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Secant spaces and Clifford's theorem

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Introduction

The following theorem is basic for the results of this paper.

THEOREM A. *Any reduced irreducible non-degenerate and linearly normal curve C of degree $d \geqslant 4r-7$ in $\mathbf{P}^r$ $(r \geqslant 2)$ has a $(2r-3)$-secant $(r-2)$-plane.*

This theorem is a special case of a more general theorem which we prove in the first part of this paper. By examples, we will show that the bound on the degree of C seems to be the best possible bound only for $r \leqslant 4$.

In the second part we first use Theorem A to clarify the relation between two invariants of a smooth, irreducible projective curve C of genus $g \geqslant 4$: the *gonality* k of C and the *Clifford index* c of C. In fact, we usually have $c = k-2$ but there are counterexamples belonging to smooth curves in $\mathbf{P}^r$ without any $(2r-2)$-secant $(r-2)$-planes, cf. [9]. But according to Theorem A these curves C (for which $c \neq k-2$) always have infinitely many $(2r-3)$-secant $(r-2)$-planes inducing (by projection) an infinite number of pencils g^1_{c+3} on C. In particular, then, $c = k-3$ for these "exceptional curves". As a consequence, we see that for a k-gonal curve C having only finitely many g^1_k the Clifford index is given by $c = k-2$. This applies to the general k-gonal curve of genus $g \geqslant 2(k-1)$. We thus recover Ballico's result [4] that every possible value for the Clifford index of a curve of given genus really occurs.

Another application of Theorem A is an *improvement of Clifford's classical theorem.* Recall that Clifford's theorem states that on a curve C of genus g any linear system g^r_d of degree $0 \leqslant d \leqslant 2g-2$ fulfills $2r \leqslant d$. More precisely we will prove

THEOREM B ("refined Clifford"): *On a k-gonal curve C $(k \geqslant 3)$ of genus g any g^r_d of degree $k-3 \leqslant d \leqslant 2g-2-(k-3)$ satisfies $2r \leqslant d-(k-3)$.*

We should note that (by Riemann–Roch) Theorem B applies to the set G, say, of all g_d^r on C with $d \leqslant g-1$ and $r \geqslant 1$ and that in this case the equality $2r = d-(k-3)$ implies that C is one of the "exceptional curves" mentioned above.

For g_d^r in G we also prove another Clifford-like result which implies $3r \leqslant d$ if k is odd and which improves Theorem B for linear systems in G if k is small with respect to g.

In part 3 of this paper we use Theorem A to determine the maximal degree of all linear systems of degree $d \leqslant g-1$ on C which compute the Clifford index c of C. Our main result is

THEOREM C. *Any g_d^r $(d \leqslant g-1)$ on C computing c has degree $d \leqslant 2(c+2)$ unless C is hyperelliptic or bi-elliptic.*

For every $c \geqslant 1$, this bound on d is the best possible. Moreover, we will show that for $g > 2c+5$ we have the better bound $d \leqslant 3(c+2)/2$. Finally, we apply Theorem C to give a new proof of the fact that on the general k-gonal curve of genus $g > 2k$ $(k \geqslant 3)$ there is only one linear system of degree at most $g-1$ computing c, namely the unique g_k^1. This fact was proved before by Ballico [4] (even for $g > 2k-2$) using degeneration theory of linear systems. Again, our proof is more concrete.

Notations and conventions

A variety (curve, resp. surface) X always means here an integral projective scheme over $\mathbf{C}$ (smooth of dimension 1, resp. 2). However we consider it in the more classical context looking only to the $\mathbf{C}$-closed points. If F is a coherent $\mathcal{O}_X$-module then $h^i(F) = \dim_{\mathbf{C}}(H^i(X, F))$, F_x $(x \in X)$ is the stalk of F at x and $F(x) = F_x/\mathcal{M}_{X,x} \cdot F_x$. If F' is another coherent $\mathcal{O}_X$-module and $\varphi: F \to F'$ a homomorphism then $\varphi(x)$: $F(x) \to F'(x)$ is the induced map. For a Cartier divisor D on X, $\mathcal{O}_X(D)$ is the associated invertible sheaf on X. Clearly, $h^i(D) = h^i(\mathcal{O}_X(D))$.

C always denotes a smooth irreducible projective curve of genus $g \geqslant 1$. For C, we adopt most of our notations from [3]. Specifically, if $d > 1$, $C^{(d)}$ is the set of effective divisors of C of degree d, g_d^r is a linear system of degree d and projective dimension r (a pencil if $r=1$), and $g_d^r(-D) = \{E - D \colon E \in g_d^r \text{ such that } E \geqslant D\}$ if D is an effective divisor of C. Note that for a complete g_d^r the linear system $g_d^r(-D)$ is complete, too. A g_d^r is classically called a simple system if the induced rational map $C \to \mathbf{P}^r$ is birational onto its image.

We identify $J(C)$, the jacobian of C, with $\mathrm{Pic}^0(C)$. For an invertible sheaf L on C of degree 0 we denote by [L] the corresponding point on $J(C)$. Conversely, if $x \in J(C)$ then L_x is an invertible sheaf on C representing x.

Fixing some base point P_0 on C we denote the important morphism

$$C^{(d)} \to J(C)\colon D \mapsto [\mathcal{O}_C(D - dP_0)]$$

by $I(d)$. If $x \in J(C)$ then $g_d(x)$ is the complete linear system on C associated to $L_x(dP_0)$. Recall that we have the well-known Zariski-closed subsets of $J(C)$

$$W_d^r = \{x \in J(C)\colon \dim(g_d(x)) \geqslant r\} = \{x \in J(C)\colon \dim((I(d))^{-1}(x)) \geqslant r\}.$$

They also have a natural scheme structure. If A and B are subsets of $J(C)$ we use the notation

$$A \oplus B = \{x + y : x \in A \quad \text{and} \quad y \in B\}.$$

1. The Secant theorem

In this section we will prove Theorem A. At first, we mention the general problem.

Let C be a smooth curve of genus g, and let g_d^r $(r \geqslant 2)$ be a linear system on C.

1.1 DEFINITION. Let $n \in \mathbf{Z}$ with $n \leqslant r-1$ and let $e \in \mathbf{Z}$ with $e \geqslant n+1$. Then $D \in C^{(e)}$ is called an *e-secant n-space divisor* for g_d^r if and only if $\dim(g_d^r(-D)) \geqslant r-n-1$ (i.e. if D imposes at most $n+1$ conditions on g_d^r). □

Consider

$$V_e^n(g_d^r) = \{D \in C^{(e)}\colon D \text{ is an } e\text{-secant } n\text{-space divisor for } g_d^r\}.$$

Let Z be an irreducible component of $V_e^n(g_d^r)$. Using a determinantal description for $V_e^n(g_d^r)$ one finds (see [3], p. 345)

$$\dim(Z) \geqslant (n + 1 - e)(r - n) + e.$$

In particular, in general one expects that $V_e^n(g_d^r)$ is not empty if $(n+1-e)(r-n)+e \geqslant 0$. In this section we prove:

1.2 THEOREM. *If* g_d^r *is complete; if* $d \geqslant 2e-1$ *and if* $(n+1-e)(r-n)+e \geqslant r-n-1$ *then* $V_e^n(g_d^r)$ *is not empty.*

Taking $n=r-2$, $e=2r-3$ we obtain Theorem A. If $n=r-2$, $e=2r-2$ and if $V_e^n(g_d^r)$ is not empty then one deduces the existence of a linear system g^1_{d-2r+2} on C. This remark is essential in the study of curves of a given Clifford index ([9]; see part 2 for the definition). If g_d^r is the canonical linear system on C then the study of $V_e^n(g_d^r)$ is very closely related to the study of special divisors on C. In fact, our investigation is similar to that of the Brill–Noether existence problem. This problem has originally been solved by means of an enumerative argument (see [15]; [16]; [17]). From ideas developed in [11] a much shorter solution has been found (see also [3], p. 311). We will use these methods to prove Theorem (1.2). The main ingredient of the proof is the following lemma, which is a slight generalization of [10] (cf. [7], Theorem 11).

1.2.1 LEMMA. *If Z is a closed irreducible subset of W_d^r satisfying* $\dim(Z) \geq r+1$ *then Z intersects W_{d-1}^r.*

Proof. Assume that $Z \cap W_{d-1}^r$ is empty (then also $Z \cap W_d^{r+1}$ is empty). Let P be the Poincaré invertible sheaf on $J(C) \times C$ and let P_Z be its inverse image under the embedding $Z \times C \hookrightarrow J(C) \times C$. Hence we have the following diagram

$$\begin{array}{ccccc} & & P_Z & & P \\ & & Z \times C & \hookrightarrow & J(C) \times C \\ & \swarrow^{q} & \downarrow^{p} & & \downarrow \\ C & & Z & \hookrightarrow & J(C) \end{array}$$

Consider the exact sequence

$$0 \to \underbrace{P_Z \otimes q^*(\mathcal{O}_C((d-1)P_0))}_{E_1} \to \underbrace{P_Z \otimes q^*(\mathcal{O}_C(dP_0))}_{E_2} \to \underbrace{P_Z \otimes q^*(\mathcal{O}_C(dP_0)) \otimes \mathcal{O}_{Z \times P_0}}_{F} \to 0.$$

Because $R^1p_*(F)=0$ (see e.g. [12], p. 279), we have an exact sequence of $\mathcal{O}_Z$-modules

$$0 \to p_*(E_1) \to p_*(E_2) \xrightarrow{\phi} p_*(F) \to R^1p_*(E_1) \xrightarrow{g} R^1p_*(E_2) \to 0. \qquad (*)$$

Let x be a point on Z. We write $P_{Z,x}$ to denote the inverse image of P_Z under the embedding of the fibre of p at x into $Z \times C$ (i.e. $P_{Z,x} \simeq \mathcal{O}_C(D-dP_0)$ if $x = I(d)(D)$). We have

$$\begin{array}{lll} h^0(P_{Z,x}(dP_0)) = r+1 & \text{because } x \in W_d^r \setminus W_d^{r+1} & \\ h^1(P_{Z,x}(dP_0)) = r-d+g & \text{(Riemann–Roch)} & \\ h^0(P_{Z,x}((d-1)P_0)) = r & \text{because } x \notin W_{d-1}^r & (1) \\ h^1(P_{Z,x}((d-1)P_0)) = r-d+g & \text{(Riemann–Roch)} & \\ h^0(P_{Z,x}(dP_0) \otimes \mathcal{O}_{P_0}) = 1 & & \end{array}$$

Hence, we can use Grauert's theorem (see e.g. [12], p. 288) to conclude that $(*)$ is a sequence of vector bundles. Consider the induced exact sequence

$$0 \to \mathrm{Ker}(g) \to R^1p*(E_1) \to R^1p_*(E_2) \to 0.$$

Because $R^1p_*(E_2)$ is locally free, tensoring with the residue field $\mathcal{O}_Z(x)=\mathcal{O}_{Z,x}/\mathcal{M}_{Z,x}$ and using Grauert's theorem again, we obtain the exact sequence

$$0\to(\operatorname{Ker} g)(x)\to H^1(C, P_{Z,x}((d-1)P_0))\xrightarrow{g(x)} H^1(C, P_{Z,x}(dP_0))\to 0.$$

But $g(x)$ is an isomorphism, hence $(\operatorname{Ker} g)(x)=0$. From Nakayama's lemma we obtain $(\operatorname{Ker} g)_x=0$ (stalk!) hence $\operatorname{Ker} g=0$. So, we have an exact sequence

$$0\to p_*(E_1)\to p_*(E_2)\xrightarrow{\phi} p_*(F)\to 0.$$

Consider the cartesian diagram

$$\begin{array}{ccc} C_Z^{(d)} & \xrightarrow{I_Z(d)} & Z \\ \cap\downarrow & & \cap\downarrow \\ C^{(d)} & \xrightarrow{I(d)} & J(C) \end{array}$$

and define

$$C_Z^{(d)}(P_0)=\{E\in C_Z^{(d)}: E\geqslant P_0\}.$$

From [3], p. 309, Proposition 2.1 (recall that $Z\cap W_d^{r+1}$ is empty) we conclude that $I_Z(d)$ can be identified with the natural morphism

$$\mathbf{P}(p_*(E_2))\to Z$$

and $\mathcal{O}_{\mathbf{P}(p_*(E_2))}(1)\simeq\mathcal{O}_{C_Z^{(d)}}(C_Z^{(d)}(P_0))$. As is explained in [3], p. 310, Proposition 2.2, this implies that the dual vector bundle $p_*(E_2)^D$, and also $p_*(E_2)^D\otimes p_*(F)$, are ample vector bundles on Z. Since rank $(p_*(E_2))=r+1$, $\operatorname{rank}(p_*(F))=1$ and $\dim(Z)\geqslant r+1$ we obtain from [3], p. 307, Proposition 1.3 that there exists a point z on Z such that rank $(\phi(z))=0$. This is a contradiction to the surjectivity of ϕ. □

1.2.2 *Proof of Theorem 1.2.* Let us write V_f^n instead of $V_f^n(g_d^r)$. We proceed by induction for f. It is clear that $V_{n+1}^n=C^{(n+1)}\neq\phi$.

Let $f\in\mathbf{Z}$ with $e>f\geqslant n+1$ and assume that V_f^n is not empty. Let Z be an irreducible component of V_f^n and consider the map

$$i: Z\to J(C): E\mapsto l-I(f)(E)$$

with $l \in J(C)$ defined by $g_d(l) = g_d^r$. (Note that g_d^r is complete, by assumption). Clearly $i(Z) \subset W_{d-f}^{r-n-1}$.

Suppose that the general non-empty fibre of i has dimension at most $r-n-2$. Since $f<e$ it follows from the hypothesis of the theorem that $\dim(Z) \geqslant (n+1-f)(r-n)+f \geqslant 2r-2n-2$. Thus $i(Z)$ is then an irreducible closed subset of W_{d-f}^{r-n-1} of dimension at least $r-n$. From Lemma (1.2.1) it follows that $i(Z)$ intersects $W_{d-(f+1)}^{r-n-1}$. Let $x \in i(Z) \cap W_{d-(f+1)}^{r-n-1}$ and let $E \in Z$ such that $i(E) = x$.

Suppose that $x \notin W_{d-f}^{r-n}$. Then P_0 is a fixed point of $g_d^r(-E)$. Thus we have $\dim(g_d^r(-E-P_0)) \geqslant r-n-1$, whence $E+P_0 \in V_{f+1}^n$.

Suppose that $x \in W_{d-f}^{r-n}$. Then we have $\dim(g_d^r(-E)) \geqslant r-n$, i.e. $E \in V_f^{n-1}$. Hence $E+P \in V_{f+1}^n$ for each $P \in C$.

Altogether, we proved that V_{f+1}^n is not empty if the general non-empty fibre of i has dimension at most $r-n-2$. Now, suppose that the general non-empty fibre of i has dimension at least $r-n-1$. In that case $I(f)(Z) \subset W_f^{r-n-1}$. Let $E \in Z$ and let $F \in g_d^r(-E)$. Since g_d^r is complete we have that $F+|E| \subset g_d^r$. It follows that $F \in V_{d-f}^n$. Since $d-f \geqslant f+1$ (we assumed $2e \leqslant d+1$) we see again that V_{f+1}^n is not empty, and Theorem (1.2) is thereby proved. □

Applying Theorem (1.2) to a base point free and simple g_d^r on C we obtain Theorem A if $n=r-2$ and $e=2r-3$. Finally, we are going to discuss the bound $d \geqslant 4r-7$ of Theorem A a little bit more closely.

1.3 EXAMPLE. (a) For $r=3$ the bound is sharp: an elliptic curve of degree 4 in $\mathbf{P}^3$ has no 3-secant line.

(b) For $r=4$ the bound is sharp: a general canonically embedded curve C of genus 5 in $\mathbf{P}^4$ has degree 8 and no 5-secant 2-plane since a general curve of genus 5 has no g_3^1.

(c) For $r=5$ the bound is not sharp: Indeed, any linearly normal curve C of degree 12 in $\mathbf{P}^5$ has a 7-secant 3-space. This follows from the fact that, if such a curve has a linear system g_5^1, then this has to be obtained from a pencil of hyperplanes in $\mathbf{P}^5$ containing a 7-secant 3-space divisor of C. By Brill–Noether theory, C has a g_5^1 if $g \leqslant 8$. So let $g \geqslant 9$. Castelnuovo's genus bound ([3], p. 116) gives us $g \leqslant 10$. If $g=9$ (resp. $g=10$), then C has a g_4^1 (resp. a g_6^2) residual to the simple g_{12}^5, and we see that C likewise has a g_5^1. Thus, for $r=5$ we have the better bound $d \geqslant 12$ in Theorem A, and this bound is sharp. In fact, if C is a general curve of genus 7 and if P is a general point on C, then $|K_C-P|$ is a very ample linear system g_{11}^5 on C. Since C has no linear systems g_4^1, the associated embedding of C in $\mathbf{P}^5$ has no 7-secant 3-plane.

(d) Using case by case analysis we checked that $3r-3$ is the best bound for the degree d in Theorem A, for $4 \leqslant r \leqslant 7$. □

1.4 PROBLEM. Is Theorem A valid also for curves in $\mathbf{P}^r$ which are not linearly normal, or do we have to change the bound?

2. On Clifford's theorem

A famous theorem in the theory of special divisors on curves is *Clifford's theorem* (1878) which is an easy consequence of the Riemann–Roch theorem and reads as follows ([6], p. 329):

CLIFFORD'S THEOREM. *Let C be a curve of genus g and let $D \in C^{(d)}$. If* $\dim(|D|) > d - g$ *then* $2\dim(|D|) \leqslant d$. □

Motivated by this theorem, Martens [22] introduced in 1968 a new invariant of C which he called the Clifford index of C.

2.1 DEFINITION. Let $D \in C^{(d)}$. The *Clifford index* of D is defined by

$$\mathrm{cliff}(D) := d - 2h^0(D) + 2.$$

D (or $|D|$) is said to *contribute to the Clifford index* if both $h^0(D) \geqslant 2$ and $h^1(D) \geqslant 2$. Finally, the *Clifford index of C* is defined by

$$\mathrm{cliff}(C) = \min(\{\mathrm{cliff}(D) : D \text{ contributes to the Clifford index}\})$$

if $g \geqslant 4$. □

In terms of these definitions, the essence of Clifford's theorem may simply be stated as $\mathrm{cliff}(C) \geqslant 0$. Furthermore, it is classically known (due to M. Noether, Bertini, C. Segre and often included in the statement of Clifford's theorem) that $\mathrm{cliff}(C) = 0$ if and only if C is hyperelliptic. As a consequence of (the closing lines in) [9] all curves of a given Clifford index $c \leqslant 33$ are classified. (The main conjecture in [9] states such a classification for every Clifford index.) This classification indicates a close connection between the Clifford index c and the gonality k of C, given by the inequalities $c+2 \leqslant k \leqslant c+3$. We will prove now that these inequalities are in fact true. Let us first recall the definition of the old invariant "gonality" of C.

2.2 DEFINITION. A smooth curve C is called *k-gonal* (and k its gonality) if C possesses a pencil g^1_k but no g^1_{k-1}. □

Clearly, $\mathrm{cliff}(C) = c$ implies $W^1_{c+1} = \varnothing$ whence C has gonality $k \geqslant c+2$. On the other hand, the following theorem tells us that we also have the non-trivial relation $k \leqslant c+3$.

2.3 THEOREM. *If* $\mathrm{cliff}(C) = c$ *then* $\dim(W^1_{c+3}) \geqslant 1$.

Proof. If C is $(c+2)$-gonal then $W^1_{c+2} \oplus W^0_1 \subset W^1_{c+3}$, hence $\dim(W^1_{c+3}) \geqslant 1$. Suppose that C is not $(c+2)$-gonal. Then there must exist a divisor D on C satisfying $h^0(D) = r+1 \geqslant 3$, $\deg(D) = c+2r$, $h^1(D) \geqslant 2$. Choose D such that r is minimal. Then $|D|$ is very ample ([9], Lemma 1.1). If $2r = c+3$ we have

$\dim(W^1_{c+3}) \geq 1$ according to [9], §3. Let $2r \neq c+3$. Then $4r \leq c+6$ (see [9], Corollary 3.5), i.e. $d = \deg(D) = c + 2r \geq 6r - 6$. Thus Theorem A implies that $V^{r-2}_{2r-3} = V^{r-2}_{2r-3}(|D|)$ is not empty. Let Z be an irreducible component of V^{r-2}_{2r-3}. We know that $\dim(Z) \geq 1$. More geometrically, if we embed C in $\mathbf{P}^r$ via $|D|$ we obtain infinitely many $(2r-3)$-secant $(r-2)$-planes for C. Let S be such a plane. Then the projection of C onto $\mathbf{P}^1$ with center S gives a g^1_{c+3} on C. If the $2r-3$ points of C on S vary in a non-trivial linear system, then $2r-5 \geq c = d-2r$ (since this linear system contributes to the Clifford index), and we obtain the contradiction $d \leq 4r-5$. Thus different $(2r-3)$-secant $(r-2)$-planes induce (by projection) different g^1_{c+3} on C. □

2.3.1 COROLLARY. *If* $\dim(W^1_d) = 0$ *then* $\mathrm{cliff}(C) = d-2$.

Proof. Since W^1_d is not empty, one has $\mathrm{cliff}(C) \leq d-2$. If $\mathrm{cliff}(C) = d-2-\varepsilon$ for some $\varepsilon \geq 1$ then it follows from Theorem (2.3) that $\dim(W^1_{d+1-\varepsilon}) \geq 1$ and therefore $\dim(W^1_d) \geq 1 + (\varepsilon - 1) \geq 1$. This is a contradiction. □

2.3.2 COROLLARY (Ballico's theorem [4]). *If C is a general k-gonal curve, then* $\mathrm{cliff}(C) = k-2$.

Proof. For $g = 2k-3$ this follows from Brill–Noether theory. Assume $k < (g+3)/2$. According to an old theorem of B. Segre a general k-gonal curve then has only a finite number of linear systems g^1_k ([24], see also [2]), hence we can apply Corollary (2.3.1). □

Note that Corollary (2.3.2) implies that each integer c, $0 \leq c \leq (g-1)/2$, occurs as the Clifford index of a smooth curve of genus g. The three other proofs of Ballico's theorem known to us ([4], [13], [23]) are not concrete: they do not indicate which k-gonal curves have Clifford index $k-2$. Just to give a concrete example, recall that a non-degenerate curve C in $\mathbf{P}^r$ is called *extremal* if the genus of C is maximal with respect to the degree of C (cf. [3], p. 117 or [8]).

2.3.3 EXAMPLE. Let C be an extremal curve of degree $d > 2r$ in $\mathbf{P}^r$ $(r \geq 3)$. There are two cases [1]:

(i) C lies on a rational normal scroll X in $\mathbf{P}^r$. Write $d = m(r-1) + 1 + \varepsilon$ where $\varepsilon = 1, 2, \ldots, r-1$. C has only finitely many pencils of degree $m+1$ (in fact, only one for $r > 3$, one or two if $r = 3$); these pencils are swept out by the rulings of X. Thus $\mathrm{cliff}(C) = m-1$ by Corollary (2.3.1).

(ii) C is the image of a smooth plane curve C' of degree $d/2$ under the Veronese map $\mathbf{P}^2 \to \mathbf{P}^5$. Then $r = 5$ and $\mathrm{cliff}(C) = \mathrm{cliff}(C') = (d/2) - 4$ (e.g. [19]). Note that in this case $W^1_{c+2} = \varnothing$, $\dim W^1_{c+3} = 1$ if $c = \mathrm{cliff}(C)$, so Corollary (2.3.1) cannot be applied. □

The main conjecture in [9] states that every k-gonal curve C has Clifford index $c = k-2$ unless C is a smooth plane curve of degree $d \geq 5$ or one of those "exceptional" curves constructed and studied in [9].

Next we want to prove Theorem B of the Introduction.

2.4 *Proof of Theorem B.* Assume that C is a k-gonal curve ($k \geqslant 3$) and assume that $g^r_d = |D|$ is a complete linear system on C satisfying $k-3 \leqslant d \leqslant 2g-2-(k-3)$ and $2r > d-(k-3)$. Clearly, $h^0(D) = r+1 \geqslant 2$, and using the Riemann–Roch theorem, we obtain from our numerical conditions for d and r that

$$h^1(D) = h^0(D) + g - d - 1 \geqslant 2.$$

Hence $|D|$ contributes to the Clifford index. Therefore

$$\mathrm{cliff}(C) \leqslant \mathrm{cliff}(D) = d - 2r < k - 3.$$

From Theorem (2.3) we obtain that $\dim(W^1_{k-1}) \geqslant 1$, a contradiction to the fact that C is k-gonal. □

For curves of large genus with respect to the gonality we have the following improvement of Theorem B (closely related to [3], p. 138, Exercise B-7). For short, let us call C here a *double curve* if there exists a curve C' and a ramified covering $\pi: C \to C'$ of degree 2.

2.4.1 PROPOSITION. *Let g^r_d be a linear system on C satisfying $0 \leqslant d \leqslant g-1$. If $3r > d$ then we have one of the following two possibilities*:

(i) *$d = 3r-1$ and g^r_d embeds C in $\mathbf{P}^r$ as an extremal curve*; *or*
(ii) *C is a double curve of even gonality k, and one has $2r \leqslant d - 2(k-3)$.*

Proof. Let g^r_d be a complete linear system on C satisfying $0 \leqslant d \leqslant g-1$ and $3r > d$. There are two cases:

(i) g^r_d is simple. From Castelnuovo's bound we obtain

$$g \leqslant \pi(d, r) = m(d - 1 - (m + 1)(r - 1)/2) \quad \text{where} \quad m = \left[\frac{d-1}{r-1}\right]$$

(cf. [3], p. 116 or [8]). Using the facts $d \leqslant g-1$ (hypothesis) and $2r \leqslant d < 3r$ (by assumption and by Clifford's theorem) a straightforward calculation shows that $d = 3r-1$ and $g = \pi(d, r) = 3r$ is the only possibility.

(ii) Suppose g^r_d is not simple. We are going to prove that the second claim of our proposition holds. We can assume that g^r_d is complete and has no fixed points. Consider the map $C \to \mathbf{P}^r$ associated to g^r_d; let C' be the normalization of the image curve in $\mathbf{P}^r$ and assume that $\deg(\varphi: C \to C') = n \geqslant 2$. Then C' possesses a complete linear system $g'^r_{d/n}$ such that $\varphi^*(g'^r_{d/n}) = g^r_d$. If $g'^r_{d/n}$ would be a special linear system on C', then—because of Clifford's theorem—$2r \leqslant d/n < 3r/n$, which gives us a contradiction. Let g' be the genus of C'. By Riemann–Roch, then, $g' = (d/n) - r < (3r/n) - r = (3-n) \cdot (r/n)$. Since $g' \geqslant 0$ one obtains $n = 2$. Thus

C is a double curve, and $g' < r/2$. From $g' = (d/2) - r = (d - 2r)/2$ we see that $g > d = 2g' + 2r > 6g'$. Assume that C is k-gonal and consider a map $\psi: C \to \mathbf{P}^1$ of degree k. If ψ does not factor through C', then—according to a genus bound of Castelnuovo for curves with morphisms (see [19], §1 or [25])—one has $g \leqslant k - 1 + 2g'$. Since $k \leqslant (g+3)/2$ (by Brill–Noether theory) we obtain $g \leqslant 4g' + 1$, contradicting $g > 6g'$. Thus ψ factors, and k is twice the gonality of C'. But then (by Brill–Noether applied to C') $k \leqslant g' + 3 = [(d - 2r)/2] + 3$. This gives us the bound stated in the proposition. □

2.4.2 REMARK. The bound in Proposition (2.4.1)(ii) is sharp if and only if C is a double covering of a curve C' of odd genus g' which is $(g'+3)/2$-gonal (and if the genus g of C is large enough). Indeed, assume that we have equality in Proposition (2.4.1)(ii). Then the curve C' in the above proof has genus $g' = (d - 2r)/2 = k - 3$ and gonality $k/2 = (g'+3)/2$. Conversely, assume that $\varphi: C \to C'$ is a double covering with C' a curve of genus g' and gonality $(g'+3)/2$. Let r be such that $g \geqslant 3r > 2g' + 2r$. Then $\varphi^*(g'^r_{g'+r})$ is a linear system of degree $d = 2g' + 2r$ and dimension r on C for which equality holds in (ii) since the proof of (ii) shows that the gonality k of C is twice the gonality $(g'+3)/2$ of C'. □

2.4.3 COROLLARY. *Let C be a curve of odd gonality and let g^r_d be a linear system on C with $0 \leqslant d \leqslant g - 1$. Then $3r \leqslant d$.*

Proof. Indeed, according to Example (2.3.3), a curve satisfying (i) of Proposition (2.4.1) is 4-gonal or 6-gonal. □

2.4.4 EXAMPLE. For trigonal curves C the bound in Corollary (2.4.3) is sharp. Of course, multiples of the linear system g^1_3 attain the bound. The only other possibility is the case in which $g = 3r + 1$ and g^r_{3r} is residual to rg^1_3 (see [20], §1).

Assume there is a g^r_{3r}, $6 \leqslant 3r < g$, on a curve C of genus g which is neither trigonal nor a double curve of even gonality. Along the lines of the proof of Proposition (2.4.1) it can be shown that the g^r_{3r} on C is a complete base point free and simple linear system. According to [8], (3.15), if we view C via g^r_{3r} as a curve of degree $3r$ in $\mathbf{P}^r$ it must lie on a surface of degree r or less, i.e. on a scrollar resp. on a del Pezzo surface. Consequently, it is not hard to check then that C has gonality $k \leqslant 6$ or $k = 8$. (In the latter case $r = 9$, and C is the image of a smooth plane nonic under the Veronese embedding $\mathbf{P}^2 \to \mathbf{P}^9$.) In particular, we see that the bound in Corollary (2.4.3) is not sharp for curves of odd gonality $k \geqslant 7$.

However, for $r \leqslant 5$ there are some 5-gonal curves admitting a g^r_{3r}: Clearly, a smooth plane sextic ($g = 10$) has a g^2_6. Adopting the notation of [12], V, 2 any smooth member of the linear system $|5C_0 + 7f|$ ($g = 14$) on the rational normal scroll $X_1 \subset \mathbf{P}^4$ has a g^4_{12}. Similarly, a smooth member of $|5C_0 + 5f|$ on $X_0 \subset \mathbf{P}^5$ (resp. $|5C_0 + 10f|$ on $X_2 \subset \mathbf{P}^5$) is a smooth curve of degree 15 in $\mathbf{P}^5$ of genus 16. This curve can also be identified as an extremal space curve of degree 10 thus lying on a smooth quadric surface (resp. on a quadric cone) in $\mathbf{P}^3$. □

3. On linear systems computing the Clifford index

3.1 DEFINITION. Let C be a smooth curve of genus g with $\mathrm{cliff}(C)=c$. Let g_d^r be a linear system on C contributing to the Clifford index. We say that g_d^r *computes the Clifford index if* $d \leqslant g-1$ and $d-2r=c$; note that such a linear system is complete and base point free. Moreover ([14]), for $r \geqslant 3$ it is simple unless C is hyperelliptic or bi-elliptic (i.e. a double covering of an elliptic curve). □

Before proving Theorem C of the Introduction, we have to prove some preliminary results. We start by recalling part of a lemma in [9] (cf. [9], Lemma 3.1) whose proof is an application of the base-point free pencil trick.

3.1.1 LEMMA. *Let D be a divisor of C computing the Clifford index of C. Let M be a divisor of C of degree m such that $|M|$ is base point free. If* $\deg(D)=g-1$ *we assume that* $m \neq 2h^0(D)-1$. *Then we have* $h^0(D-M) \geqslant h^0(D)-(m/2)$. □

3.1.2 COROLLARY. *Assume g_d^r ($d \leqslant g-1$) is a linear system on C computing the Clifford index c of C. Then any complete base point free linear system on C of degree $0<m<2r$ computes the Clifford index and has even degree.*

Proof. Let $D \in g_d^r$ and let E be an effective divisor on C of degree $m<2r$ such that $|E|$ is base point free.

Claim. $|D-E|$ computes the Clifford index of C.

Indeed, from Lemma (3.1.1) we obtain that

$$2h^0(D-E) \geqslant 2h^0(D)-m=2r+2-m>2,$$

hence $h^0(D-E) \geqslant 2$. Since $h^1(D) \geqslant 2$ we certainly have $h^1(D-E) \geqslant 2$, hence $|D-E|$ contributes to the Clifford index. By definition of the Clifford index, we have

$$d-m-2h^0(D-E)+2 \geqslant c=d-2r.$$

Comparing it with the above lower bound on $2h^0(D-E)$ we obtain

$$2h^0(D-E)=2r+2-m,$$

i.e. $|D-E|$ computes the Clifford index and m is even.

Now, we are ready to prove that $|E|$ computes the Clifford index. From the fact that $|E|$ is base point free (hence $h^0(E) \geqslant 2$) and $m<2r \leqslant d \leqslant g-1$, we obtain that $|E|$ contributes to the Clifford index. It follows that $m-2 \geqslant c=d-2r$, and therefore $d-m<2r$. Thus in our claim above we may replace E by $F \in |D-E|$, which implies that $|D-F|=|E|$ computes the Clifford index. □

3.2 *Proof of Theorem C.* Let C be a curve of genus g which is not hyperelliptic or bi-elliptic and let g_d^r $(d \leqslant g-1)$ be a linear system on C computing the Clifford index c of C.

3.2.1 *Claim.* If C has a base point free linear system g_{c+3}^1, then $d \leqslant 2c+3$. Indeed, from Corollary (3.1.2) we obtain that $c+3 \geqslant 2r$, hence

$$2c+3 \geqslant c+2r=d.$$

Because of Theorem (2.3) this completes the proof of Theorem C if C is not $(c+2)$-gonal.

3.2.2 *Claim.* If C has a base-point free linear system g_{c+2}^1 and if c is odd, then $d \leqslant 2c+1$.

Indeed, in this case $c+2$ is also odd, hence from Corollary (3.1.2) it follows that $c+2 \geqslant 2r$, whence

$$2c+2 \geqslant c+2r=d.$$

Since c is odd, so is d, and we obtain our claim. This completes the proof of Theorem C for odd Clifford index c.

Suppose c is even and C has a linear system g_{c+2}^1 (of course, being base point free). Again, if $c+2 \geqslant 2r$, then we obtain $2c+2 \geqslant d$, so we can assume that $c+2<2r$. From the claim in the proof of Corollary (3.1.2) we see

3.2.3 *Claim.* If $d \geqslant 2c+4$ and if $D \in g_d^r$, $E \in g_{c+2}^1$ then $|D-E|$ computes the Clifford index.

It follows that $|D-E|$ is a linear system g_{d-c-2}^s satisfying $(d-c-2)-2s=c$, hence $d=2c+2+2s$. It is enough to prove the following

3.2.4 *Claim.* $\dim(|D-E|)=1$ (i.e $s=1$).

First, assume $s \geqslant 3$. Since we assumed C not to be hyperelliptic or bi-elliptic, we know that $|D-E|$ is simple (see [14]). Let F be a general element of $C^{(s-1)}$. Because of the General Position theorem ([3], p. 109), $|D-E-F|$ is a linear system $g_{d-(c+1+s)}^1$ on C without fixed points. But $d-(c+1+s)=d/2$. From the assumption $c+2<2r$ it follows that $d=c+2r<4r-2$, hence $d/2<2r$. From Corollary (3.1.2) we obtain that $|D-E-F|$ computes the Clifford index, i.e. $(d/2)-2=c$. But then we have the contradiction $2c+4=d=2c+2+2s \geqslant 2c+8$.

Assume $s=2$. If $|D-E|$ is simple then we obtain a contradiction as before. Hence $|D-E|$ is not simple. Consider the associated morphism $\varphi' : C \rightarrow \mathbf{P}^2$ and let C' be the normalization of $\varphi'(C)$. Let $\varphi : C \rightarrow C'$ be the associated ramified covering. Then $n=\deg(\varphi) \geqslant 2$, and C' has a complete linear system $g_{d'}^2$ with $d'=(d-c-2)/n=(c+4)/n$ and $|D-E|=\varphi^*(g_{d'}^2)$. If $g_{d'}^2$ is not very ample on C', then C' has a linear system $g_{d'-2}^1$ and $\varphi^*(g_{d'-2}^1)$ is a linear system g_{c+4-2n}^1 on C

contributing to the Clifford index. Hence $c+2-2n \geqslant c$, which is a contradiction. Thus C' is a smooth plane curve of degree d'. Consider a linear system $g^1_{d'-1}$ on C'. Then $\varphi^*(g^1_{d'-1})$ is a linear system g^1_{c+4-n} on C contributing to the Clifford index. Hence $c+2-n \geqslant c$ which implies $n=2$. In this case $\varphi^*(g^1_{d'-1})$ computes the Clifford index. Since C' has infinitely many linear systems $g^1_{d'-1}$, C has infinitely many linear systems g^1_{c+2}. On all these linear systems one can apply Claim (3.2.3), all giving rise to the same value for s—assumed to be 2 here. So, we find an infinite number of linear systems g^2_{c+4} on C giving rise to double coverings of C over smooth plane curves C' of degree $(c/2)+2$. Our assumptions imply that C' is not rational and not elliptic. But then, the induced linear system $g'^2_{(c/2)+2}$ on every C' is unique. All together, this shows that there exists an infinite number of double coverings $C \to C'$ with $g(C') > 1$. This is impossible (e.g. cf. [18], Lemma 4), and we have proved Claim (3.2.4) and Theorem C. □

Our next result (which is in fact equivalent to Theorem C) improves a result in [14].

3.2.5 COROLLARY. *Let C be a curve of genus $g > 2c+4$ resp. $g > 2c+5$ if c is odd resp. even. Then, for a linear system g^r_d $(d \leqslant g-1)$ computing c, we have $d \leqslant 3(c+2)/2$ unless C is hyperelliptic or bi-elliptic.*

Proof. Let $g^r_d = |D|$. Leaving aside the discussion for $c \leqslant 2$ (see [14]) we assume $c \geqslant 3$. Then $d \leqslant 2c+4$ by Theorem C. But $3(c+2)/2 < d < 2(c+2)$ implies $g \leqslant 2c+4$ ([14], Cor. 1) contradicting our hypothesis on g. Thus $d \leqslant 3(c+2)/2$ or $d = 2c+4$. Assume that $d=2c+4$. Then c is even (since $d \equiv c \bmod 2$) and $g < 3(c+1)$ (see [14], Cor. 2). From Claim (3.2.1) we conclude that C has a g^1_{c+2}, $|M|$ say, and from Claim (3.2.3) we know that $|D-M|$ is again a g^1_{c+2}. Thus

$$h^0(D+M) \geqslant 2h^0(D) - h^0(D-M) = (2r+2)-2 = 2r = d-c = c+4,$$

hence

$$\text{cliff}(D+M) = 3c+6-2h^0(D+M)+2 \leqslant c.$$

If $h^1(D+M) \leqslant 1$, then by Riemann–Roch we have $c+4 \leqslant h^0(D+M) \leqslant 3c+8-g$, whence the contradiction $g \leqslant 2c+4$. Therefore, $|N| = |K_C - (D+M)|$ computes c and $h^0(D+M) = c+4$. By Riemann–Roch, then,

$$g = \deg(D+M) + 1 - h^0(D+M) + h^1(D+M) = 2c+3+h^0(N).$$

Since $g > 2c+5$, we see that $h^0(N) \geqslant 3$.

Assume that $|N|$ is simple. By the Uniform Position principle we then have

$$h^0(D+N) + h^0(D-N) \geqslant 2h^0(D) + h^0(N) - 2$$

provided that $n := \deg(N) \geqslant h^0(D) + h^0(N) - h^0(D-N) - 1$ ([3], III, Ex. B-6, note the misprint there). But the inequality for n is clearly satisfied since

$$\begin{aligned} 2h^0(D-N) &\geqslant 0 \geqslant 2 - c = (c+6) - 6 - 2c + 2 \geqslant 2h^0(D) - 2h^0(N) - 2c + 2 \\ &= 2h^0(D) - 2h^0(N) - 2(n - 2h^0(N) + 2) + 2 \\ &= 2h^0(D) + 2h^0(N) - 2n - 2. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} d + n - 2h^0(D+N) + 2 &= \mathrm{cliff}(D+N) = \mathrm{cliff}(K_C - M) = \mathrm{cliff}(M) = c \\ &= \mathrm{cliff}(D) = d - 2h^0(D) + 2, \text{ i.e. } 2h^0(D+N) = 2h^0(D) + n. \end{aligned}$$

Thus

$$\begin{aligned} 2h^0(D-N) &\geqslant 4h^0(D) + 2h^0(N) - 4 - 2h^0(D+N) \\ &= 2h^0(D) - n - 4 + 2h^0(N) = 2(h^0(D) - 1) - (n - 2h^0(N) + 2) \\ &= c + 4 - c = 4, \text{ i.e. } h^0(D-N) \geqslant 2. \end{aligned}$$

But

$$|D - N| = |2D + M - K_C| = |M - (K_C - 2D)|$$

and $\deg(K_C - 2D) > 0$ because $g > 2c + 5 = d + 1$. Since C has no g^1_{c+1}, this is a contradiction.

This contradiction shows that $|N|$ is not simple. This implies that $|N| = g^2_{c+4}$ and that C is a double covering of a smooth plane curve C' of degree $d' = (c/2) + 2$. (Cf. the proof of Theorem C.) Clearly C' has infinitely many linear systems $g'^1_{d'-1}$ which induce on C an infinite number of $|M| = g^1_{c+2}$ and thus infinitely many linear systems $|N| = |(K_C - D) - M|$. To get a contradiction we now may proceed as in the last part of the proof of Theorem C (replacing D by its dual $K_C - D$ there). □

The results given in Theorem C and its Corollary are best possible. This is shown by the following examples.

3.2.6 EXAMPLE (c even). Let X be a general $K3$ surface in $\mathbf{P}^r$ ($r \geqslant 3$). Then $\mathrm{Pic}\, X$ is generated by (the class of) a hyperplane section H, and $\deg X = (H^2) = 2r - 2$. Let C be a smooth irreducible curve on X contained in the linear system $|nH|$ of X, for $2 \leqslant n \in \mathbf{N}$. Then C is a $(1/n)$-canonical curve (i.e. $\mathcal{O}_C(n)$ is the canonical bundle of C) of genus $g = (nH)^2/2 + 1 = n^2(r-1) + 1$ and degree $d = (nH \cdot H) = 2n(r-1)$.

According to Green's and Lazarsfeld's method of computing the Clifford index of smooth curves on a $K3$ surface (cf. [21]) there has to be a smooth curve B on X such that $(B\cdot C)\leqslant g-1$ and $\mathcal{O}_X(B)\otimes\mathcal{O}_C$ computes the Clifford index of C. But since $\operatorname{Pic} X\cong \mathbf{Z}\cdot H$ we clearly have $B\in|H|$. Thus $\mathcal{O}_C(1)$ computes c, and we have $c=d-2r=2(n-1)(r-1)-2$. In particular,

$$g=\frac{n^2}{2(n-1)}(c+2)+1 \quad \text{and} \quad d=\frac{n}{n-1}(c+2).$$

Now we specialize to the cases $n=2$ and $n=3$. Then we obtain

$$d=2c+4=g-1 \quad \text{for } n=2;\ d=3(c+2)/2,\ g=(9(c+2)/4)+1 \quad \text{for } n=3,$$

whence Theorem C and its Corollary are best possible for even c. The simplest examples ($r=3$) are smooth complete intersections of X with a quadric resp. a cubic surface Y in $\mathbf{P}^3$. If Y is quadric C clearly has two resp. only one g^1_4 (computing c) according to Y is smooth resp. a cone. Let C be a smooth complete intersection of X with a cubic Y in $\mathbf{P}^3$ ($c=6$; $d=12$; $g=19$). Then C has a quadrisecant line (Cayley's formula—see e.g. [3], p. 351—is nonzero in this case), and the projection $C\to\mathbf{P}^1$ with center a quadrisecant line gives a g^1_8 (computing c) on C. Moreover, the g^1_8 on C are in 1-1-correspondence with the quadrisecant lines of C: We have

$$\dim(g^3_{12}+g^1_8)\geqslant 2\dim(g^3_{12})-\dim(g^3_{12}-g^1_8)=6-\dim(g^3_{12}-g^1_8)$$

and

$$-1\leqslant \dim(g^3_{12}-g^1_8)\leqslant 0$$

since C has no g^1_4. If $\dim(g^3_{12}-g^1_8)=-1$ we have $\dim(g^3_{12}+g^1_8)\geqslant 7$ whence there is a g^7_{20} on C inducing a g^5_{16}, by duality. But the g^5_{16} computes the Clifford index $c=6$ of C, and $16=2c+4$. Thus, by Corollary (3.2.5), we obtain a contradiction. Therefore, $\dim(g^3_{12}-g^1_8)=0$, and we see that every g^1_8 on C comes from a projection with center a quadrisecant line of C. Clearly, the quadrisecant lines of C all lie on the unique cubic surface Y containing C. If Y is smooth (for example Clebsch' diagonal surface) there are exactly 27 lines on Y all of which are easily seen to be quadrisecant lines of C. This is in accordance with Cayley's formula computing—with multiplicities—the number m of quadrisecant lines of a smooth space curve of given genus and degree, provided that m is finite. Note that C is the strict transform of a plane curve of degree 12 with six singular points $P_1,\ldots,P_6$ of multiplicity 4, under the natural map $Y\to\mathbf{P}^2$ defined by blowing up $P_1,\ldots,P_6$ ([12], p. 402). To see the other types of smooth complete intersections of X with a cubic surface Y in $\mathbf{P}^3$ we move these six points in special positions in $\mathbf{P}^2$ (including the consideration of infinitely near points). The number of g_8^1 on C depends then on the speciality of this situation. This will be described in terms of the resulting singularities of Y. So assume that Y is not smooth. Then Y can only have isolated singularities. In fact, a cubic surface in $\mathbf{P}^3$ with a double curve either is reducible or rationally ruled with a double line. The first case clearly is impossible, and in the second case the ruling would define a g_4^1 on C. Now, if Y has no triple point it is a classical fact that Y has at most four rational double points, and the number of lines on Y is then determined by the type and the number of the rational double points. We have the 20 possibilities presented in the following table ([5]) where the type of the singularities is expressed in terms of Coxeter-diagrams (A-D-E-singularities).

However, if Y has a triple point (type $\tilde{E}_6$) Y is an elliptic cone whence C is a (4 : 1)—covering of an elliptic curve and carries infinitely many g_8^1.

Type and number of rational double points of the cubic Y	Number of g_8^1 on C
A_1 (ordinary double point)	21
$2A_1$	16
$3A_1$	12
$4A_1$	9
A_2	15
A_2, A_1	11
A_2, $2A_1$	8
$2A_2$	7
$2A_2$, A_1	5
$3A_2$	3
A_3	10
A_3, A_1	7
A_3, $2A_1$	5
A_4	6
A_4, A_1	4
A_5	3
A_5, A_1	2
D_4	6
D_5	3
E_6	1

□

3.2.7 EXAMPLE (c odd). Let X be a $K3$ surface in $\mathbf{P}^r$ ($r \geqslant 3$) containing a single line E such that $\operatorname{Pic} X \simeq \mathbf{Z} \cdot H \oplus \mathbf{Z} \cdot E$, $\deg X = (H^2) = 2r-2$, $(E^2) = -2$, $(H \cdot E) = 1$. (This is possible, see [9], Lemma 4.2.) Let C be a smooth element of $|2H+E|$. Then C is a half-canonical curve of genus $g = 1 + ((2H+E)^2/2) = 4r-2$ and degree $d = ((2H+E)\cdot H) = 4r-3$. In [9], Theorem 4.3 it is proved that $\mathcal{O}_C(1)$ computes the Clifford index c of C. Hence $c = d - 2r = 2r - 3$ and we obtain

$d=2c+3$, the maximal number for odd c. From Claim (3.2.2) we know that C has no linear system g^1_{c+2}. (Even stronger, in [9], Theorem 3.7, it is proved that $\mathcal{O}_C(1)$ is the only bundle on C computing the Clifford index.)

The simplest example is a smooth complete intersection of two cubics in $\mathbf{P}^3(r=3;\ d=9;\ g=10;\ c=3)$. For details cf. [9], §4. □

3.2.8 EXAMPLE (small genus). Let X be a smooth cubic surface in $\mathbf{P}^3$. Adopting the notation of [12], V, 4, Pic X is generated by l and (the classes of) six lines $e_1,\ldots,e_6$ such that $(l^2)=1$, $(e_i^2)=-1$, $(l\cdot e_i)=0$, $(e_i\cdot e_j)=0$ $(i\neq j)$. Consider a smooth irreducible member C of $|11l-4\Sigma_{i=1}^5 e_i-3e_6|$. Then C has degree $d=10$ in $\mathbf{P}^3$ and genus $g=12$. In accordance with Cayley's formula ([3], p. 351) C has exactly 10 quadrisecant lines (given by e_i, $l-e_i-e_6$ for $i=1,\ldots,5$) but no lines cutting C in at least 5 points. Therefore, it is easy to see that C is 6-gonal (with exactly 10 g^1_6) and of Clifford index $c=4$. Thus the embedding g^3_{10} computes c, and $g=2(c+2)>d=10>3(c+2)/2$. This is in accordance with Corollary (3.2.5). □

3.2.9 REMARK. Assume C has a g^r_d, $r\geqslant 4$, $d\leqslant 3(c+2)/2$, computing the Clifford index c of C. Since $c=d-2r$ we have $d\geqslant 6r-6$ and according to [14] C may be viewed as a linearly normal curve of degree d in $\mathbf{P}^r$ not lying on a quadric of rank $\leqslant 4$. By the proof of [9], Proposition 5.1, then, C cannot be contained in a surface of degree $2r-3$ or less. □

3.3 CONSEQUENCES. Note that a curve C of Clifford index c which is not hyperelliptic and not bi-elliptic and which admits a linear system computing c of maximal degree $d=2c+3$ (c odd) resp. $d=2c+4$ (c even) must have genus $g=d+1$, by Corollary (3.2.5). The existence of such curves is settled by our previous examples, for every $c\geqslant 2$. If $c=1$ take a smooth plane quintic. For odd c these curves are studied in [9]. Here we want to make some closing remarks on these curves for even c. We will prove a "recognition theorem" for them (cf. Proposition (3.3.2)) which will then be used to deduce some criteria for curves whose Clifford index can only be computed by pencils.

3.3.1 EXAMPLE. Assume that C is not hyper- or bi-elliptic. If $|D|$ is a linear system on C of degree $2c+4$ computing c it follows from the Claims (3.2.1) and (3.2.4) that C possesses a pencil g^1_{c+2}, say $|M|$, such that $|D-M|$ is a g^1_{c+2}, too. Assume that $|D-M|=|M|$. Then $\dim(|2M|)$ is as large as possible since $|2M|$ computes the Clifford index. Consider W^1_{c+2} and let $m=I(c+2)(M)\in W^1_{c+2}$. It is well-known (see e.g. [3]) that the embedding dimension $d(m):=\dim T_m(W^1_{c+2})$ of W^1_{c+2} at the point m is given by $h^0(2M)-3$. But $h^0(2M)=(c/2)+3$, hence $d(m)$ attains its maximal value $c/2$.

Conversely, let C be a smooth curve of Clifford index $c\geqslant 2$ and gonality $k\leqslant(g-1)/2$. Let $|M|=g^1_k$, $m=I(k)(M)\in W^1_k$, and assume that $d(m)$ is maximal. Since $2k\leqslant g-1$ we have $h^0(2M)\leqslant k+1-c/2$, and since $d(m)$ is maximal if and only if $h^0(2M)$ is, we have $d(m)=k-2-c/2$. In this case $2M$ computes c whence $2k\leqslant 2c+4$, by Theorem C. Clearly, $k\geqslant c+2$. Therefore, $2k=2c+4$ and $d(m)=c/2$.

The simplest example is a complete intersection of a quartic surface and a quadric cone in $\mathbf{P}^3$. □

3.3.2 PROPOSITION. *Suppose C is a k-gonal curve ($k\geqslant 3$) admitting only finitely many base-point free g^1_k and g^1_{k+1}. Let $r\geqslant 2$ and assume that C has a g^r_d ($d\leqslant g-1$) computing the Clifford index c of C. Then $c=k-2$ is even, $d=2c+4$, and the g^r_d is the only linear system on C computing c which is not a pencil.*

Proof. By Corollary (2.3.1), $c=\text{cliff}\,(C)=k-2$. Suppose that $d<2c+3$. Then $k-2+2r=d\leqslant 2(k-2)+2$, i.e. $2r\leqslant k$.

Since the g^r_d is complete and $d=k-2+2r\geqslant 4r-2$ we can use Theorem A. We adopt the terminology of the proof of Theorem (1.2). Let Z be an irreducible component of $V^{r-2}_{2r-3}(g^r_d)$ and consider $i{:}Z\to J(C)$. Clearly $i(Z)\subset W^1_{d-2r+3}=W^1_{k+1}$. One has $\dim(Z)\geqslant 1$, hence if $\dim(i(Z))=0$ then C has a linear system g^1_{2r-3}. But $2r-3\leqslant k-3$, so this is impossible. Therefore, the assumptions on C give us the existence of $x\in W^1_k$ such that $i(Z)\supset x\oplus W^0_1$. Hence, for each $P\in C$ there exists $D_P\in Z$ such that

$$g_k(x) + P + D_P \subset g^r_d.$$

Thus $P+D_P\in|g^r_d-g_k(x)|$ and we obtain $\dim(|P+D_P|)\geqslant 1$. But $\deg(P+D_P)=d-k\leqslant k-2$. Again we obtain a contradiction. Thus $d\geqslant 2c+3$. From Claim (3.2.2) we see that c is even, whence $d=2c+4$. (Note that C cannot be hyper- or bi-elliptic.) Suppose $D\in g^r_d$ and D' is an effective divisor of degree d' with $k<d'\leqslant d$ computing c. We have already proved that $d'=d$. Take $E\in g^1_k$ on C. Because of Claim (3.2.3) we can assume $\inf(D,D')=F\geqslant E$ but $F\neq E$. Copying the proof of Theorem 3.7(ii) in [9] we obtain that F computes c. But $k<\deg(F)<d$, hence we obtain a contradiction. This proves that g^r_d is the unique linear system on C computing c which is not a pencil. □

3.3.3 COROLLARY. *Let C be a k-gonal curve ($k\geqslant 3$) such that W^1_k and $W^1_{k+1}\backslash(W^1_k\oplus W^0_1)$ are finite. Assume that one of the following conditions holds:*

(i) *k is odd; or*
(ii) *k is even, and in case of genus $g=2k+1$ we have $W^1_{k+1}\neq W^1_k\oplus W^0_1$; or*
(iii) *$W^1_k=\{x\}$, and $2x\notin W^3_{2k}$.*

Then the Clifford index c of C is only computed by pencils (corresponding to the elements of W^1_k).

Proof. (i) is an immediate consequence of Proposition (3.3.2).

(ii) holds because of Proposition (3.3.2), Claim (3.2.1) and Corollary (3.2.5).

(iii) Assume C has a linear system g^r_d, $r\geqslant 2$, computing c. By Proposition

(3.3.2) and Example (3.3.1) we conclude that $|2g_k(x)| = g^r_d$. Hence $k-2=c=d-2r=2k-2r$, so $2r=k+2 \geqslant 5$ contradicting $2x \notin W^3_{2k}$. □

3.3.4 EXAMPLE. From Corollary (3.3.3), (iii) we deduce that on a general k-gonal curve C of genus $g > 2k \geqslant 6$ there is only one linear system computing the Clifford index: the unique pencil g^1_k. In fact, by [2] we have $W^1_k = \{x\}$, $W^1_{k+1} = \{x\} \oplus W^0_1$, and by [24] we have $2x \notin W^3_{2k}$.

Note that this proof makes the meaning of "general" much more transparent than Ballico's original proof of this fact ([4]). □

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