

COMPOSITIO MATHEMATICA

GEORGE R. KEMPF

FRANK-OLAF SCHREYER

**A Torelli theorem for osculating cones to
the theta divisor**

Compositio Mathematica, tome 67, n° 3 (1988), p. 343-353

http://www.numdam.org/item?id=CM_1988__67_3_343_0

© Foundation Compositio Mathematica, 1988, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

A Torelli theorem for osculating cones to the theta divisor

GEORGE R. KEMPF¹ & FRANK-OLAF SCHREYER^{2,*}

¹*Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218, USA;*

²*Fachbereich Mathematik der Universität, Erwin-Schrödinger-Str., D-6750 Kaiserslautern, Fed. Rep. of Germany*

Received 11 November 1987; accepted in revised form 13 February 1988

Let C be a smooth complete curve of genus $g \geq 5$ over an algebraically closed field k of characteristic $\neq 2$. Let Θ be the theta divisor on the Jacobian J . Let x be a double point of Θ . Then we may expand a local equation $\theta = 0$ of Θ near x as

$$\theta = \theta_2 + \theta_3 + \text{higher order terms}$$

where θ_i is homogenous of degree i in the canonical flat structure [K3] on J . It is well-known that the tangent cone $R = \{\theta_2 = 0\}$ is a quadric of rank ≤ 4 in the canonical space \mathbb{P}^{g-1} which contains the canonical curve.

If C is not hyperelliptic, trigonal or a plane quintic then for a general double point x the quadric θ_2 has rank 4 and the two rulings of R cut out on C a pair of residual base-point-free distinct g_{g-1} 's, $|D|$ and $|K - D|$, (apply Mumford's refinement of Martens' theorem [M]. In the bi-elliptic case x should be a general point of the component of the singular locus of Θ which does not arise from g_4 's by adding base points or its residual component). Moreover the morphism $\varphi_{|D|} \times \varphi_{|K-D|}: C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ maps C birationally onto its image C' . Taking the composition with the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ we may regard $\varphi_{|D|} \times \varphi_{|K-D|}$ as the projection of C from the $(g - 5)$ -dimensional vertex $V \subseteq \mathbb{P}^{g-1}$ of R .

In this paper we study the geometry of the osculating cone $S = \{\theta_2 = \theta_3 = 0\}$. It is well-known and easy to prove that $C \subseteq S$. Thus if \sim denote the strict transform after blowing up V we have a diagram

$$\begin{array}{ccccccc} C & \hookrightarrow & \tilde{S} & \hookrightarrow & \tilde{R} & \hookrightarrow & \tilde{\mathbb{P}}^{g-1} \rightarrow \mathbb{P}^{g-1} \\ \downarrow & & \alpha \downarrow & & \downarrow \pi & & \downarrow \\ C' \subseteq \mathbb{P}^1 \times \mathbb{P}^1 & = & \mathbb{P}^1 \times \mathbb{P}^1 & \subseteq & \mathbb{P}^3 & & \end{array}$$

* Supported by the DFG.

where the vertical maps are induced by projecting from V . An important but simple observation is that S contains V and hence that α is a quadric bundle contained in the \mathbb{P}^{g-4} bundle π . We will prove

THEOREM 1. *Suppose that $x \in \Theta \subseteq J$ corresponds to a pair of base-point free residual g_{g-1} 's, $|D|$ and $|K - D|$, such that $\varphi_D \times \varphi_{K-D}: C \rightarrow C' \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image. Then the fibers of α over $\mathbb{P}^1 \times \mathbb{P}^1 - C'$ are smooth and for a smooth point c' of C' the corresponding point c of C is the only singular point of the fiber of α over c .*

Thus we have a straightforward way to recover the canonical curve from $S \subseteq R \subseteq \mathbb{P}^{g-1}$. Explicitly C is the component of the singular locus of the fibers which projects non-trivially into $\mathbb{P}^1 \times \mathbb{P}^1$.

If $\text{char}(k) \neq 2, 3$ then $\{\theta_2 = \theta_3 = \theta_4 = 0\}$ is defined. We note

PROPOSITION 2. *C is not contained in $\{\theta_2 = \theta_3 = \theta_4 = 0\}$.*

This work was done during a visit of UNAM in Mexico. We thank Sevin Recillas for his hospitality and for bringing this question to our attention.

§1. Infinitesimal calculations on the Jacobian via cohomological obstructions

Intrinsically x corresponds to a point of the $g - 1^{\text{st}}$ Picard variety $\text{Pic}_{g-1}(C) \cong J$. The point is the isomorphism class of the invertible sheaf $\mathcal{L} = \mathcal{O}_C(D)$. In our case $\Gamma(C, \mathcal{L})$ is two-dimensional. By the general procedure of [K2] we may locally around x find a 2×2 matrix (f_{ij}) of regular functions vanishing at x such that

$$\theta = \det(f_{ij}).$$

The equation of the tangent cone R is

$$\theta_2 = \det(df_{ij}|_x) = 0.$$

The equations of the vertex V are $df_{ij}|_x = 0$ for $1 \leq i, j \leq 2$.

Using the flat structure on J we may expand

$$f_{ij} = x_{ij} + q_{ij} + \text{higher order terms}$$

where $x_{ij} = df_{ij}|_x$ and q_{ij} are thought of as linear and quadratic functions on the tangent space J at x . Expanding θ as a determinant we have

$$\begin{aligned} \theta_2 &= x_{11}x_{22} - x_{12}x_{21} \text{ and} \\ \theta_3 &= x_{11}q_{22} + q_{11}x_{22} - x_{12}q_{21} - q_{12}x_{21}. \end{aligned}$$

Thus we get:

PROPOSITION 3. *The vertex V is contained in $S = \{\theta_2 = \theta_3 = 0\}$.*

Proof. $V = \{x_{11} = x_{22} = x_{12} = x_{21} = 0\}$. So θ_3 vanishes on V . □

To get deeper results we will have to use the obstruction theory from [K3]. First we will give a cohomological interpretation of the previous material.

The tangent space to J is canonically isomorphic to $H^1(C, \mathcal{O}_C)$. The matrix (x_{ij}) describes the cup product action

$$\cup: H^1(C, \mathcal{O}_C) \rightarrow \text{Hom}(\Gamma(C, \mathcal{L}), H^1(C, \mathcal{L})), \alpha \mapsto \cup \alpha$$

where $H^1(C, \mathcal{L}) \cong \Gamma(C, \Omega_C \otimes \mathcal{L}^{-1})^*$ is also two dimensional. The cone over V is the kernel of \cup and the cone over R corresponds to cohomology classes $\alpha \in H^1(C, \mathcal{O}_C)$ such that $\cup \alpha: \Gamma(C, \mathcal{L}) \rightarrow H^1(C, \mathcal{L})$ has rank ≤ 1 .

The punctured line over a point c in the canonical curve C consists of the cohomology classes $[\pi_c]$ of the principal part π_c of a rational function with a simple pole at c and otherwise zero. For practice (because these ideas are needed later in a more complicated situation) let us see cohomologically why C is contained in R and let us compute $C \cap V$. Let $\eta_c \in \Gamma(C, \mathcal{L})$ be a section which vanishes at c . Then the principal part $\eta_c \pi_c$ is zero, so $\eta_c \cup [\pi_c] = 0$ and $\cup[\pi_c]: \Gamma(C, \mathcal{L}) \rightarrow H^1(C, \mathcal{L})$ has rank ≤ 1 . Thus $C \subseteq R$. Also if c is a base-point of $\Gamma(C, \mathcal{L})$ then $[\pi_c] \in \ker(\cup)$ and so $\{\text{base-points of } \mathcal{L}\} \subseteq C \cap V$. If c is not a base-point then there is a section $\gamma_c \in \Gamma(C, \mathcal{L})$ which does not vanish at c . The principal part $\gamma_c \pi_c$ of \mathcal{L} has a pole of order 1 at c and is otherwise zero. Its cohomology class is zero if and only if \mathcal{L} has a rational section σ with a single simple pole at c . By duality σ exists iff c is a base-point of $\Gamma(C, \Omega_C \otimes \mathcal{L}^{-1})$. Thus

$$C \cap V = \{\text{base-points of } \mathcal{L}\} \cup \{\text{base-points of } \Omega_C \otimes \mathcal{L}^{-1}\}.$$

The idea behind the above calculations is the relationship between the matrix (f_{ij}) and the vanishing of cup product. The matrix (f_{ij}) controls

the cohomology of all local deformations of \mathcal{L} . The above calculation involves a deformation over the infinitesimal scheme $D_1 = \text{Spec}(k[\varepsilon]/(\varepsilon^2))$. Such a deformation is given by

$$0 \rightarrow \varepsilon\mathcal{L} \rightarrow \mathcal{L}_2 \rightarrow \mathcal{L} \rightarrow 0$$

where \mathcal{L}_2 is an invertible sheaf on $C \times D_1$. As it is well-known the isomorphism classes of such extensions correspond to cohomology classes $\alpha \in H^1(C, \mathcal{O}_C)$ and a section η of \mathcal{L} lifts to a section of \mathcal{L}_2 if and only if $\eta \cup \alpha$ is zero. We will consider similar lifting problems to higher order deformations of \mathcal{L} .

Let $D_i = \text{Spec}(A_i)$ with $A_i = k[\varepsilon]/(\varepsilon^{i+1})$. We want to describe the deformation of \mathcal{L} corresponding to a flat curve $D_i \hookrightarrow J$ with support $(D) = x$ [K3]. Such a deformation of \mathcal{L} is determined by its velocity which is a tangent vector of J ; i.e., an element in $H^1(C, \mathcal{O}_C)$. Let $\beta = (\beta_c)$ be a $k(C)$ -valued function on C such that β_c is regular at c except for finitely many c 's. Then β determines a cohomology class $[\beta]$ in $H^1(C, \mathcal{O}_C)$. We want to write the deformation of \mathcal{L} in terms of β . Let $\mathcal{L}_{i+1}(\beta)$ be the sheaf whose stalk at $c \in C \times D_i (= C \text{ as sets})$ is given by rational sections $f = f_0 + f_1\varepsilon + \dots + f_i\varepsilon^i$ of $\mathcal{L} \otimes A_i$ such that $f \exp(\varepsilon\beta_c)$ is regular at c . If $\text{char}(k) \nmid i!$ this expression makes sense as

$$\exp(\varepsilon\beta_c) = \sum_{0 \leq j \leq i} \beta_c^j \varepsilon^j / j!$$

Thus if $i = 2$ for $f = f_0 + f_1\varepsilon + f_2\varepsilon^2$ to be a global section of $\mathcal{L}_3(\beta)$ we need (0) f_0 is a regular section of \mathcal{L} , (1) $f_1 + f_0\beta_c$ is regular at c for every $c \in C$ and (2) $f_2 + f_1\beta_c + f_0\beta_c^2/2$ is regular at c for every $c \in C$.

Our main tool to study the osculating cone $S = \{\theta_2 = \theta_3 = 0\}$ is the following:

LEMMA 4. *A cohomology class $[\beta] \in H^1(C, \mathcal{O}_C)$ is contained in the cone cover $\{\theta_2 = \theta_3 = \dots = \theta_{i+1} = 0\} \subseteq \mathbb{P}^{g-1}$ if and only if*

$$\text{length}_{A_i}(\Gamma(C \times D_i, \mathcal{L}_{i+1}(\beta))) \geq i + 2.$$

In particular if $k[\beta] \in \mathbb{P}^{g-1}$ is a point which does not lie in V then $k[\beta] \in \{\theta_2 = \theta_3 = \dots = \theta_{i+1} = 0\}$ if and only if there exists a section f of $\mathcal{L}_{i+1}(\beta)$ such that $f_0 \neq 0$.

Proof. The cohomology of $\mathcal{L}_{i+1}(\beta)$ is controlled by the pullback φ of the matrix (f_{ij}) via $D_i \rightarrow J \cong \text{Pic}_{g-1}(C)$, $\varepsilon \mapsto \exp([\beta]\varepsilon)$. Since D_i is just a fat

point we have an exact sequence

$$0 \rightarrow \Gamma(C \times D_i, \mathcal{L}_{i+1}(\beta)) \rightarrow A_i^{\oplus 2} \xrightarrow{\varphi} A_i^{\oplus 2} \rightarrow H^1(C \times D_i, \mathcal{L}_{i+1}(\beta)) \rightarrow 0$$

of A_i -modules. The matrix φ is equivalent to a matrix

$$\begin{pmatrix} \varepsilon^a & 0 \\ 0 & \varepsilon^b \end{pmatrix}$$

with $1 \leq a \leq b \leq i + 1$. So $\text{length } \Gamma(C \times D_i, \mathcal{L}_{i+1}(\beta)) = a + b$. On the other hand $D_i \hookrightarrow J$ is a flat curve for a non-trivial class $[\beta]$. Hence $k[\beta]$ is contained in $\{\theta_2 = \theta_3 = \dots = \theta_{i+1} = 0\}$ if and only if $a + b \geq i + 2$. If $k[\beta] \notin V$ then $a = 1$ and hence $k[\beta] \in \{\theta_2 = \theta_3 = \dots = \theta_{i+1} = 0\}$ if and only if $b = i + 1$. But $\varepsilon^{i+1} = 0$ and so there exists a section f of $\mathcal{L}_{i+1}(\beta)$ with $f_0 \neq 0$. □

Thus $k[\beta]$ is contained in R if and only if there is a section $f_0 + f_1\varepsilon$ of $\mathcal{L}_2(\beta)$ with $f_0 \neq 0$. Moreover if $k[\beta] \in R - V$ then f_0 is uniquely determined up to a scalar factor. Also $k[\beta] \in S - V$ if and only if $f_0 + (f_1 + \eta)\varepsilon$ lifts to $\mathcal{L}_3(\beta)$ for a suitable choice of $\eta \in \Gamma(C, \mathcal{L})$. One works out that $[f_0\beta]$ is zero in $H^1(C, \mathcal{L})$ if and only if f_0 lifts to the first order. A second order lifting is possible if and only if $[f_1\beta + f_0\beta^2/2]$ is zero in $H^1(C, \mathcal{L})/\Gamma(C, \mathcal{L}) \cup [\beta]$. (The last division is required because we have to consider η).

Similarly one can compute tangent vectors to a point $k[\beta] \in S$ by computing the sections of a deformation obtained via $\exp(([\beta] + t[\gamma])\varepsilon)$ over $k[\varepsilon, t]/(\varepsilon^3, t^2)$.

Using this machinery we will prove:

PROPOSITION 5. (a) *The canonical curve C is contained in S .* (b) *If the rational maps φ_L and $\varphi_{\Omega_C \otimes \mathcal{L}^{-1}} : C \rightarrow \mathbb{P}^1$ are distinct, then S is smooth of dimension $g - 3$ at a general point of C .*

Proof. For (a) if c is not a base-point of \mathcal{L} then with the previous notation, π_c the principal part of a rational function with a simple pole at c and η_c a section of \mathcal{L} which vanishes at c , we have that $\eta_c + 0\varepsilon$ is a section of $\mathcal{L}_2(\pi_c)$ and $[\eta_c\pi_c^2/2] \in \Gamma(C, \mathcal{L}) \cup [\pi_c]$. Thus the second obstruction vanishes. So $C - C \cap V \subseteq S$. Hence $C \subseteq S$.

For (b) we will find a tangent vector to R at a general point c which is not tangent to S . Let $T = k[t]/(t^2)$ and d another point of $C - C \cap V$, so

$$c \text{ and } d \text{ are not base-points of } \mathcal{L} \text{ or } \Omega_C \otimes \mathcal{L}^{-1}. \tag{1}$$

$[\pi_c + \pi_d t]$ represents a tangent vector to $k[\pi_c] = c \in \mathbb{P}^{g-1}$. For it to be tangent to R we need $[(\eta_c + rt)(\pi_c + \pi_d t)] = 0$ in $H^1(C, \mathcal{L}) \otimes_k T$ for some regular section r of \mathcal{L} ; i.e., $[r\pi_c + \eta_c \pi_d] = 0$ in $H^1(C, \mathcal{L})$. If we assume that

$$\varphi_{\mathcal{L}}(c) = \varphi_{\mathcal{L}}(d) \tag{2}$$

then $\eta_c(d) = 0$ and we can take r to be an arbitrary multiple of η_c . Thus we have found a tangent vector to R .

To see that this vector is not tangent to S we consider the obstruction to the second order lifting $[(\eta_c + rt)(\pi_c + \pi_d t)^2/2]$ in $H^1(C, \mathcal{L}) \otimes T/(\Gamma(C, \mathcal{L}) \otimes T) \cup [\pi_c + \pi_d t]$. The question is whether there are sections $f_2 + g_2 t \in \Gamma(C, \mathcal{L}(c + d)) \otimes T$ and $s + s' t \in \Gamma(C, \mathcal{L}) \otimes T$ such that $(\eta_c + rt)\pi_c^2/2 - (s + s' t)(\pi_c + \pi_d t) - (f_2 + g_2 t)$ is regular at c and d . If we assume that

$$\varphi_{\Omega_C \otimes \mathcal{L}^{-1}}(c) \neq \varphi_{\Omega_C \otimes \mathcal{L}^{-1}}(d) \tag{3}$$

then $\Gamma(C, \mathcal{L}) = \Gamma(C, \mathcal{L}(c + d))$ by duality and the last term is regular in any case. If we assume further that

$$c \text{ is not a ramification point of } \varphi_{\mathcal{L}} \tag{4}$$

then $\eta_c \pi_c^2/2$ has a simple pole at c and one needs $s(c) \neq 0$ to make the expression regular at c . But then $s(d) \neq 0$ by (2) and consequently the term $(r\pi_c^2/2 - s\pi_d - s'\pi_c)t$ has a simple pole at d . So a second lifting is impossible. For a general point $c \in C$ the conditions (1), . . . , (4) are satisfied for every point $d \in \varphi_{\mathcal{L}}^{-1}(\varphi_{\mathcal{L}}(c)) - \{c\}$. □

Next we need to check a simpler fact. Let $c + V$ denotes the linear span of c and V in \mathbb{P}^{g-1} .

PROPOSITION 6. *If $c \in C - C \cap V$ then c is a singular point of $S \cap (c + V)$.*

Proof. As $c + V \subseteq R$ we need to compute the derivative of θ_3 along $c + V$ at c . We want to show that any tangent vector in $c + V$ at c is contained in S . Let $[\beta_1], \dots, [\beta_{g-4}]$ be a basis of $\ker(\cup)$ where the $\beta_{i,c}$ are regular at c . This is possible because each cohomology class is equivalent to one supported off any given point as $C - \{\text{point}\}$ is affine. Let $\beta = \pi_c + \sum t_i \beta_i$ where the t_i are indeterminates, $T = k[t_1, \dots, t_{g-4}]/(t_1, \dots, t_{g-4})^2$. Clearly η_c lifts to the first order as $\eta_c \cup [\beta] = 0$ in $H^1(C, \mathcal{L}) \otimes T$. Let $\eta_c + \gamma \varepsilon$ with $\gamma = 0 + \sum \gamma_i t_i$ be a lifting to a section of $\mathcal{L}_2(\beta)$. So the γ_i are

regular at c . The obstruction to the second order lifting is $[\eta_c \beta^2/2 + \gamma \beta]$ in $H^1(C, \mathcal{L}) \otimes T/(\Gamma(C, \mathcal{L}) \otimes T) \cup [\beta]$. This has the form $[\eta_c \pi_c^2/2 + \sum t_i(\eta_c \pi_c \beta_i + \gamma_i \pi_c)]$. We already saw that $\eta_c \pi_c^2/2 = \varrho \pi_c$ for some $\varrho \in \Gamma(C, \mathcal{L})$. $\pi_c \beta_i = 0$ as they have different support. $\gamma_i \pi_c$ has at most a simple pole at c , so it does not give an obstruction since there is a section which does not vanish at c . Consequently

$$[\eta_c \pi_c^2/2 + \sum t_i(\eta_c \pi_c \beta_i + \gamma_i \pi_c)] = [-\sum t_i \varrho \beta_i]$$

in $H^1(C, \mathcal{L}) \otimes T/(\Gamma(C, \mathcal{L}) \otimes T) \cup [\beta]$. Since $\beta_i \in \ker(\cup)$ we have $\varrho \cup [\beta_i] = 0$ in $H^1(C, \mathcal{L})$ and the obstruction vanishes for all tangent directions in $c + V$. □

We will now prove Proposition 2. We have to consider a third order lifting problem in $\mathcal{L}_4(\pi_c)$ for a point c of C . Assume that c is not a base-point of \mathcal{L} and $\Omega_c \otimes \mathcal{L}^{-1}$ and that c is not a ramification point of $\varphi_{\mathcal{L}}$ and $\varphi_{\Omega_c \otimes \mathcal{L}^{-1}}$. A first order lifting has the form $\eta_c + \varrho \varepsilon$ where ϱ is a regular section of \mathcal{L} . A second order lifting has the form $\eta_c + \varrho \varepsilon + \sigma \varepsilon^2$ where σ is regular and $\eta_c \pi_c^2/2 + \varrho \pi_c$ is regular at c . The obstruction to lift to the third order is $[\eta_c \pi_c^3/6 + \varrho \pi_c^2/2 + \sigma \pi_c] \in H^1(C, \mathcal{L})/\Gamma(C, \mathcal{L}) \cup [\pi_c]$. But this is cohomologous to $[-\eta_c \pi_c^3/12]$ which has a double pole at c . Thus the obstruction does not vanish. This proves Proposition 2 and shows that

$$C \cap \{\theta_2 = \theta_3 = \theta_4 = 0\} \subseteq \text{base-points } (\mathcal{L}) \cup \text{base-points } (\Omega_c \otimes \mathcal{L}^{-1}) \\ \cup \text{ramification points } (\mathcal{L}) \cup \text{ramification points } (\Omega_c \otimes \mathcal{L}^{-1})$$

The reversed inclusion is easily seen as the obstruction clearly vanishes. This means that for a (C, \mathcal{L}) general the divisor $C \cap \{\theta_4 = 0\}$ is the sum of the ramification divisors of \mathcal{L} and $\Omega_c \otimes \mathcal{L}^{-1}$ which is in $|4K|$ as it should be. We leave the determination of the multiplicities in special cases open.

§2. Global description of the quadric bundle defined by a cubic hypersurface containing a linear subspace

Let $\mathbb{P}^{h-d} \subseteq \mathbb{P}^h$ be a linear subspace of codimension d . Let $\tilde{\mathbb{P}}$ be \mathbb{P}^h blown up along \mathbb{P}^{h-d} . Then we have the projection $\pi: \tilde{\mathbb{P}} \rightarrow \mathbb{P}^{d-1}$ and an exceptional divisor E in $\tilde{\mathbb{P}}$. Let H be the inverse image of the hyperplane class in \mathbb{P}^h to $\tilde{\mathbb{P}}$ and let L be the inverse image of a hyperplane in \mathbb{P}^{d-1} under π . Then by

examining a hyperplane in \mathbb{P}^h which contains the center \mathbb{P}^{h-d} we deduce that

$$H \sim E + L. \tag{A}$$

The next well-known fact describes the \mathbb{P}^{h-d+1} -bundle π .

$$\tilde{\mathbb{P}} \cong \mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{E}) \tag{B}$$

where

$$\mathcal{E} = \pi_*(\mathcal{O}_{\tilde{\mathbb{P}}}(E)) \cong \mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(-1)^{\oplus h-d+1}.$$

Proof. Clearly E gives a hyperplane in each fiber of π . It remains to compute a basis for \mathcal{E} . To get a generator e_0 of the first summand we may take π_* of the section 1 of $\mathcal{O}_{\tilde{\mathbb{P}}}(E)$. For the other direct summands we may take $e_i = \pi_*(x_i) \in \Gamma(\mathbb{P}^{d-1}, \mathcal{E}(1))$ where $x_1, \dots, x_{h-d+1} \in \Gamma(\tilde{\mathbb{P}}, \mathcal{O}(E + L)) \cong \Gamma(\mathbb{P}^h, \mathcal{O}(1))$ are linear forms which induce homogeneous coordinates on \mathbb{P}^{d-h} . Looking at the fibers of π we see that these sections generate \mathcal{E} everywhere. \square

Let A be a cubic hypersurface in \mathbb{P}^h which contains \mathbb{P}^{h-d} . We can write inverse image in $\tilde{\mathbb{P}}$ as $E + \tilde{A}$ where \tilde{A} is an effective divisors. Then

$$\tilde{A} \subseteq \tilde{\mathbb{P}} \xrightarrow{\pi} \mathbb{P}^{d-1} \text{ is a quadric bundle} \tag{C}$$

since $\tilde{A} \sim 3H - E = 2E + 3L$. The equation of \tilde{A} is very simple. Under the identification $\Gamma(\mathbb{P}^h, \mathcal{O}(1)) \cong \Gamma(\tilde{\mathbb{P}}, \mathcal{O}(H)) \cong \Gamma(\mathbb{P}^{d-1}, \mathcal{E}(1))$

$$x_1 = e_1, \dots, x_{h-d+1} = e_{h-d+1} \text{ and}$$

$$x_{h-d+2} = y_0 e_0, \dots, x_{h+1} = y_{d-1} e_0$$

are homogeneous coordinates on \mathbb{P}^h , if $y_0, \dots, y_{d-1} \in \Gamma(\mathbb{P}^{d-1}, \mathcal{O}(1))$ are homogeneous coordinates in \mathbb{P}^{d-1} . Substituting these expressions into the cubic equation $f = f(x_1, \dots, x_h)$ of A we find

$$f = e_0 \tilde{f} \text{ with } \tilde{f} = \sum_{i \leq j} a_{ij} e_i e_j.$$

so \tilde{f} is a quadratic form in e_0, \dots, e_{h-d+1} with coefficients

$$a_{00} \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(3)),$$

$$a_{0j} = a_{j0} \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(2)) \quad \text{if } j \geq 1 \quad \text{and}$$

$$a_{ij} = a_{ji} \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(1)) \quad \text{of } i, j \geq 1.$$

Of course the $(h - d + 2) \times (h - d + 2)$ matrix $a = (a_{ij})$ is just what we obtain from the equation of \tilde{A} under the isomorphism $\Gamma(\tilde{\mathbb{P}}, \mathcal{O}(2E + 3L)) \cong \Gamma(\mathbb{P}^{d-1}, \text{Sym}^2(\mathcal{E})(3)) \subseteq \Gamma(\mathbb{P}^{d-1}, \mathcal{H}om(\mathcal{E}^*(-3), \mathcal{E}))$ using the splitting of \mathcal{E} . The fact we will use is

For any point $p \in \mathbb{P}^{d-1}$ the codimension of the singular locus of $\tilde{A} \cap \pi^{-1}(p)$ in $\pi^{-1}(p)$ is the rank of the matrix (a_{ij}) at p . (D)

Thus for a general cubic A containing \mathbb{P}^{h-d} we expect the image of all singular fibers is a divisor of degree $h - d + 4 = \text{deg det } (a)$.

Let B be a smooth projective variety together with a morphism $\tau: B \rightarrow \mathbb{P}^{d-1}$. By base change we obtain a quadric bundle

$$\tilde{A}_\tau \hookrightarrow \tilde{\mathbb{P}}_\tau \xrightarrow{\pi'} B.$$

The next result gives what we need abstractly about this quadric bundle.

PROPOSITION 7. *Let X be a closed subvariety of \tilde{A}_τ such that*

- (a) $\tilde{A}_\tau \cap \pi'^{-1}(\pi'(p))$ is singular at p for all points $p \in X$,
- (b) \tilde{A}_τ is smooth at a general point of X and
- (c) $\pi'(X)$ is a divisor on B with $\text{deg}(\pi'(X)) \geq (h - d - 4) \text{deg}(\tau^{-1}(\text{hyperplane}))$

Then (d) $\tilde{A}_\tau \cap \pi'^{-1}(q)$ is smooth for all $q \in B - \pi'(X)$ and

- (e) p is the only singular point of $\tilde{A}_\tau \cap \pi'^{-1}(\pi'(p))$ for a point p of X such that $\pi'(p)$ is smooth on $\pi'(X)$.

Proof. Let $q \in B$ be a point. The codimension of the singular locus of $\tilde{A}_\tau \cap \pi'^{-1}(q)$ is equal to the rank of $\tau^*(a_{ij}) \otimes k(q)$. The singular locus simply is

$$\mathbb{P}(\text{coker}(\tau^*(a_{ij}) \otimes k(q)) \subseteq \mathbb{P}(\tau^*(\mathcal{E}) \otimes k(q))$$

if we regard $a = (a_{ij})$ as an homomorphism $\mathcal{E}^*(-3) \rightarrow \mathcal{E}$. For the generic point of B , this singular locus has a dense subset of $k(B)$ -rational points. In each of them \tilde{A}_τ is singular. Hence the closure of the singular locus of the generic fiber is contained in the singular locus of \tilde{A}_τ . Therefore by (b) the closure does not contain X .

Next we prove that the singular locus of the generic fiber is empty. Otherwise $\tau^*(a_{ij})$ would have some rank $r \leq h - d + 1$. Let x be the generic point of X . Since x is not contained in the closure of the singular locus of the generic fiber and on the other hand the singular locus of $\tilde{A}_\tau \cap \pi'^{-1}(\pi'(x))$ is a linear space which contains x by (a) we have $\text{rank}(\tau^*(a_{ij}) \otimes k(\pi'(x))) < r$. Hence all $r \times r$ minors of τ^*a vanish at $\pi'(x)$. Since not all of them vanish identically on B and all of them are pullbacks of polynomials of degree $< h - d + 4$ on \mathbb{P}^{d-1} this is impossible by (c).

Thus the singular locus of a general fiber is empty and moreover we get that $\pi'(X)$ is scheme-theoretically the zeros of $\det(\tau^*(a_{ij}))$. This proves (d). For (e) just note that

$$\text{rank}(\tau^*(a_{ij}) \otimes k(p)) = h - d + 1$$

if $\pi'(p)$ is smooth in $\pi'(X)$ since otherwise $\pi'(p)$ would be multiple point of $\det(\tau^*(a_{ij})) = 0$. This proves (e). □

Proof of Theorem 1.

Let \mathbb{P}^h be the canonical space \mathbb{P}^{g-1} and $\mathbb{P}^{h-d} = V$ the vertex of $R = \{\theta_2 = 0\}$. Let $A = \{\theta_3 = 0\}$. Thus A contains \mathbb{P}^{h-d} by Lemma 3. Next let $B = \mathbb{P}^1 \times \mathbb{P}^1$ and τ the embedding into \mathbb{P}^3 . Then we have that $\tilde{\mathbb{P}}_\tau$ is R blown up along V and \tilde{A}_τ is the strict transform \tilde{S} of $S = \{\theta_2 = \theta_3 = 0\}$. Then take $X = C$ the canonical curve. Proposition 6 gives (a) and Proposition 5 gives (b). For (c) we note that $\pi'(C) = C'$ has class $(g - 1, g - 1)$. Hence Theorem 1 follows now from Proposition 7. □

REMARKS. (1) For singular points $c' \in C'$ the singular locus of the fiber is higher dimensional. As C' is the zero divisor of the determinant

$$\dim \text{Sing } \alpha^{-1}(c') \leq \text{mult}(C', c') - 1.$$

So if $c' \in C'$ is an ordinary double point then $\text{Sing } \alpha^{-1}(c')$ is the line spanned by the two preimage points in C and equality holds in the formula above. One might guess that equality always holds. We leave this to the reader.

(2) The canonical curve $C \subseteq \mathbb{P}^{g-1}$ lies in the birational model Y of $\mathbb{P}^1 \times \mathbb{P}^1$ obtained by the rational map defined by the linear series of adjoint curves to C' . Without proving it we mention that Y is a component of the variety defined by the partial derivatives

$$\frac{\partial \tilde{f}}{\partial e_i} = 0 \quad \text{for } i = 1, \dots, g - 4$$

in R . It is the component which dominates $\mathbb{P}^1 \times \mathbb{P}^1$ and the birational map is just the projection.

(3) In case $g = 5$ Theorem 1 might be slightly misleading. S is a $K3$ surface birational to the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along C' . C' is a divisor of type $(4, 4)$ with 4 (possibly infinitesimally near) double points which lie on a divisor Δ of class $(1, 1)$. The last property holds because the two g_4 's are residual. The double cover has rational double point singularities over the double points of C' . Resolve those. The preimage of Δ has two disjoint components. Both are (-2) curves. In case there are 4 distinct double points S is obtained by contracting one of them, the singular point will give the vertex V . If two or more of the double points of C' are infinitesimally near then S has a more complicated singularity.

References

- [K1] G.R. Kempf: On the geometry of a theorem of Riemann, *Ann. of Math.* 98 (1973) 178–185.
- [K2] G.R. Kempf: *Abelian Integrals*, Monografias de Instituto de Mathematica 13, Universidad Nacional Autonoma de Mexico, Mexico City (1984).
- [K3] G.R. Kempf: The equation defining a curve of genus 4, *Proceedings of the AMS* 97 (1986) 214–225.
- [M] D. Mumford: Prym varieties I. In: (L.V. Ahlfors, I. Kra, B. Maskit and L. Nirenberg (eds) *Contribution to Analysis*. Academic Press, New York (1974) pp. 325–350.