramond and Neveu–Schwarz superalgebras.

We recall that the Virasoro algebra *Vir* is the complex Lie algebra with basis $\{c, L_n (n \in \mathbb{Z})\}$ and commutation relations

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{(m^3 - m)}{12} \delta_{m,-n} c$$

$$[c, L_n] = 0. \tag{0.1}$$

It is the universal central extension of the Witt algebra, i.e. the Lie algebra of Laurent polynomial vector fields on the circle S^1. A representation V of *Vir* is said to be unitary with respect to a conjugate-linear anti-involution θ of *Vir* if there is a positive-definite sesqui-linear form $\langle, \rangle$ on V such that

$$\langle x . v, w \rangle = \langle v, \theta(x) . w \rangle$$

for all $x \in Vir$, v, $w \in V$. In fact, we shall see that it is enough to consider the conjugate-linear anti-involution given by

$$\theta(c) \ = \ c, \quad \theta(L_n) \ = \ L_{-n}, \quad \text{for all} \quad n \in \mathbb{Z} \tag{0.2}$$

(see Proposition 3.4 for a precise statement and proof). We shall assume in addition that V satisfies the following conditions:

(a) L_0 acts semisimply on V, and
(b) the eigenspaces of L_0 are finite-dimensional.

Assumption (a) is natural, for it is easy to see that $\mathbb{C} \cdot c \oplus \mathbb{C} \cdot L_0$ is the unique maximal abelian subalgebra of Vir which acts semisimply in the adjoint representation; it also justifies calling the eigenspaces of L_0 the weight spaces of V. Assumption (b) guarantees that c has an eigenvector in V and hence, by Schur's lemma, that c acts by a scalar, say z, if V is irreducible. Note that there are irreducible, unitary representations of Vir with infinite-dimensional weight spaces, such as the space of symmetric (or anti-symmetric) $\frac{1}{2}$-densities on the torus $S^1 \times S^1$, but they will not be considered in this paper (see [12] for a general discussion).

Two essentially different families of irreducible, unitary representations of Vir with finite-dimensional weight spaces are known. First of all, there are the lowest weight representations, i.e. the representations V generated by a vector $v \in V$ such that

   (i) $L_0 v = av$ for some $a \in \mathbb{C}$, and
   (ii) $L_n v = 0$, for all $n < 0$.

(There are, of course, highest weight representations, but their properties are in every way parallel to those of the lowest weight representations.) Such a representation is determined uniquely by the pair (z, a). It is obvious that $V = V(z, a)$ can be unitary only if $z \geqslant 0$ and $a \geqslant 0$ (see [8]). Conversely, it follows easily from the Kac determinant formula [7] that $V(z, a)$ is indeed unitary if $z \geqslant 1$ and $a \geqslant 0$, but the case $0 \leqslant z < 1$ is more difficult. It was proved by Friedan, Qiu and Shenker [1, 3] that, in this region, unitarity is possible only for the pairs $(z_n, a_n^{(p,q)})$, where

$$z_n \ = \ 1 - \frac{6}{n(n+1)}, \quad a_n^{(p,\,q)} \ = \ \frac{(np+q)^2 - 1}{4n(n+1)} \tag{0.3}$$

and the integers n, p, q satisfy $n \geqslant 2$ and $0 \leqslant p < q < n$ (see also [15]). Finally, Goddard, Kent and Olive [4, 5], Kac and Wakimoto [9], and

Tsuchiya and Kanie [16] showed that each of these "discrete series" representations are, in fact, unitary.

The other known unitary representations of *Vir* are spaces of λ-densities on S^1, where $\lambda \in \frac{1}{2} + i\mathbb{R}$. To be precise, for any $\lambda, a \in \mathbb{C}$, let $W(\lambda, a)$ be the space of densities with basis $\{w_n\}_{n \in \mathbb{Z}}$ given by $w_n = e^{i(n+a)\theta} |d\theta|^\lambda$. The natural action of the Witt algebra on $W(\lambda, a)$ by Lie differentiation is given by

$$L_k . w_n = (n + a + k\lambda)w_{n+k};$$

the action extends to *Vir* by setting $c . w_n = 0$. It is clear that $W(\lambda, a)$ is an irreducible representation of *Vir* unless $\lambda = 0$ or 1 and $a \in \mathbb{Z}$.

We can now state our main result.

THEOREM 0.5. *Let V be an irreducible, unitary representation of* ***Vir*** *with finite-dimensional weight spaces. Then either V is highest or lowest weight, or V is isomorphic to a space $W(\lambda, a)$ of λ-densities on the circle, for some $\lambda \in \frac{1}{2} + i\mathbb{R}$, $a \in \mathbb{R}$.*

We note that this result proves the following conjecture of Kac [8] in the case of unitary representations:

CONJECTURE 0.6. *Every irreducible representation of the Virasoro algebra with finite-dimensional weight spaces is either highest or lowest weight, or has all its weight spaces of dimension less than or equal to one.*

The motivation for this conjecture is the following result of Kostrikin [14]. Let W_1 be the subalgebra of the Witt algebra spanned by the L_k for $k \geqslant -1$.

THEOREM 0.7. *Let* ***g*** *be a simple $\mathbb{Z}$-graded Lie algebra of Cartan type other than W_1 and let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be an irreducible $\mathbb{Z}$-graded representation of* ***g*** *with dim $V_n < \infty$ for all n. Then, either $V_n = 0$ for $n \gg 0$ ("highest weight") or $V_n = 0$ for all $n \ll 0$ ("lowest weight").*

The $W(\lambda, a)$ show that this result is false for W_1. Conversely, for any Lie algebra of Cartan type there are analogues of the $W(\lambda, a)$ but, except for W_1, their graded components are infinite-dimensional.

We remark that in [11] it is proved that the $W(\lambda, a)$ are the only irreducible representations of the Witt algebra with one-dimensional weight spaces. We shall not need to use this result; in fact, we shall re-derive it, under the assumption of unitarity.

Although our proof of Theorem 0.5 is quite elementary, it is fairly complicated, so we shall end this introduction with a brief survey of the contents of the paper.

The representation V of **Vir** under consideration can be viewed as a representation of the **sl(2, C)**-subalgebra g_1 of **Vir** spanned by L_0, L_1, and L_{-1}, and if V is unitary for the anti-involution θ defined in (0.2), then it is unitary for the real form **su(1, 1)** of g_1. Thus, as a representation of g_1, V breaks up as a direct sum of discrete series and continuous series representations. This allows us to choose a "good" basis of V.

The unitary representations of **su(1, 1)** with finite-dimensional weight spaces are described in §2, after introducing some notation in §1. The necessary background on the Virasoro algebra is contained in §3. In particular, we classify all the conjugate-linear anti-involutions of **Vir**, and show that if a representation of **Vir** is unitary for some conjugate-linear anti-involution, then it is unitary for the anti-involution θ defined in (0.2).

The basic idea of our approach, taken from [10, 11], is to write the action of **Vir** on V, with its "good" basis, as a perturbation of the action (0.4). We prove in §4 that the dimensions of the weight spaces of V are independent of the weight, provided the weight is sufficiently large or sufficiently small. This allows an asymptotic analysis of the commutation relations in **Vir**, which is carried out in §5. In particular, this shows that, if V is not highest or lowest weight, then the centre of **Vir** must act trivially on V.

The remainder of the proof considers separately the cases where V does, in §6, or does not, in §7, contain a discrete series representation of g_1. In the first case, it is proved that all the weight spaces of V must have dimension exactly one; this allows us to analyze the commutation relations exactly, not just asymptotically, and we deduce that V is isomorphic to $W(\frac{1}{2}, \frac{1}{2})$. In the second case, it is obvious that all the weight spaces of V have the same dimension and again the commutation relations can be analyzed exactly; we then deduce that V must be isomorphic to some $W(\lambda, a)$.

We conclude in §8 with the corresponding results for the Virasoro superalgebras:

THEOREM 0.8. *Let V be an irreducible unitary representation of the Ramond or Neveu-Schwarz Lie superalgebras. Then V is highest or lowest weight.*

This follows without difficulty from Theorem 0.5. We remark that there are analogues of the $W(\lambda, a)$ for the Virasoro superalgebras; they are of the form $W = W_{(0)} \oplus W_{(1)}$, where the even and odd parts are given by

$$W_{(0)} = W(\lambda, a), \quad W_{(1)} = W(\lambda \pm \tfrac{1}{2}, a + \kappa),$$

where λ, $a \in \mathbb{C}$ and $\kappa = 0$ for the Ramond superalgebra, $\frac{1}{2}$ for the Neveu-Schwarz superalgebra. However, these modules are never unitary.

§1. Generalities

We begin with some basic notation and terminology. Let g be a complex Lie algebra (possibly infinite-dimensional) and let h be an abelian subalgebra of g. For any g-module V and any $\lambda \in h^*$, the linear dual of h, set

$$V_\lambda = \{v \in V : h \cdot v = \lambda(h)v \; \forall h \in h\}.$$

DEFINITION 1.1. A g-module V is called a (g, h)-module if

  (i) $V = \bigoplus_{\lambda \in h^*} V_\lambda$, and
  (ii) dim $V_\lambda < \infty, \; \forall \lambda \in h^*$.

Note that, by Schur's lemma, the centre of the universal enveloping algebra $U(g)$ of g acts by scalars on any irreducible (g, h)-module V.

Now let θ be a conjugate-linear anti-involution of g, i.e. θ is a map $g \rightarrow g$ such that

$$\theta(x + y) = \theta(x) + \theta(y), \quad \theta(\alpha x) = \bar{\alpha}\theta(x),$$

$$\theta[x, y] = [\theta(y), \theta(x)], \quad \theta^2 = \text{identity},$$

for all $x, y \in g$, $\alpha \in \mathbb{C}$. The real Lie algebra

$$g_0 = \{x \in g : \theta(x) = -x\}$$

is called the real form of g corresponding to θ. The complexification of g_0 is the original Lie algebra g: as a vector space we have

$$g = g_0 \oplus ig_0.$$

DEFINITION 1.2. A g-module V is said to be unitary for a conjugate-linear anti-involution θ of g, or for the corresponding real form g_0 of g, if V admits a positive-definite sesqui-linear form $\langle, \rangle$ such that

$$\langle x \cdot v, w \rangle = \langle v, \theta(x) \cdot w \rangle$$

for all $x \in g$, $v, w \in V$. (This is equivalent to requiring that g_0 acts on V by skew-adjoint operators.)

It is clear that every unitary (g, h)-module is completely reducible.

§2. Unitary representations of su(1, 1)

In this section we collect some basic facts about *sl(2, ℂ)* and the unitary representations of its real forms. For proofs see [13] or [17].

Let $\{x, y, h\}$ be the standard basis of *sl(2, ℂ)*. Thus

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

The subalgebra $h = ℂ.h$ is maximal abelian in *sl(2, ℂ)* and acts semisimply on *sl(2, ℂ)* in the adjoint representation.

Every conjugate-linear anti-involution of *sl(2, ℂ)* is conjugate, by an element of the group $SL(2, ℂ)$, to one of the following:

$$\theta_c(x) = y, \quad \theta_c(y) = x, \quad \theta_c(h) = h \quad \text{(compact anti-involution)}$$

$$\theta_n(x) = -y, \quad \theta_n(y) = -x, \quad \theta_n(h) = h$$
$$\text{(non-compact anti-involution).}$$

The real form corresponding to θ_c is the compact real form *su(2)*. Its unitary representations are described in the next result.

PROPOSITION 2.1. *Let V be an irreducible (sl(2, ℂ), h)-module which is unitary for su(2). Then V is finite-dimensional and (hence) V contains a vector $v \neq 0$ such that $x.v = 0$ and $h.v = av$ for some $a \in ℝ$. The eigenvalue $a \geqslant 0$ and $a = 0$ if and only if V is trivial.*

The situation for θ_n is a little more complicated; the corresponding real form is *su(1, 1)*. If V is an irreducible (*sl(2, ℂ)*, *h*)-module which is unitary for *su(1, 1)*, then it is clear that the eigenvalues of h on V are all real and that any two eigenvalues differ by an even integer. Thus,

$$V = \bigoplus_{n \in ℤ} V_n$$

where

$$V_n = \{v \in V : h.v = 2(a + n)v\}$$

and a is some real number, depending only on V, which can be assumed to satisfy $0 \leqslant a < 1$.

The following result gives a complete description of the irreducible (*sl(2, ℂ)*, *h*)-modules which are unitary for *su(1, 1)*.

PROPOSITION 2.3. *Let $0 \leqslant a < 1$ and let λ be a complex number such that either*

(i) $a < \lambda < 1 - a$ or $1 - a < \lambda < a$ or $\lambda \in \frac{1}{2} + i\mathbb{R}$, or
(ii) $\lambda \in a + \mathbb{Z}_+$ or $\lambda \in (1 - a) + \mathbb{Z}_+$.

($\mathbb{Z}_+$ denotes the non-negative integers).

(a) *There exists an irreducible $(sl(2, \mathbb{C}), h)$-module which is unitary for $su(1, 1)$ and which has a spanning set $\{v_n\}_{n \in \mathbb{Z}}$ such that*

$$h \cdot v_n = 2(a + n) v_n$$

$$x \cdot v_n = (\lambda + a + n) v_{n+1}$$

$$y \cdot v_n = (\lambda - a - n) v_{n-1}$$

for all $n \in \mathbb{Z}$. In case (i), the representation is denoted by $C(\lambda, a)$ and $\{v_n\}_{n \in \mathbb{Z}}$ is a basis of it. In case (ii), the representation is denoted by $D^+(\lambda)$ (resp. $D^-(\lambda)$) if $\lambda \in a + \mathbb{Z}_+$ (resp. $\lambda \in (1 - a) + \mathbb{Z}_+$) and then $\{v_n\}_{n \geqslant \lambda - a}$ (resp. $\{v_n\}_{n \leqslant -\lambda - a}$) is a basis of the representation and the remaining v_n are zero.
(b) *$C(\lambda, a)$ is isomorphic to $C(1 - \lambda, a)$ and there are no other isomorphisms between the representations described in (a).*
(c) *Every non-trivial, irreducible $(sl(2, \mathbb{C}), h)$-module which is unitary for $su(1, 1)$ is isomorphic to one of the representations described in (a).*

REMARK 2.4. The representations $C(\lambda, a)$ are called continuous series representations; the $D^{\pm}(\lambda)$ are called discrete series representations. If $\lambda \in a + \mathbb{Z}_+$ (resp. $\lambda \in (1 - a) + \mathbb{Z}_+$) then $D^+(\lambda)$ (resp. $D^-(\lambda)$) is a lowest (resp. highest) weight representation with lowest (resp. highest) weight λ (resp. $-\lambda$).

We isolate the following consequence of Proposition 2.3 for later use.

COROLLARY 2.5. *Let V be an $(sl(2, \mathbb{C}), h)$-module which is unitary for $su(1,1)$. Then the map $x: V_n \to V_{n+1}$ (resp. $y: V_n \to V_{n-1}$) is injective if $a + n > 0$ (resp. if $a + n < 0$). In particular, $\dim V_n \leqslant \dim V_{n+1}$ if $a + n > 0$, and $\dim V_n \leqslant \dim V_{n-1}$ if $a + n < 0$.*

§3. The Virasoro algebra

The Virasoro algebra *Vir* was defined in the introduction (see (0.1)). Set $d = \mathbb{C} \cdot c \oplus \mathbb{C} \cdot L_0$. It is clear that *d* is a maximal abelian subalgebra of *Vir* which acts semisimply on *Vir*. Conversely:

PROPOSITION 3.1. *The subalgebra d is the unique maximal abelian subalgebra of Vir which acts semisimply on Vir in the adjoint representation.*

Proof. (See also [11].) Let $L \in Vir$ be any semisimple element, say

$$L = \lambda c + \sum_{n=k}^{l} \lambda_n L_n,$$

where $\lambda, \lambda_n \in \mathbb{C}, k \leqslant l$ and $\lambda_k \neq 0, \lambda_l \neq 0$. Suppose that $l > 0$. There exists an eigenvector of L in the adjoint representation of the form

$$M = \mu c + \sum_{n=p}^{q} \mu_n L_n$$

where $q > 0$ and $\mu_q \neq 0$. Thus,

$$[L, M] = \nu M$$

for some $\nu \in \mathbb{C}$. The left-hand side of this equation is of the form $\lambda_l \mu_q L_{l+q} +$ (a linear combination of c and the L_n for $n < l + q$), while the right-hand side does not involve L_{l+q}. Therefore, $\lambda_l \mu_q = 0$, a contradiction. Thus, $l \leqslant 0$. Similarly, we can prove that $k \geqslant 0$.

The next result classifies the conjugate-linear anti-involutions of *Vir*.

PROPOSITION 3.2. *Any conjugate-linear anti-involution of Vir is of one of the following types*:

(i) $\theta_\alpha^+ (L_n) = \alpha^n L_{-n}, \theta_\alpha^+ (c) = c$, *for some* $\alpha \in \mathbb{R}^\times$;
(ii) $\theta_\alpha^- (L_n) = -\alpha^n L_{-n}, \theta_\alpha^- (c) = -c$, *for some* $\alpha \in S^1$, *the set of complex numbers of modulus one.*

Proof. It is easy to check that the formulae in the statement of the proposition do indeed define conjugate-linear anti-involutions of *Vir*. Conversely, let θ be any conjugate-linear anti-involution of *Vir*. From Proposition 3.1 and the fact that $\mathbb{C}.c$ is the centre of *Vir*, it follows that

$$\theta(c) = \lambda c, \quad \theta(L_0) = \lambda_0 c + \mu_0 L_0$$

for some $\lambda_0 \in \mathbb{C}, \lambda, \mu_0 \in S^1$. Write, for $n \neq 0$,

$$\theta(L_n) = \lambda_n c + \sum_{m \in \mathbb{Z}} \mu_{n,m} L_m$$

where, for fixed n, all but finitely many $\mu_{n,m}$ are zero. By applying θ to the equation

$$[L_0, L_n] = nL_n$$

we find that

$$-\mu_0 \sum_{m \in \mathbb{Z}} \mu_{n,m} (mL_m + \tfrac{1}{12}(m^3 - m)\delta_{m,0}c) = n\left(\sum_{m \in \mathbb{Z}} \mu_{n,m}L_m + \lambda_n c\right)$$

for $n \neq 0$. It follows that $\mu_0 = \pm 1$, and that $\mu_{n,m} = 0$ unless $m = \mp n$. It then follows that $\lambda_n = 0$ for $n \neq 0$.

Consider the case $\mu_0 = +1$ and set $\mu_n = \mu_{n,-n}$; thus $\theta(L_n) = \mu_n L_{-n}$ for all $n \neq 0$. Next, applying θ to the equation

$$[L_{-n}, L_n] = 2nL_0 + \tfrac{1}{12}(n^3 - n)c$$

for $n \neq 0$, we find

$$\mu_{-n}\mu_n(2nL_0 + \tfrac{1}{12}(n^3 - n)c) = 2n(\lambda_0 c + \mu_0 L_0) + \tfrac{1}{12}(n^3 - n)\lambda c.$$

Equating coefficients of L_0 gives $\mu_{-n}\mu_n = 1$; equating coefficients of c now gives $\lambda_0 = 0$, $\lambda = 1$. The equation $\theta^2(L_n) = L_n$ for all n shows that $\mu_{-n}\bar{\mu}_n = 1$, hence μ_n is real for all n. Finally, applying θ to the commutation relation for $[L_n, L_m]$ shows that $\mu_n \mu_m = \mu_{n+m}$, hence $\mu_n = \alpha^n$ for some $\alpha = \mu_1 \in \mathbb{R}$. Thus, $\theta = \theta_\alpha^+$.

Similarly, the case $\mu_0 = -1$ leads to $\theta = \theta_\alpha^-$ for some $\alpha \in S^1$.

Proposition 3.1 makes it natural to consider (*Vir*, *d*)-modules. We are going to show that, in considering the possible unitarity of such a module, it is enough to consider the single conjugate-linear anti-involution θ_1^+. To do so we shall need to use the results of §2: the connection with *sl(2, $\mathbb{C}$)* is the following.

For any integer $n > 0$ consider the elements in *Vir* given by

$$x_n = \frac{1}{n} L_n, \quad y_n = -\frac{1}{n} L_{-n}, \quad h_n = \frac{1}{n}(2L_0 + \tfrac{1}{2}(n^2 - 1)c). \qquad (3.3)$$

They generate a subalgebra g_n of *Vir* isomorphic to *sl(2, $\mathbb{C}$)* (the isomorphism is given by $x \to x_n$, etc.).

PROPOSITION 3.4. *Let V be a non-trivial, irreducible (**Vir**, d)-module.*

(a) *If V is unitary for some conjugate-linear anti-involution θ of **Vir**, then*
$\theta = \theta_\alpha^+$ *for some $\alpha > 0$.*

(b) *If V is unitary for θ_α^+ for some $\alpha > 0$, then V is unitary for θ_1^+.*

Proof. Suppose first that V is unitary for θ_α^- for some $\alpha \in S^1$. Let $v \neq 0$ be an eigenvector for L_0 in V with eigenvalue a. The relation

$$\langle L_0 v, v \rangle = -\langle v, L_0 v \rangle$$

shows that $a \in i\mathbb{R}$. The vector $L_n v$ has eigenvalue $n + a$ for L_0; on the other hand, the previous argument shows that $n + a \in i\mathbb{R}$, a contradiction for $n \neq 0$. Thus, $L_n v = 0$ for $n \neq 0$. The equation

$$[L_{-n}, L_n]v = 0 = (2nL_0 + \tfrac{1}{12}(n^3 - n)c)v$$

for all $n \neq 0$ shows that $L_0 v = cv = 0$. Since V is irreducible, v generates V, so V is trivial.

Suppose next that V is unitary for θ_α^+ for some $\alpha \in \mathbb{R}^\times$ and let $\langle\ ,\ \rangle_\alpha$ be the invariant sesqui-linear form on V. The argument above, together with the irreducibility of V, shows that the L_0-eigenvalues on V are of the form $a + n$ for $n \in \mathbb{Z}$, where $a \in \mathbb{R}$ is fixed. Define a new form $\langle\!\langle\ ,\ \rangle\!\rangle$ on V by

$$\langle\!\langle v, w \rangle\!\rangle = |\alpha|^{-n}\langle v, w \rangle_\alpha$$

if v and w have L_0-eigenvalue $a + n$; note that distinct L_0-eigenspaces are orthogonal. One verifies easily that this form makes V unitary for $\theta_{\alpha/|\alpha|}^+$.

It remains to prove that V cannot be unitary for θ_{-1}^+. Note first that the subalgebra g_1 of **Vir** is preserved by θ_{-1}^+ and that the corresponding real form of g_1 is **su(2)**. By Proposition 2.1, V is a direct sum of finite-dimensional g_1-modules. We can therefore find non-zero L_0-eigenvectors $v_\pm \in V$ such that $L_1 v_+ = L_{-1} v_- = 0$. Since $L_{\pm 1}^n(L_{\pm 2}.v_\pm) = 0$ for $n \gg 0$, it follows that $L_n v_+ = L_{-n} v_- = 0$ for $n \gg 0$.

Let $L_0 v_+ = a_+ v_+$. By Proposition 2.1, $a_+ \geq 0$. On the other hand, the subalgebra g_{2n} of **Vir** is preserved by θ_{-1}^+ and the corresponding real form is **su(1, 1)**. It follows from Proposition 2.3 that the eigenvalue of h_{2n} (see (3.3)) on v_+ is ≤ 0 for $n \gg 0$. This implies that the central charge $z \leq 0$. Suppose that $z = 0$. Then v_+ is annihilated by L_0 and hence, by Proposition 2.1, also by $L_{\pm 1}$. It follows from Proposition 2.3 that v_+ is annihilated by $L_{\pm n}$ for all $n \gg 0$; hence V is trivial. Thus, one must have $z < 0$.

Similarly, by considering v_-, we find that $z > 0$, a contradiction.

In the rest of this paper we shall only consider representations of **Vir** which are unitary for θ_1^+, which we shall henceforth denote simply by θ.

§4. Dimensional stability of weight spaces

For the rest of this paper, V will denote a non-trivial irreducible (**Vir**, **d**)-module which is unitary for the conjugate-linear anti-involution θ defined at the end of the previous section. Regarding V as a representation of **g**, we have a decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

where $V_n = \{v \in V : L_0 v = (a + n)v\}$ and a is a fixed real number, depending only on V, such that $0 \leqslant a < 1$. Let z be the eigenvalue of c on V; z is necessarily a real number.

PROPOSITION 4.1. *If $z \leqslant 0$ (resp. $z \geqslant 0$) then* $\dim V_n = \dim V_{n+1}$ *for all $n \geqslant 0$ (resp. for all $n \ll 0$).*

Proof. Suppose that $z \leqslant 0$; the proof for $z \geqslant 0$ is similar. We shall show that

$$\dim V_n \leqslant \dim V_0 + \dim V_1, \quad \text{for all } n \geqslant 0;$$

in view of Corollary 2.5, this implies the result.

Let K_{n+1} denote the kernel of the map $L_{-n} : V_{n+1} \to V_1$. It suffices to prove that $L_{-(n+1)} : K_{n+1} \to V_0$ is injective if $n \geqslant 0$. If $K_{n+1} = 0$ there is nothing to prove; otherwise, suppose that $v \in K_{N+1}$ is a non-zero vector annihilated by $L_{-(N+1)}$, for some $N > 0$. Then $L_{-(2N+1)n} v = 0$ for all $n > 0$. The elements $L_{(2N+1)n}$, for $n \in \mathbb{Z}$, generate a subalgebra Vir_{2N+1} of **Vir** isomorphic to **Vir**: the isomorphism is given by

$$L_n \to \frac{1}{2N+1} (L_{(2N+1)n} - \tfrac{1}{24} c \delta_{n,0})$$

$$c \to (2N+1)c.$$

The (Vir_{2N+1}, **d**)-module $U(Vir_{2N+1}) . v$ is unitary, lowest weight and the central element $(2N+1)c$ acts by a non-positive number on it. By the Friedan–Qiu–Shenker result (0.3), it must be trivial. In particular, v is

annihilated by L_{2N+1}, as well as by L_{-N} and $L_{-(N+1)}$. It follows easily that v is annihilated by the whole of **Vir**, and hence that V is trivial, a contradiction.

§5. Asymptotic analysis of the commutation relations

From now on we shall assume that V is neither highest nor lowest weight as a representation of **Vir**. In this section we shall assume that $z \leqslant 0$; all the arguments carry over, with obvious modifications, to the case $z \geqslant 0$. (In fact, we shall prove later in this section that $z = 0$.)

We have already observed that V, regarded as a g_1-module, is unitary for the non-compact real form. It therefore decomposes as a direct sum of discrete series and continuous series modules, as described in Proposition 2.3. Further, Proposition 4.1 implies that only finitely many continuous series and lowest weight discrete series modules can occur. Set

$$k = \max \left\{ n \in \mathbb{Z}: \begin{array}{l} V \text{ contains a lowest weight discrete series} \\ g_1\text{-module with lowest weight } a + n \end{array} \right\}$$

or

$$k = 0, \text{ if no lowest weight discrete series modules occur.} \tag{5.1}$$

Note that $\dim V_n = \dim V_k$ for all $n \geqslant k$.

PROPOSITION 5.2. *There exists a basis $\{v_{i,n}: n \geqslant k, 1 \leqslant i \leqslant r\}$ of the subspace $\bigoplus_{n \geqslant k} V_n$ of V, for some positive integer r, such that*

(i) $L_1 . v_{i,n} = (n + a + \lambda_i)v_{i,n+1}$ *for* $n \geqslant k$
    $L_{-1} . v_{i,n} = (n + a - \lambda_i)v_{i,n-1}$ *for* $n > k$,
    for some $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$. Moreover, $n + a \pm \lambda_i \neq 0$ for $n > k$.
(ii) $\langle v_{i,n}, v_{j,m} \rangle = 0$ *unless $i = j$ and $n = m$.*

Proof. Choose a decomposition of V, as a g_1-module, into discrete and continuous series modules, such that the irreducible components are orthogonal with respect to the form $\langle \, , \, \rangle$ on V. Let $V^{(1)}, \ldots, V^{(r)}$ be the continuous series and lowest weight discrete series modules which occur. If we choose a basis of each $V^{(i)}$ as in Proposition 2.3, we easily obtain the result of the proposition.

Let λ be the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_r)$ and define $r \times r$ matrices $A(n)$, for $n \geqslant k$, and $B(n)$, for $n \geqslant k + 2$, such that, with respect to the

bases chosen in Proposition 5.2, we have

$$L_2|_{V_n} = n + a + 2\lambda + A(n), \quad L_{-2}|_{V_n} = n + a - 2\lambda + B(n) \quad (5.3)$$

(c.f. (0.4)). Here, n denotes n times the $r \times r$ identity matrix, and similarly for a. Expressing the commutation relation $[L_{-1}, L_2] = 3L_1$, applied to V_n, in matrix form gives

$$A(n) = (n + a + 2 - \lambda)^{-1} A(n - 1)(n + a - \lambda),$$

which implies that

$$A(n) = \frac{\Gamma(k + a + 3 - \lambda)}{\Gamma(n + a + 3 - \lambda)} A(k) \frac{\Gamma(n + a + 1 - \lambda)}{\Gamma(k + a + 1 - \lambda)} \quad (5.4)$$

for $n \geqslant k$. The gamma functions here denote the obvious diagonal matrices. (For the definition and properties of the gamma function, see [18], Chapter 12.) Similarly,

$$B(n) = \frac{\Gamma(n + a - 2 + \lambda)}{\Gamma(k + a + \lambda)} B(k + 2) \frac{\Gamma(k + a + 2 + \lambda)}{\Gamma(n + a + \lambda)} \quad (5.5)$$

for $n \geqslant k + 2$. Next, the commutation relation $[L_{-2}, L_2] = 4L_0 + \frac{1}{2}c$, applied to V_n, gives

$$A(n - 2)(n + a - 2\lambda) + (n + a - 2 + 2\lambda)B(n)$$

$$- (n + a + 2 - 2\lambda)A(n) - B(n + 2)(n + a + 2\lambda)$$

$$= B(n + 2)A(n) - A(n - 2)B(n) - \frac{1}{2}z \quad (5.6)$$

for $n \geqslant k + 2$. Substituting from (5.4) and (5.5) into (5.6), multiplying on the left by $\Gamma(n + a + 1 - \lambda)\Gamma(k + a + 3 - \lambda)^{-1}$, on the right by $\Gamma(k + a + 1 - \lambda)\Gamma(n + a + 1 - \lambda)^{-1}$ and simplifying, we find:

$$\left\{ A(k) \frac{n + a - 2\lambda}{(n + a - \lambda)(n + a - 1 - \lambda)} \right.$$

$$\left. - \frac{n + a + 2 - 2\lambda}{(n + a + 2 - \lambda)(n + a + 1 - \lambda)} A(k) \right\}$$

$$+ \frac{\Gamma(n + a + \lambda)\Gamma(n + a + 1 - \lambda)}{\Gamma(k + a + \lambda)\Gamma(k + a + 3 - \lambda)}$$

$$\times \left\{ \frac{n + a - 2 + 2\lambda}{(n + a - 1 + \lambda)(n + a - 2 + \lambda)} B(k + 2) \right.$$

$$\left. - B(k + 2) \frac{n + a + 2\lambda}{(n + a + 1 + \lambda)(n + a + \lambda)} \right\}$$

$$\times \frac{\Gamma(k + a + 2 + \lambda)\Gamma(k + a + 1 - \lambda)}{\Gamma(n + a + \lambda)\Gamma(n + a + 1 - \lambda)}$$

$$= \frac{\Gamma(n + a + \lambda)\Gamma(n + a + 1 - \lambda)}{\Gamma(k + a + \lambda)\Gamma(k + a + 3 - \lambda)} B(k + 2)$$

$$\times \frac{\Gamma(k + a + 2 + \lambda)\Gamma(k + a + 3 - \lambda)}{\Gamma(n + a + 2 + \lambda)\Gamma(n + a + 3 - \lambda)} A(k)$$

$$- A(k) \frac{\Gamma(n + a - 1 - \lambda)\Gamma(n + a - 2 + \lambda)}{\Gamma(k + a + 1 - \lambda)\Gamma(k + a + \lambda)} B(k + 2)$$

$$\times \frac{\Gamma(k + a + 2 + \lambda)\Gamma(k + a + 1 - \lambda)}{\Gamma(n + a + \lambda)\Gamma(n + a + 1 - \lambda)}$$

$$- \tfrac{1}{2}z (k + a + 2 - \lambda)^{-1} (k + a + 1 - \lambda)^{-1}. \tag{5.7}$$

To interpret this equation, we need the following lemma.

LEMMA 5.8. *For any* $\mu, v \in \mathbb{C}$, *we have*

$$\lim_{n \to \infty} \frac{\Gamma(n + a + \mu)\Gamma(n + a - \mu)}{\Gamma(n + a + v)\Gamma(n + a - v)} = 1.$$

This follows easily from the asymptotic expansion of $\log \Gamma$ (see [18], §12.33). In view of the lemma, the terms in (5.7) are, respectively, $O(n^{-2})$, $O(n^{-4})$, $O(n^{-4})$ and $O(1)$ as $n \to \infty$. Letting $n \to \infty$ in (5.7) therefore gives:

PROPOSITION 5.9. *Let* V *be an irreducible* (**Vir**, d)-*module which is unitary for the conjugate-linear anti-involution* θ *(see* (0.2)). *If* V *is not highest or lowest weight, then the centre of* **Vir** *acts trivially on* V.

From Proposition 4.1 we deduce:

COROLLARY 5.10. *If V is as in Proposition 5.9, then* dim V_n *is independent of n for* $n \gg 0$ *and for* $n \ll 0$.

Note that, at this point, we do not know that the limiting values of dim V_n for $n \gg 0$ and for $n \ll 0$ are the same.

Returning to (5.7), we multiply both sides by n^2 and let $n \to \infty$. This gives

$$\Gamma(k + a + \lambda)\Gamma(k + a + 3 - \lambda)A(k)$$

$$+ B(k + 2)\Gamma(k + a + 2 + \lambda)\Gamma(k + a + 1 - \lambda) = 0. \quad (5.11)$$

This equation will be of crucial importance in the next two sections.

§6. The case where V contains a discrete series module

By Corollary 5.10, V breaks up, as a g_1-module, into a finite direct sum of discrete and continuous series modules. We shall first analyze the structure of V as a ***Vir***-module in the case where V contains a discrete series module for g_1. The case where V contains only continuous series modules is taken up in the next section.

THEOREM 6.1. *Let V be an irreducible* (***Vir***, ***d***)*-module which is unitary for the conjugate-linear anti-involution* θ *(see* (0.2))*, and assume that V is neither highest nor lowest weight. If V contains a discrete series representation of* g_1, *then V is isomorphic to* $W(\frac{1}{2}, \frac{1}{2})$ *as a representation of* ***Vir***.

The first step in proving Theorem 6.1 is the following.

PROPOSITION 6.2. *Suppose that V satisfies the assumptions of Theorem 6.1 and that V contains a non-zero vector v of positive* (*resp. negative*) L_0*-weight such that* $L_{-1}v = 0$ (*resp.* $L_1v = 0$)*. Then* dim $V_n \leqslant 1$ *for all* $n > 0$ (*resp. for all* $n < 0$)*.*

Proof. Suppose that $0 \neq v \in V$ is such that

$$L_0v = (a + k)v, \quad L_{-1}v = 0,$$

where $a + k > 0$ (the proof in other case is similar).

LEMMA 6.3. $L_m L_1^n v \in \mathbb{C} \cdot (L_1^{n+m}v)$ *whenever* $n \geqslant 0$, $n + m \geqslant 0$.

Assuming the lemma for the moment, let $Vir_{\pm}$ be the subalgebras of Vir spanned by the L_n such that $\pm n > 0$. Then, by the Poincaré–Birkhoff–Witt theorem,

$$V \;=\; U(Vir_-) \,.\, U(Vir_+) \,.\, v.$$

From the lemma,

$$U(Vir_+) \,.\, v \;=\; \sum_{n > 0} \mathbb{C} \,.\, (L_1^n \,.\, v).$$

It follows that V_k is spanned by the elements of the form

$$L_{-n_1} L_{-n_2} \ldots L_{-n_s} L_1^n v$$

where $n = n_1 + n_2 + \cdots + n_s$ and each $n_i > 0$. But, again by the lemma, this is a multiple of v. Hence, dim $V_k \leqslant 1$. The result now follows from Corollary 2.5.

Proof of Lemma 6.3. Suppose first that $n = 0$. We prove the result by induction on m. If $m = 0$ or 1 there is nothing to prove. Assuming the result is true up to $m \geqslant 1$, we compute

$$L_{-1} L_{m+1} v \;=\; (m + 2) L_m v$$

$$L_{-1} L_1^{m+1} v \;=\; (m + 1)(2a + 2k + m) L_1^m v.$$

By the induction hypothesis, $L_m v = c_m L_1^m v$ for some constant c_m. Then L_{-1} annihilates the vector

$$L_{m+1} v \;-\; \frac{(m+2) c_m}{(m+1)(2a + 2k + m)} \, L_1^{m+1} v.$$

By the maximality of k (5.1), this vector must be zero, which completes the induction step.

Suppose now that $m \geqslant 0$ and $n \geqslant 0$. The result in this case follows easily by induction on n: the case $n = 0$ has just been dealt with and the induction step uses the relation

$$L_m L_1^{n+1} v \;=\; L_1 L_m L_1^n v \;-\; (m - 1) L_{m+1} L_1^n v. \tag{6.4}$$

Suppose finally that $m = -l < 0$. The proof is by induction over l, with $0 < l \leqslant n$. For $l = 1$ the result is clear. For $l = 2$ it is enough to do the

case $n = 2$, by induction over n using (6.4) with $m = -2$. In the notation of Proposition 5.2, we can assume $v = v_{1,k}$. Then $L_1^2 v$ is a non-zero multiple of $v_{1,k-2}$ and, from above, $L_2 v$ is a multiple of $L_1^2 v$. Hence, $A_{1j}(k) = 0$ if $j \neq 1$ (see (5.3)). By (5.11), $B_{1j}(k + 2) = 0$ if $j \neq 1$, i.e. $L_{-2} v_{1,k+2} \in \mathbb{C} \cdot v_{1,k}$. This implies that $L_{-2} L_1^2 v$ is a multiple of v, as required.

The proof is now completed by induction on l, assuming $2 \leqslant l \leqslant n$. The induction step uses the relation

$$(l - 1)L_{-l-1}L_1^n v = L_{-l}L_{-1}L_1^n v - L_{-1}L_{-l}L_1^n v.$$

The proof of Theorem 6.1 is in four steps. Let V be as in the statement of the theorem.

Step 1: dim $V_n \leqslant 1$ for all $n \in \mathbb{Z}$

Since V contains a lowest weight discrete series module, it follows from Proposition 6.2 that dim $V_n \leqslant 1$ for all $n > 0$. If we prove that dim $V_0 \leqslant 1$, then it follows easily that dim $V_n \leqslant 1$ for all $n < 0$. Indeed, suppose that dim $V_l > 1$ for some $l < 0$ and that dim $V_n \leqslant 1$ for $l < n \leqslant 0$. Then V contains a non-zero vector annihilated by L_1; this contradicts Proposition 6.2.

We prove that dim $V_0 \leqslant 1$ by considering the cases $a = 0$ and $a > 0$ separately. If $a > 0$, then $L_1: V_0 \to V_1$ is injective by Corollary 2.5 and the result is clear. Now suppose that $a = 0$. Let K_0 be the kernel of the map $L_1: V_0 \to V_1$. Since V is not highest weight for **Vir**, it follows that $L_2: K_0 \to V_2$ is injective. If $K_0 = 0$ or V_0, we are through. Otherwise, choose $w \in V_0 \backslash K_0$, $0 \neq w' \in K_0$. Then $L_1^2 w$ and $L_2 w'$ are non-zero elements of V_2 and, since dim $V_2 \leqslant 1$, it follows that $L_1^2 w = \lambda L_2 w'$ for some $\lambda \in \mathbb{C}$. Now apply L_{-1} to both sides. By Proposition 2.1, $L_{-1} w' = 0$, so $L_{-1} L_2 w' = 0$. It follows that $L_{-1} L_1^2 w = 0$. But this is impossible since

$$\langle L_{-1} L_1^2 w, L_1 w \rangle = \| L_1^2 w \|^2 \neq 0.$$

Step 2: dim $V_n = 1$ for all $n \neq 0$

We prove first that $V_n = 0$ for at most one n. Suppose, on the contrary, that $V_q = V_r = 0$ for some $q < r$. Since V is not lowest weight as a representation of **Vir**, $V_p \neq 0$ for some $p < q$. If $0 \neq w \in V_p$ then $L_{q-p} w = L_{r-p} w = 0$, so w is a highest weight vector for the Virasoro subalgebra of **Vir** generated by $L_{(q+r-2p)n}$ for $n \in \mathbb{Z}$. Since the centre acts trivially this contradicts the Friedan–Qiu–Shenker result (0.3).

It is now clear that $V_n \neq 0$ if $|n| > 1$. For example, if $V_n = 0$ for some $n > 1$, then $V_{n-1} = 0$ by Corollary 2.5, so at least two weight spaces of V must vanish.

It remains to show that V_1 and V_{-1} are non-zero. Suppose, for example, that $V_1 = 0$. Then $L_1 V_0 = 0$, so by Proposition 2.1, $L_{-1} V_0 = 0$. Since V is not lowest weight, $L_{-2} V_0 = V_{-2}$. This implies that, with obvious notation, the subspace $V_{\leq -2}$ of V is g_1-stable. Therefore, the subspace $V_{\leq -2} \oplus V_0 \oplus V_{\geq 2}$ is g_1-stable, hence so is its orthogonal complement V_{-1}. This is clearly impossible.

Step 3: $V_0 \neq 0$

Suppose, for a contradiction, that $V_0 = 0$, dim $V_n = 1$ for all $n \neq 0$. Then, as a g_1-module,

$$V \simeq D^-(1 - a) \oplus D^+(1 + a).$$

We shall prove first that $a = 0$.

Now V has a basis $\{v_n\}_{n \neq 0}$ such that

$$\left. \begin{aligned} L_1 v_n &= (n + 1 + 2a)v_{n+1} \\ L_{-1} v_n &= (n - 1)v_{n-1} \\ L_0 v_n &= (n + a)v_n \end{aligned} \right\} \quad \text{for all } n \geq 1.$$

We claim that

$$L_2 v_n = (n + 2 + 3a)v_{n+2} \quad \text{for all } n \geq 1.$$

Indeed, since $L_{-1} v_1 = 0$ we find that

$$L_{-1}(L_2 v_1 - (3 + 3a)v_3) = 0;$$

as the element in brackets has weight $> a + 1$, it cannot be annihilated by L_{-1} unless it is already zero. The claim is now established by induction on n, starting with $n = 0$. The induction step follows by computing that

$$L_{-1}(L_2 v_n - (n + 2 + 3a)v_{n+2}) = 0,$$

using the induction hypothesis.

The commutation relation

$$[L_{-2}, L_2]v_2 = 4(a + 2)v_2$$

now leads to

$$L_{-2}v_4 = \frac{4(a + 2)}{(4 + 3a)} v_2;$$

inserting this into the unitarity relation

$$\langle L_{-2}v_4, v_2 \rangle = \langle v_4, L_2 v_2 \rangle$$

gives

$$\frac{\| v_4 \|^2}{\| v_2 \|^2} = \frac{4(a + 2)}{(4 + 3a)^2}. \tag{6.5}$$

But from $\langle L_1 v_n, v_{n+1} \rangle = \langle v_n, L_{-1}v_{n+1} \rangle$ we find that

$$\frac{\| v_{n+1} \|^2}{\| v_n \|^2} = \frac{n}{n + 1 + 2a}, \quad n \geqslant 1,$$

and combined with (6.5) this gives

$$\frac{4(a + 2)}{(4 + 3a)^2} = \frac{3}{(a + 2)(3 + 2a)}. \tag{6.6}$$

It is easy to check that $a = 0$ is the only real root of equation (6.6). Thus, as a $\boldsymbol{g}_1$-module,

$$V \simeq D^-(1) \oplus D^+(1)$$

and we have

$$L_{\pm 1}v_n = (n \pm 1)v_{n \pm 1}, \quad L_0 v_n = nv_n, \quad \text{for all } n \neq 0.$$

By suitably normalizing the v_n for $n < 0$, we can arrange that

$$L_{-2}v_1 = -v_{-1};$$

note that $L_{-2}v_1 \neq 0$ as V is not highest weight for **Vir**. By computations similar to those above, we find

$$L_{-2}v_{-1} = -3v_{-3}, \quad L_2v_{-3} = -v_{-1};$$

using

$$[L_{-2}, L_2]v_{-1} = -4v_{-1}$$

gives

$$L_2v_{-1} = v_1$$

and inserting this into

$$\langle L_2v_{-1}, v_1 \rangle = \langle v_{-1}, L_{-2}v_1 \rangle$$

leads to

$$\| v_1 \|^2 = - \| v_{-1} \|^2,$$

which is impossible. (In fact, by repeatedly applying $L_{\pm 1}$ to the above equations, and using unitarity, one can prove that $L_k v_n = (n + k)v_{n+k}$ for all $n, k, n \neq 0$.)

Step 4: If dim $V_n = 1$ for all $n \in \mathbb{Z}$, then $V \simeq W(\frac{1}{2}, \frac{1}{2})$

The method is similar to that of the previous step, so we shall be brief. As a g_1-module,

$$V \simeq D^-(1) \oplus \mathbb{C} \oplus D^+(1) \quad \text{or} \quad V \simeq D^-(1 - a) \oplus D^+(a)$$

($\mathbb{C}$ here denotes the one dimensional trivial module). The first case is ruled out by repeating the argument in the last paragraph of Step 3. In the second case, V has a basis $\{v_n\}_{n \in \mathbb{Z}}$ such that

$$\left.\begin{aligned} L_1 v_n &= (n + 2a)v_{n+1} \\[1mm] L_{-1} v_n &= nv_{n-1} \\[1mm] L_0 v_n &= (n + a)v_n \end{aligned}\right\} \quad \text{for } n \geqslant 0.$$

By repeating the steps which led to (6.6), but computing

$$[L_{-2}, L_2]v_2 = 4(a + 2)v_2,$$

we find that

$$\frac{6(2 + 3a)^2}{(1 + a)(3 + 2a)} - \frac{9a}{1 + 2a} = 4(a + 2).$$

The roots of this equation are $a = -\frac{7}{8}, 0, \frac{1}{2}, 1$. The values $a = -\frac{7}{8}, 1$ lie outside the allowed range, and if $a = 0$ then v_0 is annihilated by L_0, L_1 and L_2 and so, by Proposition 2.1, also by L_{-1} and L_{-2}. This contradicts the irreducibility of V. Thus, $a = \frac{1}{2}$.

We now have

$$\left.\begin{array}{l} L_1 v_n = (n + 1)v_{n+1} \\[2mm] L_{-1}v_n = nv_{n-1} \\[2mm] L_0 v_n = (n + \frac{1}{2})v_n \end{array}\right\} \quad \text{for all } n \geqslant 0.$$

We can arrange that

$$L_{-2}v_0 = -\tfrac{3}{2}v_{-2}.$$

By repeatedly applying $L_{\pm 1}$ to the above equations, and using unitarity, it is not difficult to prove that

$$L_k v_n = (n + \tfrac{1}{2}(k + 1))v_{n+k}$$

for all $n, k \in \mathbb{Z}$. Comparing with (0.4), we see that $V \simeq W(\frac{1}{2}, \frac{1}{2})$.

This completes the proof of Theorem 6.1.

§7. The case where V does not contain a discrete series module

In this section we shall prove the following result which, together with Theorem 6.1, completes the proof of Theorem 0.5.

THEOREM 7.1. *Let V be a non-trivial, irreducible $(\textbf{Vir}, \textbf{d})$-module which is unitary for the conjugate-linear anti-involution θ (see (0.2)). If V does not contain a discrete series representation of $\textbf{g}_1$, then V is isomorphic to $W(\lambda, a)$ as a representation of $\textbf{Vir}$, for some $\lambda \in \frac{1}{2} + i\mathbb{R}$, $a \in \mathbb{R}$, $(\lambda, a) \neq (\frac{1}{2}, \frac{1}{2})$.*

REMARK 7.2. We do not have to assume that V is not highest or lowest weight for $\textbf{Vir}$, since the assumptions of Theorem 7.1 imply that dim V_n is independent of n for all n, which is never true for a highest or lowest weight representation.

Proof of Theorem 7.1. As a $\textbf{g}_1$-module, V decomposes into an orthogonal sum of continuous series representations:

$$V \simeq \bigoplus_{j=1}^{r} C(\lambda_j, a).$$

By Proposition 2.3(b), we can assume that the λ_j satisfy

$$a < \lambda_j < 1 - a \quad \text{or} \quad 1 - a < \lambda_j < a \quad \text{or} \quad \lambda_j = \tfrac{1}{2} + it_j \quad \text{with} \quad t_j \geqslant 0. \tag{7.3}$$

The considerations of §5 now apply for all $n \in \mathbb{Z}$, not just for $n \gg 0$ and for $n \ll 0$. Thus, we can take $k = 0$ in (5.11) and, using (5.5) to relate $B(2)$ to $B(0)$, we find that

$$\Gamma(a - 2 + \lambda)\Gamma(a + 3 - \lambda)A(0)$$
$$+ B(0)\Gamma(a + 2 + \lambda)\Gamma(a + 1 - \lambda) = 0. \tag{7.4}$$

We recall that $\lambda = \text{diag}(\lambda_1, \dots, \lambda_r)$ and that the gamma functions are shorthand for the obvious diagonal matrices. Equation (5.11) follows from (5.7) by letting $n \to +\infty$; we can also let $n \to -\infty$; this leads to

$$A(0)\Gamma(-a + \lambda)\Gamma(-a + 1 - \lambda)$$
$$+ \Gamma(-a + 3 - \lambda)\Gamma(-a - 2 + \lambda)B(0) = 0. \tag{7.5}$$

By considering the (i, j)-entries of equations (7.4) and (7.5), we see that $A_{ij}(0) = B_{ij}(0) = 0$ unless

$$\Gamma(a - 2 + \lambda_i)\Gamma(a + 3 - \lambda_i)\Gamma(-a + 3 - \lambda_i)\Gamma(-a - 2 + \lambda_i)$$
$$= \Gamma(a + 2 + \lambda_j)\Gamma(a + 1 - \lambda_j)\Gamma(-a + \lambda_j)\Gamma(-a + 1 - \lambda_j).$$

Simplifying this equation using the relation

$$\Gamma(\xi)\Gamma(1 - \xi) = \frac{\pi}{\sin \pi\xi} \quad \text{for } \xi \notin \mathbb{Z}$$

(see [18], §12.14), we find that

$$\sin \pi(\lambda_i - a) \sin \pi(\lambda_i + a) = \sin \pi(\lambda_j - a) \sin \pi(\lambda_j + a),$$

i.e.,

$$\cos 2\pi\lambda_i = \cos 2\pi\lambda_j.$$

This implies that $\lambda_i \equiv \lambda_j$ or $1 - \lambda_j \pmod{\mathbb{Z}}$. For λ_j in the range (7.3), it is easy to see that this is possible only if $\lambda_i = \lambda_j$. Summarizing, we have shown that $A_{ij}(0) = B_{ij}(0) = 0$ unless $\lambda_i = \lambda_j$. As V is irreducible, equations (5.3) show that we must have $\lambda_i = \lambda_j = \lambda$ (say) for all i, j.

We assert that, in this case, dim $V_n = 1$ for all $n \in \mathbb{Z}$, and that

$$A(0) = 0 \quad \text{or} \quad \frac{2\lambda(1 - \lambda)(2\lambda - 1)}{(a + 2 - \lambda)(a + 1 - \lambda)}. \tag{7.6}$$

If $A(0) = 0$, the relations (5.3) are exactly the relations (0.4) defining $W(\lambda, a)$; of course, the action of $L_{\pm 1}$ and $L_{\pm 2}$ determines that of the whole of **Vir**. If

$$A(0) = \frac{2\lambda(1 - \lambda)(2\lambda - 1)}{(a + 2 - \lambda)(a + 1 - \lambda)},$$

then V is isomorphic to $W(1 - \lambda, a)$. To see this, define non-zero constants c_n, for $n \in \mathbb{Z}$, such that

$$\frac{c_{n+1}}{c_n} = \frac{n + a + \lambda}{n + a + 1 - \lambda}$$

and set $w_n = c_n v_n$. Then it is easy to check that the w_n satisfy the relations (0.4) defining $W(1 - \lambda, a)$.

To prove our assertion, we return to (5.7); taking $k = 0$, using (5.10) and simplifying, we find that

$$
\left\{ \frac{n + a - 2\lambda}{(n + a - \lambda)(n + a - 1 - \lambda)} - \frac{n + a - 2 + 2\lambda}{(n + a - 1 + \lambda)(n + a - 2 + \lambda)} \right\} A(0)
$$

$$
- \left\{ \frac{n + a + 2 - 2\lambda}{(n + a + 2 - \lambda)(n + a + 1 - \lambda)} - \frac{n + a + 2\lambda}{(n + a + 1 + \lambda)(n + a + \lambda)} \right\} A(0)
$$

$$
= \left\{ \frac{1}{(n + a - \lambda)(n + a - 1 - \lambda)(n + a - 1 + \lambda)(n + a - 2 + \lambda)} \right.
$$

$$
\left. - \frac{1}{(n + a + 1 + \lambda)(n + a + \lambda)(n + a + 2 - \lambda)(n + a + 1 - \lambda)} \right\}
$$

$$
\times (a + 1 - \lambda)(a + 2 - \lambda) A(0)^2. \tag{7.7}
$$

The two brackets on the left-hand side are, respectively, equal to $2\lambda(1 - \lambda)(2\lambda - 1)$ times the two terms in brackets on the right-hand side. It follows that (7.7) holds for all $n \in \mathbb{Z}$ if and only if

$$
A(0)^2 = \frac{2\lambda(1 - \lambda)(2\lambda - 1)}{(a + 1 - \lambda)(a + 2 - \lambda)} A(0). \tag{7.8}
$$

If $\lambda \neq \frac{1}{2}$, this forces $A(0)$ to be diagonalizable and hence V to be reducible unless $\dim V_n = 1$ for all n. In that case, (7.6) holds as required.

To complete the proof, it remains to consider the case $\lambda = \frac{1}{2}$. Consider the unitarity equation

$$
\langle L_2 \cdot v_{i,0}, v_{j,2} \rangle = \langle v_{i,0}, L_{-2} \cdot v_{j,2} \rangle.
$$

This gives

$$
(a + 1 + A_{ij}(0)) \| v_{j,2} \|^2 = (a + 1 + \overline{B_{ji}(2)}) \| v_{i,0} \|^2. \tag{7.9}
$$

But, for $\lambda = \frac{1}{2}$, it follows easily from Proposition 2.3 that $\| v_{i,n} \|^2$ is independent of n, so we may assume that $\| v_{i,n} \|^2 = 1$ for all i, n. Then (7.9) and (5.11) give

$$
A(0)^* = -A(0).
$$

Combining this with the equation

$$A(0)^2 = 0$$

from (7.8), we find $A(0) = 0$ as desired.
   This completes the proof of Theorem 7.1.

§8. Unitary representations of the Virasoro superalgebras

A Lie superalgebra g (see [6]) is a $\mathbb{Z}_2$-graded vector space

$$g = g_{(0)} \oplus g_{(1)}$$

such that $g_{(0)}$ is a Lie algebra and $g_{(1)}$ a $g_{(0)}$-module, together with a symmetric homomorphism of $g_{(0)}$-modules

$$g_{(1)} \otimes g_{(1)} \rightarrow g_{(0)}.$$

These requirements define a bilinear pairing $[x, y]$ between any two graded elements $x, y \in g$ which satisfies

$$[x, y] = -(-1)^{\deg x \, \deg y} [y, x].$$

This pairing is extended to the whole of g by linearity. It is required to satisfy the additional condition

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for $x, y, z \in g_{(1)}$.
   For the Virasoro superalgebras Vir^κ one takes

$$g_{(0)} = Vir, \quad g_{(1)} = W(-\tfrac{1}{2}, \kappa)$$

where $\kappa = 0$ or $\tfrac{1}{2}$ for the Ramond or Neveu-Schwarz superalgebra respectively. The map $g_{(1)} \otimes g_{(1)} \rightarrow g_{(0)}$ is, apart from the central term, the natural pairing of $(-\tfrac{1}{2})$-densities to give a vector field; explicitly,

$$w_n \otimes w_m \rightarrow 2L_{n+m+2\kappa} + \tfrac{1}{3}\{\tfrac{1}{4} - (m + \kappa)^2\}\delta_{m+\kappa, -n-\kappa}. \tag{8.1}$$

The definition of a unitary representation V of a Lie superalgebra g, with respect to an anti-linear anti-involution θ of g, follows that in §1 with appropriate modifications. We note only that θ induces an anti-linear anti-involution of $g_{(0)}$ with respect to which V is unitary as a $g_{(0)}$-module.

PROPOSITION 8.2. *Let V be a non-trivial unitary representation of Vir^κ. Then V is unitary with respect to the anti-linear anti-involution θ of Vir^κ given by*

$$\theta(L_n) \;=\; L_{-n}, \quad \theta(c) \;=\; c, \quad \theta(w_n) \;=\; -w_{-n-2\kappa}.$$

Proof. From the above remarks and Proposition 3.4, we can assume that $\theta(L_n) = L_{-n}, \theta(c) = c$. It is easy to see that this forces the action of θ on the odd part of Vir^κ to be as stated.

Now let V be an irreducible representation of Vir^κ which is unitary with respect to the anti-linear anti-involution defined in Proposition 8.2. Suppose V has finite-dimensional weight spaces (as a Vir-module). By Schur's lemma ([6] §1.1.6), the centre c of Vir^κ acts by a (real) scalar z on V. If $z > 0$ then, by Theorem 0.5, V is a direct sum of a finite number of unitary lowest weight Vir-modules. By selecting a vector of minimal L_0-eigenvalue, we see that V is a lowest weight Vir^κ-module (i.e. it is generated by a vector annihilated by L_n and w_n for $n < 0$). Similarly, if $z < 0$ then V is highest weight. Suppose now that $z = 0$.

LEMMA 8.3. *There is no non-zero Vir-module map*

$$W(-\tfrac{1}{2}, \kappa) \otimes W(\lambda, a) \;\to\; W(\mu, b)$$

if $\mathrm{Re}(\lambda) = \mathrm{Re}(\mu) = \tfrac{1}{2}, a, b \in \mathbb{R}$.

Assume this for a moment. By Theorem 0.5 the even and odd parts of V are direct sums of Vir-modules of the form $W(\lambda, a)$, with $\mathrm{Re}(\lambda) = \tfrac{1}{2}$, $a \in \mathbb{R}$. By the lemma, the odd part $Vir^\kappa_{(1)}$ acts trivially on V. Since $Vir^\kappa_{(0)} = [Vir^\kappa_{(1)}, Vir^\kappa_{(1)}]$, $Vir^\kappa_{(0)}$ acts trivially too.

This completes the proof of Theorem 0.8, except for the omitted

Proof of Lemma 8.3. Consider first the Ramond case $\kappa = 0$. Suppose there is a non-zero Vir-module map

$$W(-\tfrac{1}{2}, 0) \otimes W(\lambda, a) \;\to\; W(\mu, b). \tag{8.4}$$

By considering the action of L_0, one sees that $b = a$. The crucial observation is that, as a g_2-module (see (3.3)), $W(-\tfrac{1}{2}, 0)$ contains a copy of the natural representation $\mathbb{C}^2$ of $sl(2, \mathbb{C})$, spanned by $w_{\pm 1}$. Further, as a

g_2-module,

$$W(\lambda, a) \simeq C(\lambda, \tfrac{1}{2}a) \oplus C(\lambda, \tfrac{1}{2}a + \tfrac{1}{2}).$$

Restricting the map (8.4) to g_2, we obtain *sl(2, C)*-module maps

$$\mathbb{C}^2 \otimes C(\lambda, \tfrac{1}{2}a) \to C(\mu, \tfrac{1}{2}a + \tfrac{1}{2})$$

$$\mathbb{C}^2 \otimes C(\lambda, \tfrac{1}{2}a + \tfrac{1}{2}) \to C(\mu, \tfrac{1}{2}a).$$

(8.5)

On the other hand, there are non-trivial *Vir*-module maps

$$W(-\tfrac{1}{2}, 0) \otimes W(\lambda, a) \quad \begin{array}{c} \nearrow \quad W(\lambda - \tfrac{1}{2}, a) \\ \searrow \quad W(\lambda + \tfrac{1}{2}, a) \end{array}$$

for any $\lambda, a \in \mathbb{C}$; in fact, the maps are given by

$$w_n \otimes w_m \quad \begin{array}{c} \nearrow \quad w_{n+m} \\ \searrow \quad (\lambda n + \tfrac{1}{2}m) w_{n+m} \end{array}.$$

Restricting these maps to g_2, we obtain non-zero *sl(2, C)*-module maps

$$\mathbb{C}^2 \otimes C(\lambda, \tfrac{1}{2}a) \quad \begin{array}{c} \nearrow \quad C(\lambda - \tfrac{1}{2}, \tfrac{1}{2}a + \tfrac{1}{2}) \\ \searrow \quad C(\lambda + \tfrac{1}{2}, \tfrac{1}{2}a + \tfrac{1}{2}) \end{array}$$

$$\mathbb{C}^2 \otimes C(\lambda, \tfrac{1}{2}a + \tfrac{1}{2}) \quad \begin{array}{c} \nearrow \quad C(\lambda - \tfrac{1}{2}, \tfrac{1}{2}a) \\ \searrow \quad C(\lambda + \tfrac{1}{2}, \tfrac{1}{2}a) \end{array}.$$

(8.6)

When $\mathrm{Re}(\lambda) = \tfrac{1}{2}$, the modules on the right-hand side of (8.6) are all irreducible. But then the modules on the left-hand side can have no further irreducible quotients, since all weight spaces are two-dimensional. Since $\mathrm{Re}(\mu) = \tfrac{1}{2}$, this means that the maps in (8.5) must be zero. It follows easily that the map (8.4) must also be zero. (We note that $C(\tfrac{1}{2}, \tfrac{1}{2}) = D^+(\tfrac{1}{2}) \oplus D^-(\tfrac{1}{2})$ is reducible, but this does not affect the argument.)

The Neveu-Schwarz case $\kappa = \tfrac{1}{2}$ is similar but easier. One observes that $W(-\tfrac{1}{2}, \tfrac{1}{2})$ contains a copy of the natural representation of g_1.

REMARK 8.7. The conditions for a lowest weight representation of *Vir*$^\kappa$ to be unitary can be found in [2].

Acknowledgement

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