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The cohomological dimension of the quotient field of the two dimensional complete local domain

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Let A be a complete Noetherian local domain with separably closed residue field k and with quotient field K . For a prime number p which is different from $\text{char}(K)$, let $\text{cd}_p(K)$ be the p -cohomological dimension of the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$ (cf. [13], [14], here K^{sep} denotes the separable closure of K). In this paper, we determine $\text{cd}_p(K)$ in the case $\dim(A) = 2$, $\text{char}(k) = p$ and $\text{char}(K) = 0$.

In general, if $\text{char}(k) \neq p$, a standard conjecture (Artin [2]) is that

$$\text{cd}_p(K) = \dim(A). \tag{0.1}$$

In the more delicate case where $\text{char}(k) = p$ and $\text{char}(K) = 0$, Artin suggests in [2] that the rank of the absolute differential module $\Omega_k^1 = \Omega_{k/\mathbb{Z}}^1$ should be involved in $\text{cd}_p(K)$. The precise form of his conjecture in this case should be

$$\text{cd}_p(K) = \dim(A) + \dim_k(\Omega_k^1). \tag{0.2}$$

The aim of this paper is to prove (0.2) in the case $\dim(A) = 2$.

THEOREM. *Let A be a complete Noetherian two dimensional local domain of mixed characteristic $(0, p)$ with a separably closed residue field k , and K be the quotient field of A . Then,*

$$\text{cd}_p(K) = \dim_k(\Omega_k^1) + 2.$$

The conjecture (0.1) has been proved in the case $\dim(A) \leq 2$ (the case $\dim(A) = 1$ is classical and the case $\dim(A) = 2$ is due to O. Gabber [4]). The conjecture (0.2) has been proved in the case $\dim(A) = 1$ (cf. [6]II, [13], [14]), and in the case where $\dim(A) = 2$ and k is algebraically closed (then $\Omega_k^1 = (0)$) by K. Kato ([12] §5).

Notation

A ring means a commutative ring with a unit.

For a local ring A ,

\hat{A} : the completion of A by the maximal ideal m_A ,

$A_{\mathfrak{p}}$: the localization of A at a prime ideal \mathfrak{p} ,

$\kappa(\mathfrak{p})$: the residue field of $A_{\mathfrak{p}}$,

Ω_A^i : the i th exterior power over A of the absolute differential module Ω_A^1 .

For a field k ,

$K_i(k)$: the i th Milnor's K -group of k ([11]).

For an abelian group M and a family $S_\lambda (\lambda \in \Lambda)$ of elements of M , $\langle S_\lambda; \lambda \in \Lambda \rangle$ is the subgroup of M generated by S_λ for $\lambda \in \Lambda$.

Proof of theorem. Throughout this paper, let A be a complete Noetherian two dimensional local domain with a separably closed residue field k of characteristic $p > 0$ and with the quotient field K of characteristic 0. Without loss of generality, we assume that if $p \neq 2$ (resp. $p = 2$) K contains a primitive p th (resp. 4th) root of unity ([15]).

First of all, we have

PROPOSITION 1.

$$\text{cd}_p(K) \geq \dim_k(\Omega_k^1) + 2.$$

Proof. Let \mathfrak{p} be a prime ideal of A such that $\text{ht}(\mathfrak{p}) = 1$ and $\text{char}(\kappa(\mathfrak{p})) = 0$, and let $K_{\mathfrak{p}}$ be the quotient field of the henselization of the local ring of A at \mathfrak{p} . Then we have

$$\text{cd}_p(K) \geq \text{cd}_p(K_{\mathfrak{p}}) \tag{[14]}$$

$$= \text{cd}_p(\kappa(\mathfrak{p})) + 1 \tag{[8]}$$

$$= \dim_k(\Omega_k^1) + 2. \tag{[8]}$$

Hence it remains to prove that $\text{cd}_p(K) \leq \dim_k(\Omega_k^1) + 2$. In the rest of this paper, we assume $\dim_k(\Omega_k^1) < \infty$. Let $r = \dim_k(\Omega_k^1)$ (so $[k:k^p] = p^r$).

We fix an algebraic closer \bar{K} of K , and r elements b_1, b_2, \dots, b_r of A such that the residue classes \bar{b}_i of b_i ($1 \leq i \leq r$) form a p -basis of k . Then we can pick up elements $\{b_{i,j}; 1 \leq i \leq r, j = 0, 1, 2, \dots\}$ of \bar{K} which satisfy the following conditions:

$$(b_{i,j})^p = b_{i,j-1} \quad 1 \leq i \leq r, j = 1, 2, \dots$$

$$b_{i,0} = b_i \quad 1 \leq i \leq r.$$

For integers n ($n = 0, 1, 2, \dots, \infty$), we define extensions $A^{(n)}$ (resp. $K^{(n)}$) of A (resp. K) by

$$A^{(n)} = A[b_{i,n}; 1 \leq i \leq r]$$

$$A^{(\infty)} = \bigcup_{n=0}^{\infty} A^{(n)}$$

$$K^{(n)} = K(b_{i,n}; 1 \leq i \leq r)$$

$$K^{(\infty)} = \bigcup_{n=0}^{\infty} K^{(n)}.$$

PROPOSITION 2.

$$cd_p(K) \leq r + 3.$$

Proof. In Lemma 1 below, we shall prove that $A^{(\infty)}$ is an excellent henselian local domain. Since the residue field of $A^{(\infty)}$ is algebraically closed, $cd_p(K^{(\infty)}) = 2$ (cf. [12] §5 Th. the excellence of $A^{(\infty)}$ is needed here). Let ζ_{p^∞} be a subgroup of K^* which consists of all roots of unity of p -primary orders. Then $cd_p(K^{(\infty)}(\zeta_{p^\infty})) \leq 2$ ([14]).

On the other hand, the field $K^{(\infty)}(\zeta_{p^\infty})$ is a Galois extension of K and the Galois group of $K^{(\infty)}(\zeta_{p^\infty})/K$ is isomorphic to \mathbb{Z}_p^{r+1} (\mathbb{Z}_p is the ring of p -adic integers). Then we have inequalities

$$cd_p(K) \leq cd_p(\mathbb{Z}_p^{r+1}) + cd_p(K^{(\infty)}(\zeta_{p^\infty}))$$

$$\leq r + 3. \tag{[14]}$$

LEMMA 1. $A^{(\infty)}$ is an excellent henselian two dimensional local ring.

Proof. By [9], A is finite over $R[[X]]$ where R is a complete discrete valuation ring with mixed characteristic containing b_1, b_2, \dots, b_r whose residue field is the same as that of A , and X is a variable. So we may assume that $A = R[[X]]$, $b_1, b_2, \dots, b_r \in R$. We define rings $R^{(n)} = R[b_{i,n}; 1 \leq i \leq r]$ for integers $n \geq 0$ and $R^{(\infty)} = \bigcup_{n=0}^{\infty} R^{(n)}$, and fix a prime element π of R .

First, we will prove that $A^{(\infty)}$ is Noetherian.

It is enough to show that every prime ideal \mathcal{P} of $A^{(\infty)}$ is finitely generated ([9]). Since $A^{(\infty)}$ is a two dimensional ring, every prime ideal $\neq (0)$ is either maximal or of height one. Assume \mathcal{M} is a maximal ideal of $A^{(\infty)}$. Then $\mathcal{M} \cap R^{(n)}[[X]]$ is a

maximal ideal of $R^{(n)}[[X]]$ for all integers $n \geq 0$. Hence $\mathcal{M} \cap R^{(n)}[[X]] = (\pi, X)$ for all $n \geq 0$, and this implies $\mathcal{M} = (\pi, X)$. On the other hand, if \mathcal{P} is a prime ideal of $A^{(\infty)}$ of height one, $\mathcal{P} \cap R^{(n)}[[X]]$ is (π) or $(X^m + a_1 X^{m-1} + \dots + a_m)$ (m is a positive integer and a_i are elements of the maximal ideal of $R^{(n)}$ for $1 \leq i \leq m$) for all integers $n \geq 0$. When $\mathcal{P} \cap R[[X]] = (\pi)$, $\mathcal{P} \cap R^{(n)}[[X]] = (\pi)$ also. This implies $\mathcal{P} = (\pi)$. When $\mathcal{P} \cap R[[X]] \neq (\pi)$, the degree m of the above polynomial becomes stable for sufficiently large integers n . So \mathcal{P} is generated by an element which generates $\mathcal{P} \cap R^{(n)}[[X]]$ for integers $n \gg 0$. Thus $A^{(\infty)}$ is Noetherian.

Secondly, recall that, a Noetherian local ring S is excellent, if and only if, S is a G -ring and universally catenary (cf. [9] Ch. 13, 34).

It is easily deduced that $A^{(\infty)}$ is universally catenary from the fact that $A^{(\infty)}$ is the union of subrings which are finite over the excellent ring A . And $A^{(\infty)}$ is a G -ring when $\hat{A}^{(\infty)} \otimes_{A^{(\infty)}} L$ is regular for any prime ideal \mathcal{P} of $A^{(\infty)}$ and any finite extension L of $\kappa(\mathcal{P})$. The regularity is easy and we omit the proof.

We have shown that $\text{cd}_p(K)$ is $r + 2$ or $r + 3$. To prove that $\text{cd}_p(K) = r + 2$, it is sufficient to show that the Galois cohomology groups $H^{r+3}(L, \mathbb{Z}/p\mathbb{Z})$ vanish for all finite extension fields L over K .

LEMMA 2. *The cohomology symbol map (cf. [6]II)*

$$h_k^{r+3}; K_{r+3}(K)/p \rightarrow H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

is surjective.

Proof. In the first place, we consider the fact (*).

(*) *Let k be a field, S a Galois extension of k of infinite degree, p a prime number which is invertible in k , and $q \geq 0$ an integer. Suppose that $\text{cd}_p(\text{Gal}(S/k)) \leq q$ and $\text{cd}_p(S) \leq 2$, and that for any open subgroup J of $\text{Gal}(S/k)$, the cup product*

$$\otimes^q H^1(J, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^q(J, \mathbb{Z}/p\mathbb{Z})$$

is surjective. Then, h_k^{q+2} is surjective.

Using the fact that h_k^2 is surjective for any field k ([10]), the arguments in the proof of Proposition 3 of [6]II, §1.3 can be used to prove (*) (we replace $\text{cd}_p(S) \leq 1$ (resp. h_k^{q+1}) by $\text{cd}_p(S) \leq 2$ (resp. h_k^{q+2})).

We apply (*) to $k = K$, $S = K^{(\infty)}(\zeta_{p^\infty})$ and $q = r + 1$. From the proof of Proposition 2, $\text{cd}_p(K^{(\infty)}(\zeta_{p^\infty})) \leq 2$, $\text{cd}_p(\text{Gal}(K^{(\infty)}(\zeta_{p^\infty})/K)) = r + 1$ and $\text{Gal}(K^{(\infty)}(\zeta_{p^\infty})/K) \cong \mathbb{Z}_p^{r+1}$. Any open subgroup of \mathbb{Z}_p^{r+1} is isomorphic to \mathbb{Z}_p^{r+1} and the cup product

$$\otimes^{r+1} H^1(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^{r+1}(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. Hence the assumption of (*) is satisfied. This lemma is proved.

We consider the condition.

(**) *A is regular and there exist two elements u and v of A which generate the maximal ideal of A, such that p is invertible in A[1/uv].*

For $w = u$ or v , let $\mathcal{I}_w = (w)$, $\bar{A}_w = A/\mathcal{I}_w$ and

$$f_w = \begin{cases} (p/p - 1)\text{ord}_w(p) & \text{if } \text{char } \kappa(\mathcal{I}_w) = p \\ 0 & \text{if } \text{char } \kappa(\mathcal{I}_w) = 0. \end{cases}$$

We distinguish two cases;

case (I) $\text{char}(\kappa(\mathcal{I}_u)) = \text{char}(\kappa(\mathcal{I}_v)) = p$

case (II) $\text{char}(\kappa(\mathcal{I}_u)) = 0, \text{char}(\kappa(\mathcal{I}_v)) = p.$

LEMMA 3. *Assume (**). Let*

$$\Delta = \left\langle \begin{array}{l} \{a, b_1, \dots, b_{r+2}\}; \quad b_s \in A[1/uv]^* \text{ for } 1 \leq s \leq r+2 \text{ and} \\ a \in 1 + uvA \text{ in the case (I)} \\ \text{(resp. } a \in 1 + vA \text{ in the case (II))} \end{array} \right\rangle \\ \text{in } K_{r+3}(K)/p.$$

Then $\Delta = 0$.

Proof. For integers $i \geq 0$ and $j > 0$, we define

$$\Delta_{i,j} = \left\langle \begin{array}{l} \{1 + au^i v^j, b_1, \dots, b_r, c, d\}; \\ a \in A, b_s \in A^* \text{ for } 1 \leq s \leq r \text{ and } c, d \in A[1/uv]^* \end{array} \right\rangle \text{ in } K_{r+3}(K)/p.$$

In the case (I) (resp. (II)), we deduce $\Delta = 0$ from the following facts,

- (1) $\Delta = \Delta_{1,1}$ (resp. $\Delta = \Delta_{0,1}$)
- (2) if $0 \leq i \leq f_u, 0 < j < f_v$ and either $p \nmid i$ or $p \nmid j$, we have $\Delta_{i,j} = \Delta_{i,j+1}$
- (3) $\Delta_{f_u, f_v} = 0$.

Proof of (1). This is easy and we omit the proof.

Proof of (2). We can define the homomorphism

$$\chi_1: \Omega_{\bar{A}_v}^{r+2} \rightarrow \Delta_{i,j}/\Delta_{i,j+1}$$

by

$$\bar{a} \cdot \frac{d\bar{a}_1}{\bar{a}_1} \wedge \dots \wedge \frac{d\bar{a}_{r+2}}{\bar{a}_{r+1}} \mapsto \{1 + au^i v^j, a_1, \dots, a_{r+2}\}.$$

If $p \nmid j$, by the following calculation in $K_2(K)$, the map χ_1 is surjective.

$$\begin{aligned} j\{1 + au^hv^j, v\} &= \{1 + au^hv^j, -au^h\} \\ \{1 + au^hv^j, 1 + bv\} &= \{1 + au^hv^j, 1 - abu^hv^{j+1}\} - \{1 + bv, 1 - abu^hv^{j+1}\} \\ &\text{for } a \in A^*, b \in A \text{ and } h \geq i. \end{aligned}$$

In the case $p \mid j$ and $p \mid i$, we can define the homomorphism

$$\chi_2: \Omega_{A_v}^{r+1} \rightarrow \Delta_{i,j}/\Delta_{i,j+1}$$

by

$$\bar{a} \frac{d\bar{a}_1}{\bar{a}_1} \wedge \dots \wedge \frac{d\bar{a}_{r+1}}{\bar{a}_{r+1}} \mapsto \{1 + au^hv^j, a_1, \dots, a_{r+1}, v\}.$$

Then every element of $\Delta_{i,j}/\Delta_{i,j+1}$ is a sum of elements of the images of χ_1 and χ_2 .

On the other hand, the equalities $\text{char}(\kappa(\not\phi_v)) = p$ and $[\kappa(\not\phi_v): \kappa(\not\phi_v)^p] = p^{r+1}$ imply $\Omega_{A_v}^{r+2} = 0$.

$$\Omega_{A_v}^{r+1} = \left\langle \bar{a}^p \cdot \frac{d\bar{a}_1}{\bar{a}_1} \wedge \dots \wedge \frac{d\bar{a}_{r+1}}{\bar{a}_{r+1}}; \bar{a} \in \bar{A}_v, \bar{a}_j \in \bar{A}_v^*, 1 \leq j \leq r+1 \right\rangle + d\Omega_{A_v}^r$$

([7] II). Then we have $\Delta_{i,j} = \Delta_{i,j+1}$.

Proof of (3). Let c be the element of A^* such that $p = cu^{f_u(p-1)/p}v^{f_v(p-1)/p}$. We can take a solution $x \in A$ of the equation $X^p + cX - a = 0$ for any $a \in A$. Then,

$$1 + au^{f_u}v^{f_v} = (1 + xu^{f_u/p}v^{f_v/p})^p.$$

Thus the proof of Lemma 3 is complete.

LEMMA 4. Assume (**). Let

$$\begin{aligned} \Delta' = \langle \{a, b_1, \dots, b_{r+2}\}; a \in 1 + m_A \text{ and } b_j \in A[1/uv]^* \text{ for } 1 \leq j \leq r+2 \rangle \\ \text{in } K_{r+3}(K)/p. \end{aligned}$$

where m_A is the maximal ideal of A .

Then $\Delta' = 0$.

Proof. As is easily seen, $1 + m_A$ is generated by elements of the form $1 - aw$ ($a \in A^*$) for $w = u$ or v . From this, we see that Δ' is generated by elements of the forms $\{1 - aw, b_1, \dots, b_r, c, d\}$ with a, b_1, \dots, b_r and $c \in A^*, d \in A[1/uv]^*$ and $w = u$ or v such that b_1, \dots, b_{r-1} and b_r form a p -basis of k .

Hence it suffices to prove

$$\{1 - au, b_1, \dots, b_r, c, d\} \in pK_{r+3}(K)$$

for a, b_1, \dots, b_r, c and d as above. Let

$$B = A[(au)^{1/p}, b_1^{1/p}, \dots, b_r^{1/p}] \subset \bar{K}$$

and L be the quotient field of B . Since $B/vB = (A/vA)^{1/p}$, there exist elements $c' \in B^*$ and $c'' \in B$ such that $c = (c')^p(1 + c''v)$. We apply Lemma 3 to B ,

$$\{1 - (au)^{1/p}, b_1^{1/p}, \dots, b_r^{1/p}, 1 + c''v, d\} \in pK_{r+3}(L).$$

With $N_{L/K}$ denoting the norm map, we have

$$\begin{aligned} \{1 - au, b_1, \dots, b_r, c, d\} &= N_{L/K}(\{1 - (au)^{1/p}, b_1^{1/p}, \dots, b_r^{1/p}, c, d\}) \\ &\in N_{L/K}(pK_{r+3}(L)) \subset pK_{r+3}(K). \end{aligned}$$

LEMMA 5. Assume (**). Let

$$\begin{aligned} \Delta'' &= \langle \{a_1, a_2, \dots, a_{r+3}\}; a_j \in A[1/uv]^* \text{ for } 1 \leq j \leq r+3 \rangle \\ &\text{in } K_{r+3}(K)/p. \end{aligned}$$

Then $\Delta'' = 0$.

Proof.

$$\Delta'' = \langle \{a_1, \dots, a_{r+1}, b, c\}; a_j \in A^* \text{ for } 1 \leq j \leq r+1, b, c \in A[1/uv]^* \rangle$$

Hence it suffices to prove

$$\{a_1, \dots, a_{r+1}, b, c\} \in pK_{r+3}(K)$$

for a_1, \dots, a_{r+1}, b and c as above. Since $\bar{a}_i = a_i \bmod \mathfrak{m}_A$ ($1 \leq i \leq r+1$) cannot be p -independent, there exists s such that $1 \leq s \leq r$ and $\bar{a}_{s+1} \in k^p(a_1, \dots, a_s)$. Let

$$B = A[a_1^{1/p}, \dots, a_s^{1/p}] \subset \bar{K}$$

and L be the quotient field of B . Since the residue field of B contains $\bar{a}_{s+1}^{1/p}$, there exist elements $a' \in B^*$ and $a'' \in \mathfrak{m}_B$ (\mathfrak{m}_B is the maximal ideal of B) such that

$a_{s+1} = (a')^p(1 + a'')$. By applying Lemma 4 to B , we have

$$\begin{aligned} & \{a_1, \dots, a_{r+1}, b, c\} \\ &= N_{L/K}(\{a_1^{1/p}, \dots, a_s^{1/p}, a_{s+1}, \dots, a_{r+1}, b, c\}) \\ &\in N_{L/K}(pK_{r+3}(L)) \subset pK_{r+3}(K). \end{aligned}$$

We follow the method of [12] §5.

LEMMA 6. *Let $\mathfrak{X} \rightarrow \text{Spec}(A)$ be a proper birational morphism with regular such that $Y = \mathfrak{X} \otimes_A A/\mathfrak{m}_A$ is a reduced divisor with normal crossing on \mathfrak{X} ([1], [5]). Then,*

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{x \in Y_0} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z})$$

where Y_0 denotes the set of closed points of Y and for each $x \in Y_0$, K_x denotes the quotient field of the henselization of $\mathfrak{D}_{\mathfrak{X},x}$.

Proof. Let $\lambda: \text{Spec}(K) \rightarrow \mathfrak{X}$ be the inclusion map and put $\mathfrak{F} = R\lambda_*(\mathbb{Z}/p\mathbb{Z})$. By the proper base change theorem, we have

$$H_x^q(Y, i^*\mathfrak{F}) = H^q(\mathfrak{X}, \mathfrak{F}) = H^q(K, \mathbb{Z}/p\mathbb{Z})$$

where $i: Y \rightarrow \mathfrak{X}$ is the inclusion map. From this, we obtain an exact sequence

$$\rightarrow \bigoplus_{x \in Y_0} H_x^q(Y, i^*\mathfrak{F}) \rightarrow H^q(K, \mathbb{Z}/p\mathbb{Z}) \rightarrow \bigoplus_{\eta} H^q(K_{\eta}, \mathbb{Z}/p\mathbb{Z}) \rightarrow \bigoplus_{x \in Y_0} H_x^{q+1}(Y, i^*\mathfrak{F}) \rightarrow$$

where η ranges over all generic points of Y and K_{η} denotes the quotient field of the henselization of $\mathfrak{D}_{x,\eta}$. For each $x \in Y_0$, we have an exact sequence

$$\rightarrow H_x^q(Y, i^*\mathfrak{F}) \rightarrow H^q(K_x, \mathbb{Z}/p\mathbb{Z}) \rightarrow \bigoplus_{\mathfrak{v}} H^q(K_{\mathfrak{v}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_x^{q+1}(Y, i^*\mathfrak{F}) \rightarrow$$

where \mathfrak{v} ranges over all generic points of the henselization of $\text{Spec}(\mathfrak{D}_{Y,x})$, and $K_{\mathfrak{v}}$ denotes the quotient field of the henselization of the discrete valuation ring of K_x corresponding to \mathfrak{v} .

Since $\text{cd}_p(K_{\eta}) = \text{cd}_p(K_{\mathfrak{v}}) = r + 2$ ([8]), we have

$$H^{r+3}(K_{\eta}, \mathbb{Z}/p\mathbb{Z}) = H^{r+3}(K_{\mathfrak{v}}, \mathbb{Z}/p\mathbb{Z}) = 0.$$

On the other hand, from the classical approximation theorem for a finite family of discrete valuation on K^* and K_x^* , the maps

$$K_{r+2}(K) \rightarrow \bigoplus_{\eta} K_{r+2}(K_{\eta})/p \quad \text{and} \quad K_{r+2}(K_x) \rightarrow \bigoplus_{\nu} K_{r+2}(K_{\nu})/p$$

are subjective. By [3] §5, the cohomological symbol maps

$$K_{r+2}(K_{\eta})/p \rightarrow H^{r+2}(K_{\eta}, \mathbb{Z}/p\mathbb{Z})$$

and

$$K_{r+2}(K_{\nu})/p \rightarrow H^{r+2}(K_{\nu}, \mathbb{Z}/p\mathbb{Z})$$

are subjective. Hence the maps

$$H^{r+2}(K, \mathbb{Z}/p\mathbb{Z}) \rightarrow \bigoplus_{\eta} H^{r+2}(K_{\eta}, \mathbb{Z}/p\mathbb{Z})$$

and

$$H^{r+2}(K_x, \mathbb{Z}/p\mathbb{Z}) \rightarrow \bigoplus_{\nu} H^{r+2}(K_{\nu}, \mathbb{Z}/p\mathbb{Z})$$

are also subjective.

Putting these things together,

$$\bigoplus_{x \in Y_0} H_x^{r+3}(Y, i^* \mathfrak{F}) \xrightarrow{\cong} H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

and

$$H_x^{r+3}(Y, i^* \mathfrak{F}) \xrightarrow{\cong} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z}).$$

These isomorphisms induce the isomorphism of Lemma 6.

PROPOSITION 3. For $r = \text{dom}_k(\Omega_k^1)$,

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

Proof. By Lemma 2 and 6, we have

$$\begin{array}{ccc} \bigoplus_{x \in Y_0} K_{r+3}(K_x)/p & \rightarrow & \bigoplus_{x \in Y_0} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z}) \\ & & \uparrow \cong \\ K_{r+3}(K)/p & \rightarrow & H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}). \end{array}$$

For any family of fixed elements $a_1, a_2, \dots, a_{r+3} \in K^*$, we can take \mathfrak{X} such that the union Z of Y with the supports of the divisor of a_1, \dots, a_{r+2} and a_{r+3} on \mathfrak{X} is normally crossing divisor ([5]). Then, by Lemma 5,

$$\{a_1, a_2, \dots, a_{r+3}\} \in pK_{r+3}(K_x)$$

for any $x \in Y_0$. This shows that

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

We are now in the position to complete the proof of our theorem. By Proposition 1 and 2, $\text{cd}_p(K) = r + 2$ or $r + 3$. For any finite extension field K' over K ,

$$H^{r+3}(K', \mathbb{Z}/p\mathbb{Z}) = 0$$

by Proposition 3. Hence $\text{cd}_p(K) = r + 2$ ([14]).

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