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On the moduli of curves with theta-characteristics

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0. Introduction

Let C be a smooth complete connected curve of genus g (i.e. C is a compact connected Riemann surface of genus g) over the field of complex numbers $\mathbf{C}$. We denote by K_C the canonical line bundle of C.

DEFINITION. A line bundle $\mathscr{L}$ is called a *theta-characteristic* on C if $\mathscr{L}^2 := \mathscr{L} \otimes \mathscr{L} \simeq K_C$.

If $\mathscr{L}$ is a line bundle on C then '$\deg \mathscr{L}$' denotes its degree. Since $\deg K_C = 2g - 2$, we have $\deg \mathscr{L} = g - 1$ for any theta-characteristic $\mathscr{L}$ on C. If $g \geqslant 1$, then $\mathrm{Pic}(C) \simeq \mathrm{Pic}^0(C) \times \mathbf{Z}$ (where $\mathrm{Pic}(C)$ is the Picard group of C; $\mathrm{Pic}^0(C)$ is the identity component of $\mathrm{Pic}(C)$) and $\mathrm{Pic}^0(C)$ is a complex torus of dimension g. Thus if $g \geqslant 1$, then there are 2^{2g} theta-characteristics on C.

DEFINITION. Let $\mathscr{L}$ be a line bundle on C. Then $\mathscr{L}$ is said to be even (resp. odd) theta-characteristic if $h^0(\mathscr{L}) := \dim_{\mathbf{C}} H^0(C, \mathscr{L})$ is even (resp. odd).

Among the 2^{2g} theta-characteristics on C, $2^{g-1}(2^g + 1)$ (resp. $2^{g-1}(2^g - 1)$) of them are even (resp. odd) [see [M] p. 190].

It follows from a theorem of Mumford [see [M] p. 184] and the fact about the monodromy action [see [ACGH] p. 294] that the moduli of curves with theta-characteristics (i.e., the variety parametrizing isomorphism classes of $(C, \mathscr{L})$, C a curve of genus g as above and $\mathscr{L}$ a theta-characteristic on C) has exactly two connected components $\mathscr{M}_g^+$ and $\mathscr{M}_g^-$ corresponding to even and odd theta-characteristics respectively. If $\mathscr{M}_g$ denotes the moduli of genus g curves, then we have covering projections $\mathscr{M}_g^+ \to \mathscr{M}_g$ and $\mathscr{M}_g^- \to \mathscr{M}_g$ of degree $2^{g-1}(2^g + 1)$ and $2^{g-1}(2^g - 1)$ respectively.

Let $\mathscr{M}_g^r \subset \mathscr{M}_g$ be the closure of the locus of all curves C, such that on C there is a theta-characteristic $\mathscr{L}$ with $h^0(\mathscr{L}) = r$, with its natural subscheme structure (see §1). It follows that $\mathscr{M}_g^r$ is the locus of the curves C possessing a theta-characteristic $\mathscr{L}$ with $h^0(\mathscr{L}) \geqslant r$ and $h^0(\mathscr{L}) \equiv r(2)$. Note if r is 0 or 1 $\mathscr{M}_g^r = \mathscr{M}_g$. In this note, following a suggestion of M. V. Nori, we give a method to compute Zariski tangent spaces to $\mathscr{M}_g^r$ in the moduli-stack. Using the above method we find the dimension of the Zariski tangent spaces to $\mathscr{M}_g^r$ at a hyperelliptic curve.

We give an example of a $\mathscr{M}_g^r$ which is not reduced as a scheme. Also we give example of $\mathscr{M}_g^r$ which is not irreducible.

We refer to Teixidor I Bigas. M. [T], for a detailed study of the above moduli for small r.

It is a great pleasure for me to thank Prof. Madhav. V. Nori, for his help and constant encouragement.

1. Method to compute Zariski tangent space

For the standard facts about moduli [see [T] p. 100]. Let C be a curve of genus g as in the introduction. Let U be a neighbourhood of $[C]$ in a suitable cover of the moduli space $\mathscr{M}_g$ of genus g curves. Note that the tangent space to U at $[C]$ can be identified with $H^1(C, T_C)$, where T_C is the tangent bundle of C. Let $\mathbf{C}$ be the corresponding universal curve over U, i.e., we have a proper smooth morphism

$$\pi\colon \mathbf{C} \to U,$$

such that for each point $x \in U$, $C_x := \pi^{-1}(x)$ is a smooth curve of genus g with suitable universal properties. Let L be a line bundle on $\mathbf{C}$ such that for $x \in U$, $L_x := L|_{\pi^{-1}(x)}$ is a theta-characteristic on C_x. Since $\deg L_x = g - 1$, by Riemann-Roch theorem [see [H] p. 295] we see that $h^0(L_x) = h^1(L_x)$, where $h^0(L_x) := \dim_{\mathbf{C}} H^0(C_x, L_x)$ and $h^1(L_x) := \dim_{\mathbf{C}} H^1(C_x, L_x)$. Set $\mathscr{L} = L_{[C]}$, where $[C]$ is the point of U corresponding C. If $h^0(\mathscr{L}) = r$, then by using semi-continuity theorem [see [H] p. 281–291] we get a morphism (changing U by a suitable neighbourhood of $[C]$, if necessary)

$$\theta\colon U \to \mathrm{Hom}_{\mathbf{C}}(H^0(C, \mathscr{L}), H^1(C, \mathscr{L})) \simeq M_r(\mathbf{C}),$$

such that the scheme-theoretic inverse image of the origin is the locus of curves in U corresponding to $\mathscr{M}_g^r$ defined in the introduction. We are interested in the tangent space mapping

$$\Theta\colon H^1(C, T_C) \to \mathrm{Hom}_{\mathbf{C}}(H^0(C, \mathscr{L}), H^1(C, \mathscr{L})),$$

of θ at $[C] \in U$.

First note that by Serre's duality theorem [See [H] p. 295] $H^0(C, \mathscr{L})$ (resp. $H^0(C, K_C^2)$, where $K_C^2 := K_C \otimes K_C$) is naturally dual to $H^1(C, \mathscr{L})$ (resp. $H^1(C, T_C)$).

THEOREM 1. *The mapping*

$$\Theta^\vee\colon H^0(C, \mathscr{L}) \otimes H^0(C, \mathscr{L}) \to H^0(C, K_C^2),$$

defined by

$$(fe_1, ge_1) \mapsto (f\,\mathrm{d}g - g\,\mathrm{d}f)e_1^2,$$

(where $fe_1, ge_1 \in H^0(C, \mathscr{L})$*), is dual to* Θ *(up to a scalar multiple).*

Proof. Let $t \in H^1(C, T_C)$. First we describe the homomorphism

$$\Theta(t)\colon H^0(C, \mathscr{L}) \to H^1(C, \mathscr{L}).$$

Choose an affine covering $\{U_1, U_2\}$ of C such that

$$\mathscr{L}(U_1) \simeq \mathscr{O}_C(U_1)e_1 \quad \text{and} \quad \mathscr{L}(U_2) \simeq \mathscr{O}_C(U_2)e_2$$

and also we have

$$K_C(U_1) \simeq \mathscr{O}_C(U_1)e_1^2 = \mathscr{O}_C(U_1)h\,\mathrm{d}a$$
$$K_C(U_2) \simeq \mathscr{O}_C(U_2)e_2^2 = \mathscr{O}_C(U_2)h_1\,\mathrm{d}b,$$

where $\mathscr{O}_C$ is the structure sheaf of C and a, b, h, h_1 are rational functions on C. If $\alpha_{12} \in \mathscr{O}_C(U_1 \cap U_2)^*$ is the transition function of $\mathscr{L}$, then by our assumption on $\mathscr{L}$, α_{12}^2 is the transition function for K_C, where $\mathscr{O}_C(U_1 \cap U_2)^*$ is the group of invertible elements of $\mathscr{O}_C(U_1 \cap U_2)$. Now $t \in H^1(C, T_C)$ gives an infinitesimal deformation $C_t[\varepsilon]$ as follows: let $D\colon \mathscr{O}_C(U_1 \cap U_2) \to \mathscr{O}_C(U_1 \cap U_2)$ be the derivation corresponding to 't', then $C_t[\varepsilon]$ is defined by glueing

$$\mathrm{Spec}\left(\mathscr{O}_C(U_1) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon_2)}\right) \quad \text{and} \quad \mathrm{Spec}\left(\mathscr{O}_C(U_2) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)}\right)$$

along $\mathrm{Spec}(\mathscr{O}_C(U_1 \cap U_2) \otimes \mathbf{C}[\varepsilon]/(\varepsilon^2))$ by the function

$$f \mapsto f + \varepsilon D(f).$$

If $K_{C_t[\varepsilon]}$ is the relative cononical bundle of $C_t[\varepsilon] \to \mathrm{Spec}(\mathbf{C}[\varepsilon]/(\varepsilon^2))$, then it is easy to verify that $K_{C_t[\varepsilon]}$ is given by the transition function

$$\alpha_{12}^2\left(1 + \varepsilon\left(\frac{\mathrm{d}(D(a))}{\mathrm{d}a} + \frac{D(h)}{h}\right)\right) \in \left(\mathscr{O}_C(U_1 \cap U_2) \otimes \frac{\mathbf{C}[\varepsilon]}{(\varepsilon^2)}\right)^*.$$

Then

$$\alpha_{12}\left(1 + \frac{1}{2}\varepsilon\left(\frac{\mathrm{d}(D(a))}{\mathrm{d}a} + \frac{D(h)}{h}\right)\right)$$

gives transition function for a line bundle $\mathscr{L}_1$ on $C_t[\varepsilon]$ such that $\mathscr{L}_1^2 \simeq K_{C_t[\varepsilon]}$ and $\mathscr{L}_1|_C \simeq \mathscr{L}$. Also on $C_t[\varepsilon]$ we have an exact sequence

$$0 \to \varepsilon\mathscr{L} \to \mathscr{L}_1 \to \mathscr{L} \to 0.$$

From this exact sequence we get a coboundary homomorphism

$$\psi: H^0(C, \mathscr{L}) \to H^1(C, \mathscr{L}).$$

Using Cech-cohomology with respect to the covering $\{U_1, U_2\}$, we get

$$\psi(fe_1) = \left(\frac{f}{2}\left(\frac{\mathrm{d}(D(a))}{\mathrm{d}a} + \frac{D(h)}{h}\right) + D(f)\right) e_1,$$

where $fe_1 \in H^0(C, \mathscr{L})$. But $\Theta(t)$ is nothing but ψ. Note that if $fe_1, ge_1 \in H^0(C, \mathscr{L})$ then cup product of $\psi(fe_1)$ and fe_1 gives an element

$$\psi(fe_1)ge_1 \in H^1(C, K_C).$$

But

$$\begin{aligned} \psi(fe_1)ge_1 &= \left(\frac{gf}{2}\left(\frac{\mathrm{d}(D(a))}{\mathrm{d}a} + \frac{D(h)}{h}\right) + D(f)g\right) e_1^2 \\ &= \frac{fgh}{2}\,\mathrm{d}(D(a)) + \frac{gfD(h)}{2}\,\mathrm{d}a + (ghD(f))\,\mathrm{d}a \\ &= \frac{fgh}{2}\,\mathrm{d}(D(a)) + \frac{fgD(a)}{2}\,\mathrm{d}h + ghD(a)\,\mathrm{d}f \\ &= \frac{D(a)h}{2}(f\,\mathrm{d}g - g\,\mathrm{d}f) + \tfrac{1}{2}\mathrm{d}(fghD(a)) \end{aligned}$$

(In the above equation we have used the fact $D(h)\,da = D(a)\,\mathrm{d}h$ and $D(a)\,\mathrm{d}f = D(f)\,\mathrm{d}a$). *So if* $p \in C - U_1$, then

$$\mathrm{res}_p(\psi(fe_1)ge_1) = \mathrm{res}_p\left(\frac{D(a)h}{2}(f\,\mathrm{d}f - g\,\mathrm{d}g)\right).$$

On the other hand the derivation D corresponding to $t \in H^1(C, T_C)$ induces (by cup product) a homomorphism

$$\mathrm{D}: H^0(C, K_C^2) \to H^1(C, K_C).$$

Composing the homomorphism with

$$\mathrm{res} := \sum \mathrm{res}_p \colon H^1(C, K_C) \to \mathbf{C},$$

where summation is over all $p \in C - U_1$, gives that the homomorphism

$$H^0(C, \mathscr{L}) \otimes H^0(C, \mathscr{L}) \to H^0(C, K_C^2)$$
$$(fe_1, ge_1) \mapsto (f\,\mathrm{d}g - g\,\mathrm{d}f)h\,\mathrm{d}a,$$

is dual to the homomorphism (up to a scalar multiple)

$$\Theta \colon H^1(C, T_C) \to \mathrm{Hom}_{\mathbf{C}}(H^0(C, \mathscr{L}), H^1(C, \mathscr{L})).$$

This proves the theorem.

COROLLARY 1. *Image of* Θ *is contained in the set of alternating matrices.*

Proof. The corollary follows immediately from the theorem because $\Theta^\vee$ is clearly zero on symmetric tensors.

COROLLARY 2. [See [Ha] p. 616]. *If* $\mathscr{M}_g^r \neq \varnothing$, *then every irreducible component of* $\mathscr{M}_g^r$ *has codimension at most* $(r(r-1))/2$ *in* $\mathscr{M}_g$.

Proof. From the corollary (1) above and the definition of $\mathscr{M}_g^r$, corollary (2) follows immediately.

2. Examples

First we compute the tangent space map described above, at hyperelliptic curve. If C is a hyperelliptic curve of genus g, then C is the normalization of the plane curve

$$y^2 = \prod_{i=1}^{2g+2} (x - a_i),$$

($a_i \in \mathbf{C}$ and $a_i \neq a_j$ for $1 \leqslant i, j \leqslant 2g + 2$). Then

$$H^0(C, K_C) = \mathbf{C}\frac{\mathrm{d}x}{y} \oplus \mathbf{C}x\frac{\mathrm{d}x}{y} \oplus \cdots \oplus \mathbf{C}x^{g-1}\frac{\mathrm{d}x}{y},$$

and

$$H^0(C, K_C^2) = \mathbf{C}\left(\frac{\mathrm{d}x}{y}\right)^2 \oplus \mathbf{C}x\left(\frac{\mathrm{d}x}{y}\right)^2 \oplus \cdots$$
$$\oplus \mathbf{C}x^{2g-2}\left(\frac{\mathrm{d}x}{y}\right)^2 \oplus \mathbf{C}y\left(\frac{\mathrm{d}x}{y}\right)^2 \oplus \cdots$$
$$\oplus \mathbf{C}x^{g-3}y\left(\frac{\mathrm{d}x}{y}\right)^2.$$

Given integer r $(0 \leqslant r \leqslant [(g+1)/2])$, then set $s = (g-1) - 2(r-1)$. If

$$\mathscr{L} = \pi^*\mathcal{O}_{\mathbf{P}^1}(r-1) \otimes \mathcal{O}_C\left(\sum_{k=1}^{s} t_{i_k}\right),$$

where $\pi\colon C \to \mathbf{P}^1$ is the covering ramified precisely over a_i $(1 \leqslant i \leqslant 2g+2)$ and $t_{i_k} \in \pi^{-1}(\{a_1, \ldots, a_{2g+2}\})$, then $\mathscr{L}$ is a theta-characteristic on C with $h^0(\mathscr{L}) = r$. Conversely every theta-characteristic $\mathscr{L}$ on C with $h^0(\mathscr{L}) = r$, is of the above form. Fix a theta-characteristic $\mathscr{L}$ on C with $h^0(\mathscr{L}) = r$, then

$$\Theta^\vee\colon H^0(C, \mathscr{L}) \otimes H^0(C, \mathscr{L}) \to H^0(C, K_C)$$

is induced by

$$(x^a e_1, x^b e_1) \to (x^a\,\mathrm{d}x^b - x^b\,\mathrm{d}x^a)e_1^2 = (b-a)x^{a+b-1}y\left(\frac{\mathrm{d}x}{y}\right)^2,$$

(where $x^a e_1, x^b e_1 \in H^0(C, \mathscr{L})$). Now if $r \geqslant 2$ it is easy to see that image of $\Theta^\vee$ is a $2r-3$ dimensional subspace of $H^0(C, K_C^2)$. So by the above theorem if $r \geqslant 2$ it follows that at $(C, \mathscr{L})$ the tangent mapping

$$\Theta\colon H^1(C, T_C) \to \mathrm{Hom}_{\mathbf{C}}(H^0(C, \mathscr{L}), H^1(C, \mathscr{L}))$$

has rank $(2r-3)$, hence the $\ker(\Theta)$ is of codimension $2r-3$ in $H^1(C, T_C)$. Thus we have proved the following:

THEOREM 2. *In a suitable covering space of $\mathscr{M}_g$, the Zariski tangent space to $\mathscr{M}_g^r$ $(r \geqslant 2)$* at (C, $\mathscr{L}$) *has dimension $3g-2r$, where C is an hyperelliptic curve and $\mathscr{L}$ is a theta-characteristic on C with $h^0(\mathscr{L}) = r$.*

THEOREM 3. *$\mathscr{M}_8^4$ is non-reduced scheme of dimension* 15 *and $(\mathscr{M}_8^4)_{\mathrm{red}}$ is the locus of hyperelliptic curves.*

Proof. By Theorem (2), if $(C, \mathscr{L}) \in \mathscr{M}_8^4$ is such that C is hyperelliptic curve of genus 8 and $\mathscr{L}$ a theta-characteristic on C with $h^0(\mathscr{L}) = 4$, then the Zariski tangent space at $(C, \mathscr{L})$ is of dimension 16. On the other hand we show that if C is a curve of genus 8 with a theta-characteristic $\mathscr{L}$ such that $h^0(\mathscr{L}) = 4$, then C is hyperelliptic, this will prove the theorem.

CLAIM. *If* $(C, \mathscr{L}) \in \mathscr{M}_8^4$ *then* C *hyperelliptic.*

Proof (of the claim). Suppose $\mathscr{L}$ has a base point p, then $\mathscr{L}(-p)$ is a degree 6 line bundle on a genus 8 curve with 4 linearly independent sections, hence by Clifford's theorem [see [H] p. 343] C is hyperelliptic, so claim is proved if $\mathscr{L}$ has a base point. Hence we can assume that $\mathscr{L}$ has no base point. Let

$$\phi_{\mathscr{L}}: C \to \mathbf{P}^3$$

be the corresponding morphism. Since $h^0(\mathscr{L}^2) = 8$ and $h^0(\mathscr{O}_{\mathbf{P}^3}(2)) = 10$, there are at least two linearly independent quadrics vanishing on $\phi_{\mathscr{L}}(C)$. This implies, since $\phi_{\mathscr{L}}(C)$ is not contained in any hyperplane, degree of $\phi_{\mathscr{L}}(C)$ in $\mathbf{P}^3$ is $\leqslant 4$. But $\deg \mathscr{L} = 7$, so $\mathscr{L}$ must have a base point which contradicts our assumption on $\mathscr{L}$. This proves the claim.

Since locus of hyperelliptic curves is a 15 dimensional subvariety of $\mathscr{M}_8$ theorem follows.

Next we will describe moduli $\mathscr{M}_g^3$ for small g. Note that $\mathscr{M}_g^3 = \varnothing$ for $1 \leqslant g \leqslant 4$ by Clifford's theorem. For $g \geqslant 5$ we have the following:

THEOREM [see [T] p. 113]. *The locus $\mathscr{M}_g^3$ has pure codimension in $\mathscr{M}_g$ if $g \geqslant 5$, and a generic point of any of its components is a curve C which has only one $\mathscr{L}$ with $\mathscr{L}^2 \simeq \mathscr{K}_C$ such $h^0(\mathscr{L}) = 3$ if $g \geqslant 6$. Moreover this theta-characteristic gives a birational morphism of C into $\mathbf{P}^2$ if $g \geqslant 6$.*

(1) *When $g = 5$, it follows by Clifford's theorem that $\mathscr{M}_5^3$ is precisely the locus of hyperelliptic curves.*
(2) *Next $g = 6$. Let C be a curve of genus 6 and $\mathscr{L}$ be a theta-characteristic on C with $h^0(\mathscr{L}) = 3$. If $\mathscr{L}$ has no base point then clearly $\mathscr{L}$ gives embedding of C in $\mathbf{P}^2$. Locus $(\mathscr{M}_6^3)^0$ smooth plane curves of degree 6 is locally closed in moduli $\mathscr{M}_6$ of genus 6 curves and again by Clifford's theorem it follows that $\mathscr{M}_6^3 = (\mathscr{M}_6^3)^0 \cup \mathscr{H}_6$, where $\mathscr{H}_6$ is the locus of hyperelliptic curves.*

THEOREM 4. *$\mathscr{M}_7^3$ is an irreducible subvariety of dimension 15 in the moduli-space $\mathscr{M}_7$.*

Proof. By tangent space computations it follows that $\mathscr{M}_7^3$ has dimension $\geqslant 15$. It follows by Clifford's theorem that if $C \in \mathscr{M}_7^3$ and if the corresponding theta-characteristic $\mathscr{L}$ on C has a base point then C must be a hyperelliptic curve. But moduli of hyperelliptic curves is of dimension 13, hence C cannot be a general member of $\mathscr{M}_7^3$. So on a general member $C \in \mathscr{M}_7^3$ there exists theta-characteristic $\mathscr{L}$ with $h^0(\mathscr{L}) = 3$ and $\mathscr{L}$ does not have base points. Let

$$\phi_{\mathscr{L}}: C \to \mathbf{P}^2$$

be the corresponding morphism. Then the image curve can have degree 2, 3 or 6. But again by dimension count we get that if C is general then image C is a degree 6 in $\mathbf{P}^2$, hence $\phi_{\mathscr{L}}$ is birational onto its image. Since $\mathscr{L}^2 \simeq K_C$, we see that image of C under $\phi_{\mathscr{L}}$ is a degree 6 curve having exactly three ordinary double points lying on a line and no other singularities. Now fix a line $l \subset \mathbf{P}^2$ and three distinct points p_1, p_2, p_3 on l then the exact sequence

$$0 \to \prod_{i=1}^{3} m_{\mathbf{P}^2,p_i}^2 \to \mathcal{O}_{\mathbf{P}^2} \to \bigoplus_{i=1}^{3} \frac{\mathcal{O}_{\mathbf{P}^2,p_i}}{m_{\mathbf{P}^2,p_i}^2} \to 0,$$

where $m_{\mathbf{P}^2,p_i}$ is the ideal sheaf of the point $p_i \in \mathbf{P}^2$, after tensoring with $\mathcal{O}_{\mathbf{P}^2}(6)$ gives the following cohomology exact sequence

$$0 \to H^0\left(\mathbf{P}^2, \prod_{i=1}^{3} m_{\mathbf{P}^2,p_i}^2(6)\right) \to H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(6))$$
$$\to \bigoplus_{i=1}^{3} H^0\left(\mathbf{P}^2, \frac{\mathcal{O}_{\mathbf{P}^2,p_i}}{m_{\mathbf{P}^2,p_i}^2}\right) \to H^1\left(\mathbf{P}^2, \prod_{i=1}^{3} m_{\mathbf{P}^2,p_i}^2(6)\right) \to 0.$$

But it is easy to see that

$$H^1\left(\mathbf{P}^2, \prod_{i=1}^{3} m_{\mathbf{P}^2,p_i}^2(6)\right) = 0.$$

Now using the fact that $p_i (1 \leqslant i \leqslant 3)$ lie on a line and Bertini's theorem [See [H], p. 274] we get an open set

$$U \subset \mathbf{P}\left(H^0\left(\mathbf{P}^2, \prod_{i=1}^{3} m_{\mathbf{P}^2,p_i}^2(6)\right)\right),$$

such that if $C \in U$ then C is irreducible plane curve of degree 6 and has double points at p_i $(1 \leqslant i \leqslant 3)$ and no other singularities. Note that $\dim U = 18$ and general member is a nodal curve. If we vary $l \subset \mathbf{P}^2$ and $p_i \in l$ $(1 \leqslant i \leqslant 3)$, we get a 23 dimensional irreducible locally closed subvariety W of $\mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(6)))$ such that if $C \in W$ then C has exactly three ordinary double points all of them lie on a line and has no other singularities. On W, PGL(3) acts with finite stabilizer at each of its points. Now the quotient V of W by PGL(3) gives a dense open subset of $\mathscr{M}_7^3$. Since dimension of V is 15 and V is irreducible theorem follows immediately.

THEOREM 5. *$\mathcal{M}_8^3$ is an irreducible subvariety of dimension* 18 *in the moduli space $\mathcal{M}_8$.*

Proof. By the theorem quoted above of Teixidor I Bigas, each irreducible component of $\mathcal{M}_8^3$ is 18 dimensional and whose general member C has a theta-characteristic $\mathscr{L}$ such that $\mathscr{L}$ gives a birational morphism

$$\phi_{\mathscr{L}}: C \to \mathbf{P}^2.$$

As above using the fact that $\mathscr{L}^2 \simeq K_C$, we get $\phi_{\mathscr{L}}(C)$ is a curve of degree 7 and has exactly 7 ordinary double points all of them lie on smooth conic and has no other singularities. Fix a smooth conic $E \subset \mathbf{P}^2$ and 7 distinct points $p_1, \ldots, p_7$ on it. Consider the exact sequence

$$0 \to \prod_{i=1}^{7} m_{p_i}^2 \to \mathcal{O}_{\mathbf{P}^2} \to \bigoplus_{i=1}^{7} \frac{\mathcal{O}_{\mathbf{P}^2}}{m_{p_i}^2} \to 0,$$

where m_{p_i} is the ideal sheaf of p_i in $\mathbf{P}^2$. It is easy to see that

$$H^1\left(\mathbf{P}^2, \prod_{i=1}^{7} m_{p_i}^2(7)\right) = 0.$$

Hence from the above exact sequence, after tensoring with $\mathcal{O}_{\mathbf{P}^2}(7)$ we get a cohomology exact sequence

$$0 \to H^0\left(\mathbf{P}^2, \prod_{i=1}^{7} m_{p_i}^2(7)\right) \to H^0\left(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(7)\right) \to \bigoplus_{i=1}^{7} \frac{\mathcal{O}_{\mathbf{P}^2}}{m_{p_i}^2} \to 0.$$

Again it is easy to see that there exists an open set

$$U \subset \mathbf{P}\left(H^0\left(\mathbf{P}^2, \prod_{i=1}^{7} m_{p_i}^2(7)\right)\right)$$

such that every curve parametrized by U is irreducible and has ordinary double points exactly at the points $p_1, \ldots, p_7$ and no other singularities. Now varying the conic and the 7 points on it we get a 26 dimensional irreducible locally closed sub variety W of $\mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(7))$, on which PGL(3) acts with finite stabilizer at each of its points. The quotient V, of W by PGL(3) is a dense open subset of $\mathcal{M}_8^3$. Thus $\mathcal{M}_8^3$ is a irreducible codimension 3 subvariety of $\mathcal{M}_8$.

THEOREM 6. *$\mathcal{M}_9^3$ has exactly two irreducible components each of dimension* 21 *in $\mathcal{M}_9$.*

Proof. Again by the theorem of Teixidor I Bigas, each irreducible component of $\mathscr{M}_9^3$ is 21 dimensional and whose general members is a curve C with a theta-characteristic $\mathscr{L}$ which give rise to a birational morphism

$$\phi_{\mathscr{L}}: C \to \mathbf{P}^2.$$

As above the fact that $\mathscr{L}$ is theta-characteristic gives that $\phi_{\mathscr{L}}(C)$ is a curve of degree 8 and has exactly 12 ordinary double points all of which lie on degree 3 curve and has no other singularities. Also note that the above 12 points on the degree 3 curve has the property twice the sum of these 12 points is the zeros of a section of $\mathscr{O}_{\mathbf{P}^2}(8)$ restricted to the degree 3 curve. Since we are interested in an open subset of $\mathscr{M}_9^3$, we look at curves C as above with corresponding singularities of $\phi_{\mathscr{L}}(C)$ lie on a smooth degree 3 curve. We fix a smooth degree 3 curve $E \subset \mathbf{P}^2$ and 12 distinct points $p_1, \ldots, p_{12}$ on it such that $2\sum_{i=1}^{12} p_i \in \mathbf{P}(H^0(E, \mathscr{O}_{\mathbf{P}^2}(8)|_E))$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(5)) & \to & H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(8)) & \to & H^0(E, \mathscr{O}_E(8)) & \to 0 \\ & \| & & \uparrow & & \uparrow & \\ 0 \to & H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(5)) & \to & V & \to & \mathbf{C}\sigma & \to 0 \end{array}$$

where σ is a section of $\mathscr{O}_E(8)$ corresponding to $2\sum_{i=1}^{12} p_i$, V is the inverse image of $\mathbf{C}\sigma$ in $H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(8))$, $\dim_C V = 22$. From V we have the following mapping

$$V \to \bigoplus_{i=1}^{12} \frac{m_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2}$$

whose image is the 12 dimensional subspace

$$\bigoplus_{i=1}^{12} \frac{m_{E, p_i}}{m_{E, p_i}^2}.$$

So

$$W = \ker\left(V \to \bigoplus_{i=1}^{12} \frac{m_{\mathbf{P}^2, p_i}}{m_{\mathbf{P}^2, p_i}^2}\right)$$

is a 10 dimensional subspace of $H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(8))$. Again by Bertini's theorem $\mathbf{P}(W)$ contains an open set $U_{(E,(p_i))}$ such that if C is a curve corresponding to a point of $U_{(E,(p_i))}$ then C irreducible degree 8 curve which has ordinary double points at p_i $(1 \leqslant i \leqslant 12)$ and has no other singularities. Now the variety H parametrizing $(E, \sum_{i=1}^{12} p_i)$, $E \subset \mathbf{P}^2$ degree 3 smooth curve, $p_1, \ldots, p_{12}$ distinct points on E such that $2\sum_{i=1}^{12} p_i \in \mathbf{P}(H^0(E, \mathscr{O}_{\mathbf{P}^2}(8)|_E))$ is 20 dimensional. Note that H has two connected (irreducible) components (see, Introduction) of dimension 20 corresponding to two types of points $p_1, \ldots, p_{12}$ namely whether $\sum_{i=1}^{12} p_i$ is in $\mathbf{P}(H^0(E, \mathcal{O}_{\mathbf{P}^2}(4)|_E))$ or not. The above construction gives a variety X fibred over H with fibres the 9 dimensional variety $U_{(E,(p_i))}$. On $X = X_1 \cup X_2$, PGL(3) acts with finite stabilizer at each of its point and the quotient W is a dense open subset of $\mathcal{M}_9^3$. This proves that $\mathcal{M}_9^3$ has two irreducible components.

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