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Disjointness preserving and diffuse operators

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1. Introduction

In recent years considerable attention has been given to disjointness preserving operators on Banach lattices and their spectral theory (see e.g. [Ab], [A1], [A2], [H1], [H2], [AH], [LS], [M], [P2], [Wi], and see also the references given in these papers). Typical examples of such operators are weighted composition operators in $L_p(\mu)$ and $C(K)$ spaces (i.e., operators of the form $Tf(x) = w(x)f(\varphi(x))$). Rather than studying the properties of a single disjointness preserving operator, we discuss in the present paper some properties of the collection of all such operators. To be more specific, we will consider the band $\mathscr{H}$ generated by all disjointness preserving operators and its disjoint complement $\mathscr{D}$, the operators in which we will call diffuse. For linear functionals such a decomposition has been considered by H. Gordon, who introduced and characterized in [G] positive diffuse functionals. In this connection we note that the diffuse positive functionals on a $C(K)$ space correspond precisely to the diffuse Borel measures on the compact Hausdorff space K. For positive operators in Banach function spaces on a separable metric space an analogous decomposition was first investigated by L. W. Weis in [W1] and [W2]. His approach is based on the representation of a positive operator by means of a stochastic kernel, which is then decomposed in an atomic and a diffuse part. In the present paper, however, the properties of such a decomposition are studied by different means and in a more general context.

In Section 2 we describe the band $\mathscr{D}$ of diffuse operators, whereas, in Section 3 this characterization is then used to obtain some algebraic properties of $\mathscr{H}$ and $\mathscr{D}$. These properties are reflected in certain averaging properties of the corresponding band projections. Next we present some typical classes of diffuse operators, such as kernel operators, compact operators and certain convolution operators. In Section 5 we prove that for a wide class of Banach lattices the band projection onto the band generated by a single disjointness preserving operator is contractive with respect to the operator norm, which can be regarded as an extension of the result of J. Voigt [V1] to the effect that the band projection onto the center of a Dedekind complete Banach lattice is contractive with respect to the operator norm. Finally, in the last section, it is illustrated how some of the results of Sections 3 and 5 can be used to get information about the spectrum of disjointness preserving operators. In particular it is shown that any norm bounded aperiodic disjointness preserving operator on a Banach lattice with order continuous norm has a rotationally invariant peripheral spectrum.

We end this introduction with some preliminary information. We assume that the reader is familiar with the basic terminology and theory of vector lattices and Banach lattices, as can be found in the text books [AB], [LZ], [Sch] and [Z]. All vector lattices considered in this paper are Archimedean. Most of the results in this paper hold for both real and complex vector lattices (of course, for results involving the spectrum we consider complex Banach lattices). By a complex vector lattice E we mean $E = \operatorname{Re} E \oplus i \operatorname{Re} E$, the complexification of a real vector lattice $\operatorname{Re} E$, where we assume that for each $z = x + iy \in E$ the modulus $|z| = \sup\{(\cos\theta)x + (\sin\theta)y : 0 \leqslant \theta \leqslant 2\pi\}$ exists in $\operatorname{Re} E$. Note that this is the case if $\operatorname{Re} E$ is uniformly complete ([Z], section 90), in particular if $\operatorname{Re} E$ is Dedekind complete or is a Banach lattice. Throughout we will denote $(\operatorname{Re} E)^+$ by E^+.

Finally we recall that the linear operator T from the vector lattice E into the vector lattice F is called *disjointness preserving* if $f \perp g$ in E implies $Tf \perp Tg$ in F. As is well known, if T is in addition order bounded, then the absolute value $|T|$ exists and satisfies $|Tf| = |T|(|f|)$ for all $f \in E$. Moreover, if E and F are Banach lattices, then any norm bounded disjointness preserving operator from E into F is automatically order bounded (see [Ab] and also [P2], and use [M] for adaptation to the complex situation). A positive disjointness preserving operator is a lattice (or Riesz) homomorphism.

2. The band of diffuse operators

First we fix some notation. Throughout, E denotes a vector lattice (Riesz space) and F a Dedekind complete vector lattice. By an operator from E into F we shall always mean a linear mapping from E into F. The Dedekind complete vector lattice of all order bounded (=regular) operators from E into F is denoted by $\mathscr{L}_b(E, F)$, and $\mathscr{L}_n(E, F)$ is the band in $\mathscr{L}_b(E, F)$ of all order bounded, order continuous operators from E into F. The set of all lattice homomorphisms (or Riesz homomorphisms) from E into F is denoted by $\operatorname{Hom}(E, F)$. We shall write $\mathscr{I}(E, F)$ for the (order) ideal generated by $\operatorname{Hom}(E, F)$ in $\mathscr{L}_b(E, F)$. Evidently, any $T \in \mathscr{I}(E, F)^+$ is a finite sum of lattice homomorphisms. The band generated by $\operatorname{Hom}(E, F)$ in $\mathscr{L}_b(E, F)$ is henceforth denoted by $\mathscr{H}(E, F)$ and for its disjoint complement in $\mathscr{L}_b(E, F)$ we shall put $\mathscr{D}(E, F)$. Observe that $\mathscr{I}(E, F)$ (and hence $\mathscr{H}(E, F)$) contains all order bounded disjointness preserving operators from E into F and that

$$\mathscr{D}(E, F) = \mathrm{Hom}(E, F)^d = \mathscr{I}(E, F)^d = \mathscr{H}(E, F)^d.$$

Hence we have the band decomposition $\mathscr{L}_b(E, F) = \mathscr{H}(E, F) \oplus \mathscr{D}(E, F)$. Following H. Gordon [G], who used this terminology in the case $F = \mathbb{R}$, we will call an operator in $\mathscr{D}(E, F)$ *diffuse*. In this connection we mention that L. W. Weis considered this decomposition in the special case of positive operators in Banach function spaces on separable metric spaces (see [W1] and [W2]). Furthermore, W. Arendt studied in his thesis ([A1], Kapitel 1) a related but different decomposition of $\mathscr{L}_b(E)$. Instead of the band $\mathscr{H}(E)$ he considered the band $\mathscr{L}_a(E)$ generated by all lattice *isomorphisms* of E. However, this band $\mathscr{L}_a(E)$ is in general strictly included in $\mathscr{H}(E)$.

We shall write $\mathrm{Hom}_n(E, F)$ for the set of all order continuous lattice homomorphisms from E into F, whereas $\mathscr{I}_n(E, F)$ and $\mathscr{H}_n(E, F)$ denote the ideal and the band in $\mathscr{L}_b(E, F)$ generated by all order continuous lattice homomorphisms respectively. Their disjoint complement (with respect to $\mathscr{L}_n(E, F)$) is denoted by $\mathscr{D}_n(E, F)$. Observe that $\mathscr{H}_n(E, F) = \mathscr{H}(E, F) \cap \mathscr{L}_n(E, F)$ and that $\mathscr{D}_n(E, F) = \mathscr{D}(E, F) \cap \mathscr{L}_n(E, F)$.

For any positive operator $T \in \mathscr{L}_b(E, F)^+$ we define the mapping $p_T: E \to F^+$ by

$$p_T(f) = \inf\left\{ \bigvee_{i=1}^{n} Tu_i : |f| \leqslant \bigvee_{i=1}^{n} u_i;\ u_1, \ldots, u_n \in E^+,\ n \in \mathbb{N} \right\}$$

(cf. [PW], proof of Theorem 1, where a similar expression for norms is used). Notice that $p_T(f)$ is well-defined since F is Dedekind complete and that $p_T(u) \leqslant Tu$ for all $u \in E^+$. Furthermore, if $T \in \mathrm{Hom}(E, F)$, then $p_T(u) = Tu$ for all $u \in E^+$.

We claim that p_T is an (F-valued) M-seminorm on E, i.e.,

(i) $p_T(f + g) \leqslant p_T(f) + p_T(g)$, $p_T(\alpha f) = \alpha p_T(f)$ for all $f, g \in E$ and $0 \leqslant \alpha \in \mathbb{R}$
(ii) $p_T(f) \leqslant p_T(g)$, whenever $|f| \leqslant |g|$ in E
(iii) $p_T(u \vee v) = p_T(u) \vee p_T(v)$ for all $u, v \in E^+$.

Indeed, positive homogeneity and (ii) are trivial. As for the subadditivity of p_T, take $u, v \in E^+$ and suppose that we have coverings

$$u \leqslant \bigvee_{i=1}^{n} u_i, \quad v \leqslant \bigvee_{j=1}^{m} v_j$$

for some $u_1, \ldots, u_n, v_1, \ldots, v_m \in E^+$. Then $u + v \leqslant \bigvee_{i=1}^n \bigvee_{j=1}^m (u_i + v_j)$, so

$$p_T(u + v) \leqslant \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (Tu_i + Tv_j) = \bigvee_{i=1}^{n} Tu_i + \bigvee_{j=1}^{m} Tv_j.$$

From this we may conclude immediately that $p_T(u + v) \leqslant p_T(u) + p_T(v)$. It remains to verify the M-property (iii). By (ii), $p_T(u) \vee p_T(v) \leqslant p_T(u \vee v)$ for all u, $v \in E^+$. Take $u_1, \ldots, u_n, v_1, \ldots, v_m \in E^+$ such that $u \leqslant \bigvee_{i=1}^n u_i$, $v \leqslant \bigvee_{j=1}^m v_j$. Then $u \vee v \leqslant (\bigvee_{i=1}^n u_i) \vee (\bigvee_{j=1}^m v_j)$ and hence

$$p_T(u \vee v) \leqslant \left(\bigvee_{i=1}^{n} Tu_i\right) \vee \left(\bigvee_{j=1}^{m} Tv_j\right).$$

Consequently, $p_T(u \vee v) \leqslant p_T(u) \vee p_T(v)$, so equality holds. By this, the claim is completely proved.

It is easily checked that for $R \in \mathrm{Hom}(E, F)$ the statements $0 \leqslant R \leqslant T$ and $Ru \leqslant p_T(u)$ for all $u \in E^+$ are equivalent.

PROPOSITION 2.1. *If* $T \in \mathscr{L}_b(E, F)^+$, *then*

$$p_T(u) = \max\{Ru : R \in \mathrm{Hom}(E, F),\ 0 \leqslant R \leqslant T\}$$

for all $u \in E^+$. *In particular*, $p_T \neq 0$ *if and only if* T *majorizes a non-trivial lattice homomorphism.*

Proof. Fix $u \in E^+$ and put $E_0 = \{\alpha u : \alpha \in \mathbb{R}\}$, the vector sublattice generated by u. Define $R_0 \in \mathrm{Hom}(E_0, F)$ by $R_0(\alpha u) = \alpha p_T(u)$ for all $\alpha \in \mathbb{R}$. Then $R_0 f \leqslant p_T(f)$ for all $f \in E_0$. Now it follows from a Hahn-Banach type theorem for lattice homomorphisms by G. J. H. M. Buskes and A. C. M. van Rooij ([BR1], Theorem 3.6) that there exists $R \in \mathrm{Hom}(E, F)$ such that $Rf = R_0 f$ for all $f \in E_0$ (so in particular $Ru = p_T(u)$) and $Rf \leqslant p_T(f)$ for all $f \in E$ (hence $0 \leqslant R \leqslant T$ by the remark preceding this proposition). This finishes the proof. □

We are now in a position to prove the main result of this section.

THEOREM 2.2. *Let* E *be a vector lattice and* F *a Dedekind complete vector lattice. If* $T \in \mathscr{L}_b(E, F)$, *then* $T \in \mathscr{D}(E, F)$ *if and only if*

$$\inf\left\{\bigvee_{i=1}^{n} |T|u_i : |f| \leqslant \bigvee_{j=1}^{m} u_i;\ u_1, \ldots, u_n \in E^+,\ n \in \mathbb{N}\right\} = 0$$

for all $f \in E$.

Proof. We may clearly assume that $T \geqslant 0$. Obviously $T \in \mathscr{D}(E, F)$ if and only if the only lattice homomorphism majorized by T is trivial. By the above proposition this is equivalent to $p_T = 0$ and the result follows. □

Theorem 2.2 has been proved for the case $F = \mathbb{R}$ by H. Gordon in [G]. Other characterizations of diffuse operators have been obtained by L. W. Weis in the case that E and F are Banach function spaces on separable metric spaces (see [W1] and [W2]).

For later purposes, notice that for $T \in \mathscr{L}_b(E,F)^+$ it is obvious that

$$p_T(u) = \inf\left\{\bigvee_{i=1}^{n} Tu_i : u = \bigvee_{i=1}^{n} u_i;\ u_1, \ldots, u_n \in E^+,\ n \in \mathbb{N}\right\}$$

for all $u \in E^+$. Moreover, if we assume in addition E to be Dedekind complete (actually the principal projection property for E suffices), then we may confine ourselves in this formula to disjoint coverings

$$u = \sum_{i=1}^{n} u_i (u_i \wedge u_j = 0,\ i \neq j)$$

For a proof we refer to [PW], lemma 3.

REMARK 2.3. As observed above, every $T \in \mathscr{I}(E,F)^+$ is a finite sum of lattice homomorphisms. It is a question of independent interest whether finite sums of lattice homomorphisms can be characterized in one way or another. The following remarks extend easily to any finite sum of lattice homomorphisms. If $T = T_1 + T_2$ with T_1, $T_2 \in \mathrm{Hom}(E,F)$, then the positive operator T has the property that $Tu_1 \wedge Tu_2 \wedge Tu_3 = 0$ for each disjoint set $\{u_1, u_2, u_3\}$ in E^+. In case $F = C(X)$, with X an extremally disconnected compact Hausdorff space, it can be shown that this property characterizes operators $T \in \mathscr{L}_b(E,F)^+$ which are sums of two lattice homomorphisms. So far we were unable to prove or disprove this result for arbitrary F.

3. The projection of $\mathscr{L}_b(E, F)$ onto $\mathscr{H}(E, F)$

Throughout this section E, F, G are vector lattices and F, G are Dedekind complete. Denote by $\mathscr{P}_{EF}$ the band projection of $\mathscr{L}_b(E,F)$ onto $\mathscr{H}(E,F)$, according to the band decomposition $\mathscr{L}_b(E,F) = \mathscr{H}(E,F) \oplus \mathscr{D}(E,F)$. For $\mathscr{H}(E,E)$, $\mathscr{D}(E,E)$ and $\mathscr{P}_{EE}$ we simply write $\mathscr{H}(E)$, $\mathscr{D}(E)$ and $\mathscr{P}_E$ respectively. It is our aim to present in this section some algebraic properties of these projections.

The next two propositions exhibit the algebraic structure of $\mathscr{H}(E,F)$ and $\mathscr{D}(E,F)$.

PROPOSITION 3.1. *If* $S \in \mathscr{H}(E,F)$ *and* $T \in \mathscr{H}_n(F,G)$, *then* $TS \in \mathscr{H}(E,G)$.

Proof. It is obvious that $S \in \mathscr{I}(E,F)^+$ and $T \in \mathscr{I}(F,G)^+$ entails $TS \in \mathscr{I}(E,G)^+$. Now take $S \in \mathscr{H}(E,F)^+$ and $T \in \mathscr{I}_n(F,G)^+$. There exist $S_\alpha \in \mathscr{I}(E,F)^+$ such that

$0 \leqslant S_\alpha \uparrow_\alpha S$. As observed, $TS_\alpha \in \mathscr{I}(E, G)^+$ for all α. By the order continuity of T we have $TS_\alpha \uparrow_\alpha TS$ and so $TS \in \mathscr{H}(E, G)^+$. Finally let $S \in \mathscr{H}(E, F)^+$ and $T \in \mathscr{H}_n(F, G)^+$ be given. Then $0 \leqslant T_\beta \uparrow_\beta T$ for appropriate $T_\beta \in \mathscr{I}_n(F, G)^+$. Since $T_\beta S \in \mathscr{H}(E, G)^+$ for all β and $T_\beta S \uparrow_\beta TS$, we may conclude that $TS \in \mathscr{H}(E, G)^+$ from which the assertion in the proposition follows. □

PROPOSITION 3.2. *If* $S \in \mathscr{D}(E, F)$ *and* $T \in \mathscr{L}_n(F, G)$, *then* $TS \in \mathscr{D}(E, G)$.

Proof. We may assume without loss of generality that $S, T \geqslant 0$. By Theorem 2.2, $S \in \mathscr{D}(E, F)^+$ implies that

$$\inf\left\{\bigvee_{i=1}^{n} Su_i : |f| \leqslant \bigvee_{i=1}^{n} u_i;\ u_1, \ldots, u_n \in E^+,\ n \in \mathbb{N}\right\} = 0$$

for all $f \in E$. Since the set of all such finite suprema $\bigvee_{i=1}^{n} Su_i$ is directed downwards, $T \in \mathscr{L}_n(F, G)^+$ yields

$$\inf\left\{T\left(\bigvee_{i=1}^{n} Su_i\right) : |f| \leqslant \bigvee_{i=1}^{n} u_i;\ u_1, \ldots, u_n \in E^+,\ n \in \mathbb{N}\right\} = 0$$

for all $f \in E$. Now

$$0 \leqslant \bigvee_{i=1}^{n} TSu_i \leqslant T\left(\bigvee_{i=1}^{n} Su_i\right)$$

implies

$$\inf\left\{\bigvee_{i=1}^{n} TSu_i : |f| \leqslant \bigvee_{i=1}^{n} u_i;\ u_1, \ldots, u_n \in E^+,\ n \in \mathbb{N}\right\} = 0$$

for all $f \in E$. Once more by Theorem 2.2 we get $TS \in \mathscr{D}(E, G)$. □

A remarkable consequence of the above propositions is the following result, a special case of which was proved in [W1], Corollary 2.

COROLLARY 3.3. *If E is a Dedekind complete vector lattice, then both $\mathscr{H}_n(E)$ and $\mathscr{D}_n(E)$ are subalgebras of $\mathscr{L}_n(E)$. Actually, $\mathscr{D}_n(E)$ is even a left algebra ideal of $\mathscr{L}_n(E)$.*

The results of Propositions 3.1 and 3.2 are reflected in the following properties of the band projections.

THEOREM 3.4. *If* $S \in \mathscr{L}_b(E, F)$ *and* $T \in \mathscr{L}_n(F, G)$, *then*

(i) $\mathscr{P}_{EG}(TS) = \mathscr{P}_{EG}(T\mathscr{P}_{EF}(S))$
(ii) $\mathscr{P}_{EG}(\mathscr{P}_{FG}(T)S) = \mathscr{P}_{FG}(T)\mathscr{P}_{EF}(S)$.

Proof. (i) It follows from Proposition 3.2 that $T(S - \mathscr{P}_{EF}(S) \in \mathscr{D}(E, G)$ and so $\mathscr{P}_{EG}(TS) = \mathscr{P}_{EG}(T\mathscr{P}_{EF}(S))$.

(ii) If $S \in \mathscr{L}_b(E, F)$ and $T \in \mathscr{L}_n(F, G)$, then $\mathscr{P}_{EF}(S) \in \mathscr{H}(E, F)$ and $\mathscr{P}_{FG}(T) \in \mathscr{H}_n(F, G)$, so by Proposition 3.1 $\mathscr{P}_{FG}(T).\ \mathscr{P}_{EF}(S) \in \mathscr{H}(E, G)$. Hence, by part (i),

$$\mathscr{P}_{EG}(\mathscr{P}_{FG}(T)S) = \mathscr{P}_{EG}(\mathscr{P}_{FG}(T).\ \mathscr{P}_{EF}(S)) = \mathscr{P}_{FG}(T).\ \mathscr{P}_{EF}(S)$$

and the proof is complete. □

In particular, if E is a Dedekind complete vector lattice, then

$$\mathscr{P}_E(TS) = \mathscr{P}_E(T\mathscr{P}_E(S)),\ \mathscr{P}_E(\mathscr{P}_E(T)S) = \mathscr{P}_E(T).\ \mathscr{P}_E(S)$$

for all $S \in \mathscr{L}_b(E)$, $T \in \mathscr{L}_n(E)$. The latter equality expresses that the band projection in $\mathscr{L}_n(E)$ onto $\mathscr{H}_n(E)$ satisfies the (left) averaging identity. Even if we assume that $S, T \in \mathscr{L}_n(E)$, the identities $\mathscr{P}_E(TS) = \mathscr{P}_E(\mathscr{P}_E(T)S)$, $\mathscr{P}_E(T\mathscr{P}_E(S)) = \mathscr{P}_E(T).\ \mathscr{P}_E(S)$ need not hold in general. In fact, we shall present in Section 4.5 operators T, S on $E = L_2$ ([0, 1]) for which $T \in \mathscr{D}(E)^+$, $S \in \mathrm{Hom}(E)$ such that $TS \notin \mathscr{D}(E)^+$. This example also shows that $\mathscr{P}_E$ need not be multiplicative and that $\mathscr{D}_n(E)$ is in general not a right algebra ideal of $\mathscr{L}_n(E)$.

The phenomenon of averaging band projections also occurs in other situations. By way of example we will show that the band projection of $\mathscr{L}_b(E)$ onto $\mathscr{L}_n(E)$ is averaging. Similar to Propositions 3.1 and 3.2 and Theorem 3.4 we will discuss a slightly more general situation.

As before, E, F and G are vector lattices and F, G are Dedekind complete. As usual (see [Z], Section 8.4) we have the band decomposition

$$\mathscr{L}_b(E, F) = \mathscr{L}_n(E, F) \oplus \mathscr{L}_s(E, F),$$

where $\mathscr{L}_s(E, F) = \mathscr{L}_n(E, F)^d$ is the band of singular operators. The band projection of $\mathscr{L}_b(E, F)$ onto $\mathscr{L}_n(E, F)$ will be denoted by $\mathscr{Q}_{EF}$ (and $\mathscr{Q}_E$ is the band projection of $\mathscr{L}_b(E)$ onto $\mathscr{L}_n(E)$).

PROPOSITION 3.5. (i) *If* $S \in \mathscr{L}_s(E, F)$ *and* $T \in \mathscr{L}_n(F, G)$, *then* $TS \in \mathscr{L}_s(E, G)$.

(ii) $\mathscr{Q}_{EG}(\mathscr{Q}_{FG}(T)S) = \mathscr{Q}_{FG}(T).\ \mathscr{Q}_{EF}(S)$ *whenever* $S \in \mathscr{L}_b(E, F)$ *and* $T \in \mathscr{L}_b(F, G)$.

Proof. (i) We may assume that $0 \leqslant S \in \mathscr{L}_s(E, F)$ and $0 \leqslant T \in \mathscr{L}_n(F, G)$. By a result of A. R. Schep (see [S1], Theorem 2.6), the fact that S is singular is equivalent to

$$\inf\{\sup Su_\alpha : 0 \leqslant u_\alpha \uparrow u\} = 0$$

for all $u \in E^+$. Using that the set $\{\sup Su_\alpha : 0 \leqslant u_\alpha \uparrow u\}$ is downwards directed and that T is order continuous we obtain

$$0 \leqslant \inf\{\sup(TSu_\alpha): 0 \leqslant u_\alpha \uparrow u\} \leqslant \inf\{T(\sup Su_\alpha): 0 \leqslant u_\alpha \uparrow u\} = 0$$

for all $u \in E^+$. Hence, $TS \in \mathscr{L}_s(E, G)$.

(ii) Applying (i) and observing that $\mathscr{L}_n(F, G). \mathscr{L}_n(E, F) \subset \mathscr{L}_n(E, G)$, the proof of this statement is similar to the proof of Theorem 3.4. □

It is a direct consequence of Proposition 3.5(ii) that for a Dedekind complete vector lattice E the projection $\mathscr{Q}_E$ is averaging, i.e., $\mathscr{Q}_E(\mathscr{Q}_E(T)S) = \mathscr{Q}_E(T). \mathscr{Q}_E(S)$ for all $T, S \in \mathscr{L}_b(E)$. A combination with the earlier result on $\mathscr{P}_E$ yields that the band projection of $\mathscr{L}_b(E)$ onto $\mathscr{H}_n(E)$ is averaging as well.

An interesting spin-off of the above observations is the following result.

COROLLARY 3.6. *Let E and F be Dedekind complete vector lattices.*

(i) *If $T \in \mathscr{L}_n(E, F)$ such that T^{-1} exists in $\mathscr{L}_b(F, E)$, then $T^{-1} \in \mathscr{L}_n(F, E)$.*
(ii) *If $T \in \mathscr{H}_n(E, F)$ such that T^{-1} exists in $\mathscr{L}_b(F, E)$, then $T^{-1} \in \mathscr{H}_n(F, E)$.*

In particular, $\mathscr{L}_n(E)$ and $\mathscr{H}_n(E)$ are full subalgebras of $\mathscr{L}_b(E)$.

Proof. (i) Since $TT^{-1} = I_F$, the identity mapping on F, it follows from Proposition 3.5(ii) that

$$I_F = \mathscr{Q}_F(TT^{-1}) = \mathscr{Q}_F(\mathscr{Q}_{EF}(T)T^{-1}) = \mathscr{Q}_{EF}(T)\mathscr{Q}_{FE}(T^{-1}) = T\mathscr{Q}_{FE}(T^{-1}),$$

hence $T^{-1} = \mathscr{Q}_{FE}(T^{-1}) \in \mathscr{L}_n(F, E)$.

(ii) Use (i) and Theorem 3.4(ii). □

REMARK 3.7. As is well known, another example of a full subalgebra of $\mathscr{L}_b(E)$, with E Dedekind complete, is provided by the band and subalgebra Orth(E) of all orthomorphisms on E (for the general theory of Orth(E), see [Z], Chapter 20). In fact, also in this case the band projection onto Orth(E) is averaging. It can even be shown (see [HP], Theorem 3) that Orth(E) is a full subalgebra of $\mathscr{L}^\#(E)$, the algebra of all linear operators on the vector lattice E (cf. [A2], Proposition 2.7, [BR 2], 3.7.1 and [S2], Theorem 1.8).

We end this section with a formula for $\mathscr{P}_{EF}(T)$, $T \in \mathscr{L}_b(E, F)^+$.

PROPOSITION 3.8. *If E and F are vector lattices and F is Dedekind complete, then for any $T \in \mathscr{L}_b(E, F)^+$*

$$\mathscr{P}_{EF}(T)u = \sup\left\{\sum_{i=1}^n p_T(u_i): u = \sum_{i=1}^n u_i;\ u_1, \dots, u_n \in E^+,\ n \in \mathbb{N}\right\}$$

for all $u \in E^+$ (where p_T is as in Section 2).

Proof. We confine ourselves to a rough sketch. Since there exists a maximal disjoint system in $\mathscr{H}(E, F)^+$ consisting of elements in Hom(E, F) and since each positive element in the band generated by a lattice homomorphism is itself a lattice homomorphism, any $S \in \mathscr{H}(E, F)^+$ is the supremum of a disjoint set in $\mathrm{Hom}(E, F)$. This implies that

$$\mathscr{P}_{EF}(T) = \sup\{R \in \mathrm{Hom}(E, F): 0 \leqslant R \leqslant T\}.$$

It follows from this formula that if $S \in \mathscr{L}_b(E, F)^+$ satisfies $0 \leqslant p_T(v) \leqslant Sv$ for all $v \in E^+$, then $\mathscr{P}_{EF}(T) \leqslant S$. On the other hand, an application of Proposition 2.1 shows that $p_T(v) \leqslant \mathscr{P}_{EF}(T)(v)$ for all $v \in E^+$, so

$$\mathscr{P}_{EF}(T) = \inf\{S \in \mathscr{L}_b(E, F)^+: p_T(v) \leqslant Sv \text{ for all } v \in E^+\}$$

A standard argument yields now the desired result. □

As a consequence of the M-property of p_S (with $S \in \mathscr{L}_b(E, F)^+$) we see that $p_S(u) = Su$ for all $u \in E^+$ if and only if $S \in \mathrm{Hom}(E, F)$. This shows that in general $p_T \neq \mathscr{P}_{EF}(T)$ on E^+.

4. Examples

In this section we will illustrate the previous results with some examples (and counterexamples).

4.1. *A class of diffuse operators on* ℓ_∞

Let $L: \ell_\infty \to \ell_\infty$ be the left translation defined by $(Lx)(n) = x(n+1)(n \in \mathbb{N})$ for all $x \in \ell_\infty$. An operator S from ℓ_∞ into a Dedekind complete vector lattice F is called *left translation invariant* whenever $SL = S$ (observe that S is in this case right translation invariant as well). We will show that any left translation invariant operator $S \in \mathscr{L}_b(\ell_\infty, F)$ is diffuse. Indeed, note first that $|S|L = |S|$, so we may assume that $S \geqslant 0$. Since $\mathbf{1} = (1, 1, 1, \ldots)$ is a strong order unit in ℓ_∞, it suffices by Theorem 2.2 to show that

$$\inf\left\{\bigvee_{i=1}^{n} Su_i : \mathbf{1} \leqslant \bigvee_{i=1}^{n} u_i;\ u_1, \ldots, u_n \in \ell_\infty^+,\ n \in \mathbb{N}\right\} = 0.$$

Given $m \in \mathbb{N}$, we can write $\mathbf{1} = \Sigma_{i=1}^m w_i$, where $w_i (i = 1, \ldots, m)$ are positive pairwise disjoint elements in ℓ_∞ such that $L^{i-1} w_i = w_1$ $(i = 2, \ldots, m)$. Since S is left translation invariant, it is clear that $Sw_i = (1/m)S\mathbf{1}$ $(i = 1, \ldots, m)$ and hence $\bigvee_{i=1}^m Sw_i = (1/m)S\mathbf{1}$ from which the desired result follows.

Taking $F = \mathbb{R}$ in the above we get in particular that every Banach-Mazur limit on ℓ_∞ (see e.g. [Z], Exercises 103.16 and 103.17) is necessarily diffuse.

In the next example it is shown that there do not exist non-trivial order continuous diffuse operators on ℓ_∞.

4.2. *Vector lattices with only trivial diffuse operators*

Let E be a discrete vector lattice (i.e., E contains a maximal disjoint system consisting of atoms; see [LZ], Exercise 37.22) and F a Dedekind complete vector lattice. We assert that $\mathscr{D}_n(E, F) = \{0\}$. To this end, it suffices to show that any $0 < T \in \mathscr{L}_n(E, F)$ majorizes a non-trivial lattice homomorphism. Since T is order continuous and E is discrete, there exists an atom $0 < u \in E$ such that $Tu > 0$. It is well-known that the principal band $\{u\}^{dd}$ generated by u in E is a one-dimensional projection band ([LZ], Theorem 26.4). Let P_u be the band projection of E onto $\{u\}^{dd}$. Then TP_u is evidently a non-trivial lattice homomorphism dominated by T.

Note that this result implies in particular that $\mathscr{D}(\ell_p) = \{0\}$ and $\mathscr{L}_b(\ell_p) = \mathscr{H}(\ell_p)(1 \leqslant p < \infty)$. Moreover, $\mathscr{D}_n(\ell_\infty) = \{0\}$, so $\mathscr{L}_n(\ell_\infty) = \mathscr{H}_n(\ell_\infty)$.

4.3. *Kernel operators*

Let (Y, Σ, ν) and (X, Λ, μ) be σ-finite measure spaces and let $L_0(Y, \nu)$ and $L_0(X, \mu)$ be the Dedekind complete vector lattices of all measurable functions on these measure spaces with the usual identification of almost equal functions. Let L and M be order ideals in $L_0(Y, \nu)$ and $L_0(X, \mu)$ respectively. We may assume without loss of generality that the carriers of L and M are Y and X respectively (see [Z], Section 86). It is well known that the set of all absolute kernel operators from L into M is a band ([Z], Theorem 94.5).

As another application of Theorem 2.2 we will show that in case (Y, Σ, ν) does not contain atoms, all absolute kernel operators from L into M are diffuse. Indeed, let T be a positive kernel operator from L into M with positive kernel $T(x, y)$, i.e.,

$$(Tf)(x) = \int_Y T(x, y) f(y)\,\mathrm{d}\nu(y)$$

a.e. on X. By Theorem 2.2 we have to prove that $p_T(u) = 0$ for all $u \in L^+$. Since T is order continuous, a standard argument shows that it is sufficient to prove that $p_T(\chi_A) = 0$ for all $A \in \Sigma$, $0 < \nu(A) < \infty$ and $\chi_A \in L$. Since (Y, Σ, ν) is non-atomic there exist for each $n \in \mathbb{N}$ disjoint sets $A_{nk} \in \Sigma$ with

$$A = \bigcup_{k=1}^{2^n} A_{nk}, \; \nu(A_{nk}) = 2^{-n}\nu(A) \qquad (k = 1, \ldots, 2^n).$$

We claim that $\bigvee_{k=1}^{2^n} T\chi_{A_{nk}} \downarrow_n 0$ in M. If not, there exist $B \in \Lambda$, $0 < \mu(B) < \infty$ and $\varepsilon > 0$ such that $(\bigvee_{k=1}^{2^n} T\chi_{A_{nk}})(x) \geqslant \varepsilon$ for all $x \in B$ $(n = 1, 2, \ldots)$. Fix $x_0 \in B$. It follows that there exists for each $n \in \mathbb{N}$ a natural number $k(n)$ such that $(T\chi_{A_{n,k(n)}})(x_0) \geqslant \varepsilon$, i.e.,

$$\int_Y T(x_0, y)\chi_{A_{n,k(n)}}(y)\,\mathrm{d}v(y) \geqslant \varepsilon.$$

On the other hand

$$0 \leqslant T(x_0, y)\chi_{A_{n,k(n)}}(y) \leqslant T(x_0, y)\chi_A(y)$$

a.e. on Y and $T(x_0, y)X_{A_{n,k(n)}}(y) \to 0$ a.e. on Y, which is at variance with the dominated convergence theorem, so the claim is proved. It follows immediately from the definition of p_T that $p_T(\chi_A) = 0$.

The fact that kernel operators are disjoint to lattice homomorphisms was already obtained by the second author in ([P1], Theorem 8.1) by somewhat different means.

4.4. *Compact operators*

It was shown by the second author (see [P1], Theorem 5.2) in 1979 and independently by A. W. Wickstead ([Wi], Theorem 5.3) that if E, F are Banach lattices and E^* is non-atomic, then the only compact lattice homomorphism $T: E \to F$ is 0. However, if E^* does contain atoms, then there exist non-trivial compact lattice homomorphisms. Indeed, if $0 < \varphi \in E^*$ is an atom (so φ is a non-trivial real lattice homomorphism on E) and $0 < g \in F$, then $T = \varphi \otimes g$ (i.e., $Tf = \varphi(f)g$ for all $f \in E$) is a rank one operator and hence compact. Furthermore $\varphi \in \mathrm{Hom}(E, \mathbb{R})$ implies that $T \in \mathrm{Hom}(E, F)$. It is in fact shown by the above authors that every non-zero compact $T \in \mathrm{Hom}(E, F)$ can be written in the form

$$T = \sum_{n=1}^{\infty} \varphi_n \otimes g_n$$

for suitable disjoint atoms $0 < \varphi_n \in E^*$ and disjoint elements $0 < g_n \in F$.

A related question is whether $0 \leqslant T \leqslant K$, $T \in \mathrm{Hom}(E, F)$, K compact, implies that T is compact as well. Notice that in case E^* and F have order continuous norms the answer is affirmative by the Dodds-Fremlin result ([Z], Theorem 124.10). Observe in this connection that it was shown by A. R. Schep ([S2], Theorem 1.6) and also by W. Arendt ([A1], Satz 1.25) that if E is a Banach lattice, $0 \leqslant \pi \leqslant K$, $\pi \in Z(E)$, the center of E, and K compact implies that π is compact and that $\pi = 0$ provided E does not contain atoms. In general, the answer to the above question is negative. By way of example, take $E = \ell_1$, $F = \ell_\infty$, T the imbedding of ℓ_1 into ℓ_∞ and $K: \ell_1 \to \ell_\infty$ defined by

$$(Kf)(n) = \sum_{i=1}^{\infty} f(i) \qquad (n = 1, 2, \dots)$$

for all $f \in \ell_1$.

In the next theorem we will show that the answer to the above question is positive if E^* is non-atomic.

THEOREM 4.1. *Let E, F be Banach lattices such that E^* does not contain atoms. If $T \in \mathrm{Hom}(E, F)$, $K: E \to F$ is compact and $0 \leqslant T \leqslant K$, then $T = 0$.*

In order to prove this result, we first consider a special situation to which we shall reduce the general case.

PROPOSITION 4.2. *Let E, F be Dedekind complete Banach lattices such that E is non-atomic and ${}^{\perp}(F_n^*) = \{0\}$. If $T \in \mathrm{Hom}_n(E, F)$, $K: E \to F$ compact and $0 \leqslant T \leqslant K$, then $T = 0$.*

Proof. We will show that the present hypotheses imply $0 \leqslant 2T \leqslant K$ from which the result is immediate. For this purpose, choose $0 < u \in E$, $0 < \psi \in F_n^*$ and put $\varphi = T^*\psi \in (E_n^*)^+$. We have to prove that $\langle Ku, \psi \rangle \geqslant 2\langle Tu, \psi \rangle$, so we may assume that $\langle u, \varphi \rangle > 0$. Since E is non-atomic there exist for each $n \in \mathbb{N}$ disjoint components u_{nk} of u such that $\langle u_{nk}, \varphi \rangle = 2^{-n}\langle u, \varphi \rangle$ ($k = 1, \dots, 2^n$) and $u_{n-1,k} = u_{n,2k-1} + u_{n,2k}$ ($k = 1, \dots, 2^{n-1}$) (with $u_{01} = u$). For $n = 1, 2, \dots$, we define

$$u_n = \sum_{\substack{k=1 \\ (k \text{ odd})}}^{2^n} u_{nk}.$$

A straightforward argument shows that if $n_1 < n_2 < \cdots < n_p$ then

$$\left\langle \bigvee_{i=1}^{p} u_{n_i}, \varphi \right\rangle = (1 - 2^{-p})\langle u, \varphi \rangle.$$

This implies easily that $\langle \limsup_i u_{n_i}, \varphi \rangle = \langle u, \varphi \rangle$ for every subsequence $\{n_i\}$ of $\mathbb{N}$. A similar computation yields that $\langle \limsup_i (u - u_{n_i}), \varphi \rangle = \langle u, \varphi \rangle$ or, equivalently, $\langle \liminf_i u_{n_i}, \varphi \rangle = 0$.

It follows from $0 \leqslant u_n \leqslant u$ that $\{u_n\}_{n=1}^{\infty}$ is norm bounded so that $\{Ku_n\}_{n=1}^{\infty}$ has a norm convergent subsequence, say $Ku_{n_i} \to v \in F^+$ in norm. Now, since every norm convergent sequence in a Banach lattice has an order convergent subsequence (according to [Z], Theorem 100.6) we may assume that $Ku_{n_i} \to v$ in order as well, hence $v = \limsup_i Ku_{n_i}$. Evidently, we have

$T(\limsup_i u_{n_i}) = \limsup_i Tu_{n_i}$, as T is an order continuous lattice homomorphism and hence

$$v = \limsup_i Ku_{n_i} \geqslant T\Big(\limsup_i u_{n_i}\Big).$$

It follows that

$$\langle v, \psi\rangle \geqslant \Big\langle T\Big(\limsup_i u_{n_i}\Big), \psi\Big\rangle = \Big\langle \limsup_i u_{n_i}, \varphi\Big\rangle = \langle u, \varphi\rangle.$$

In like manner, $0 \leqslant Ku - v = \limsup_i K(u - u_{n_i})$ yields that

$$\langle Ku - v, \psi\rangle \geqslant \langle u, \varphi\rangle.$$

Hence,

$$\langle Ku, \psi\rangle = \langle v, \psi\rangle + \langle Ku - v, \psi\rangle \geqslant 2\langle u, \varphi\rangle = 2\langle Tu, \psi\rangle,$$

which is the desired result. □

Proof of Theorem 4.1. Observe first that $T^*: F^* \to E^*$ is interval preserving (see e.g. [AB], Theorem 7.8). Moreover, T^{**} maps $(E^*)_n^*$ into $(F^*)_n^*$, as T^* is order continuous. By [AB], Theorem 7.7, $T^{**} \in \mathrm{Hom}_n(E^{**}, F^{**})$. If $(T^*)'$ denotes the restriction of T^{**} to $(E^*)_n^*$, then we have $(T^*)' \in \mathrm{Hom}_n((E^*)_n^*, (F^*)_n^*)$. Clearly, $0 \leqslant (T^*)' \leqslant (K^*)'$ and $(K^*)'$ is compact. Furthermore, E^* is non-atomic, so $(E^*)_n^*$ is non-atomic as well. Proposition 4.2 gives $(T^*)' = 0$ and hence $T = 0$ (since E can be considered as a subspace of $(E^*)_n^*$). □

Under the assumption that T is order continuous, the conclusion of the above theorem remains valid under the weaker hypothesis that E is non-atomic. This follows by observing that $0 \leqslant T \leqslant K: E \to F$, with T order continuous and K compact, implies $T^*(F^*) \subseteq E_n^*$. Now the reduction to Proposition 4.2 is similar to the proof of the theorem above.

Combining the above results with Theorem 2.2 the following corollary is immediate.

COROLLARY 4.3. *Let E and F be Banach lattices, with F Dedekind complete.*

(a) *If* E^* *is non-atomic and* $0 \leqslant K: E \to F$ *compact, then* $K \in \mathscr{D}(E, F)$ *and hence*

$$\inf\Big\{\bigvee_{i=1}^n Ku_i \colon |f| \leqslant \bigvee_{i=1}^n u_i;\ u_1, \ldots, u_n \in E^+,\ n \in \mathbb{N}\Big\} = 0$$

for all $f \in E$.

(b) *If E is non-atomic, then $K \in \mathscr{H}_n(E, F)^d$ for all compact $0 \leqslant K: E \to F$.*

In connection with Corollary 4.3(b) it should be mentioned that W. Arendt showed in [A1], Korollar 1.26, that any positive compact operator on a non-atomic Dedekind complete Banach lattice is disjoint to all lattice isomorphisms (which are, of course, all order continuous).

Finally note that in the situation of Corollary 4.3(b), K need not be disjoint to all σ-order continuous lattice homomorphisms. In fact, the Dedekind σ-complete Banach lattice $E = \ell_\infty/c_0$ is non-atomic and has σ-order continuous norm, whereas the real valued lattice homomorphisms separate the points of E.

4.5. *Convolution operators*

In this example we will characterize a class of diffuse convolution operators. Let $\Delta = \{-1, 1\}^{\mathbb{N}}$ be the Cantor group with Haar measure λ. For $n = 1, 2, \ldots$, let the probability measure μ_n on $\{-1, 1\}$ be defined by $\mu_n(\{1\}) = \alpha_n$, $\mu_n(\{-1\}) = 1 - \alpha_n$ with $0 \leqslant \alpha_n \leqslant 1$ and let $\mu = \bigotimes_{n=1}^\infty \mu_n$ be the corresponding product measure on Δ. Note that for the choice $\alpha_n = \frac{1}{2}(n = 1, 2, \ldots)$ we get $\mu = \lambda$. For ease of notation we shall write henceforth $\alpha_n^{(1)} = \alpha_n$ and $\alpha_n^{(-1)} = 1 - \alpha_n$ $(n = 1, 2, \ldots)$. The Rademacher functions $\{r_n\}_{n=0}^\infty$ on Δ are defined by

$$r_0 = \mathbf{1},\ r_n(t) = r_n(t) = r_n(t_1, t_2, \ldots) = t_n$$

for all $t \in \Delta$. For each finite subset F of $\mathbb{N}$ the corresponding Walsh function is defined by $w_F = \Pi_{n \in F} r_n$ $(w_\phi = \mathbf{1})$. The Walsh functions constitute an orthonormal basis of $E = L_2(\Delta, \lambda)$.

The convolution operator $T_\mu \in \mathscr{L}_b(E)^+$ is defined by $T_\mu f = \mu * f$ for all $f \in E$. It follows from

$$T_\mu w_F = \prod_{n \in F} (2\alpha_n - 1) w_F$$

that T_μ is a diagonal operator in E. In passing we note that for $0 < \alpha_n < 1$ $(n = 1, 2, \ldots)$ according to a classical result of S. Kakutani (see e.g. [HS], Theorem 22.36 and Exercise 22.38), μ is either absolutely continuous or singular with respect to λ depending on the convergence or divergence of the series $\Sigma_{n=1}^\infty (2\alpha_n - 1)^2$, respectively. Furthermore, in this case T_μ is a kernel operator if and only if $\Sigma_{n=1}^\infty (2\alpha_n - 1)^2 < \infty$ and T_μ is compact if and only if $\alpha_n \to \frac{1}{2}$ as $n \to \infty$ (for details we refer to [PS], Example 3.7). It is not difficult to show that $T_\mu \in \mathrm{Hom}(E)$ if and only if $\alpha_n \in \{0, 1\}$ $(n = 1, 2, \ldots)$, i.e. μ is a Dirac measure (cf. [A1], Satz 2.8). Our aim is to derive a necessary and sufficient condition for T_μ to be diffuse. We first need a lemma which is formulated for the space $E = L_2(\Delta, \lambda)$ although the result extends easily to Banach function spaces with order continuous norm on a wider class of measure spaces. The proof involves a standard measure theoretical argument and is therefore omitted. First we settle some notation. For $t = (t_1, \ldots, t_n) \in \{-1, 1\}^n$ we define the cylinder set

$$\Delta_t = \{(s_1, s_2, \ldots) \in \Delta : s_j = t_j, j = 1, \ldots, n\}.$$

Observe that $\Delta = \cup\{\Delta_t : t \in \{-1, 1\}^n\}$, a disjoint union. By the very definition of the product measure μ we have $\mu(\Delta_t) = \alpha_1^{(t_1)} \cdots \alpha_n^{(t_n)}$.

LEMMA 4.4. *If* $T \in \mathscr{L}_b(E)^+$, *then* $p_T(\mathbf{1}) = \inf_n \sup(T\chi_{\Delta_t} : t \in \{-1, 1\}^n)$.

THEOREM 4.5. $T_\mu \in \mathscr{D}(E)$ *if and only if*

$$\inf_n \max_{t \in \{-1,1\}^n} \alpha_1^{(t_1)} \cdots \alpha_n^{(t_n)} = 0.$$

Proof. Obviously, T_μ is diffuse if and only if $p_{T_\mu}(\mathbf{1}) = 0$. For $s, t \in \{-1, 1\}^n$ we will write $st = (s_1 t_1, \ldots, s_n t_n)$. A straightforward computation shows that $T_\mu \chi_{\Delta_t}(x) = \mu(\Delta_{st})$ for all $x \in \Delta_s$ and hence

$$T_\mu \chi_{\Delta_t} = \sum_{s \in \{-1,1\}^n} \mu(\Delta_{st}) \chi_{\Delta_s} = \bigvee_{s \in \{-1,1\}^n} \mu(\Delta_{st}) \chi_{\Delta_s}$$

for all $t \in \{-1, 1\}^n$. Hence,

$$\begin{aligned} \bigvee_{t \in \{-1,1\}^n} T_\mu \chi_{\Delta_t} &= \bigvee_{t \in \{-1,1\}^n} \bigvee_{s \in \{-1,1\}^n} \mu(\Delta_{st}) \chi_{\Delta_s} \\ &= \bigvee_{s \in \{-1,1\}^n} \bigvee_{t \in \{-1,1\}^n} \mu(\Delta_{st}) \chi_{\Delta_s} = \bigvee_{s \in \{-1,1\}^n} \bigvee_{t \in \{-1,1\}^n} \mu(\Delta_t) \chi_{\Delta_s} \\ &= \Bigg(\bigvee_{t \in \{-1,1\}^n} \mu(\Delta_t) \Bigg) \Bigg(\bigvee_{s \in \{-1,1\}^n} \chi_{\Delta_s} \Bigg) = \max_{t \in \{-1,1\}^n} \alpha_1^{(t_1)} \cdots \alpha_n^{(t_n)} \cdot \mathbf{1}. \end{aligned}$$

By Lemma 4.4,

$$p_{T_\mu}(\mathbf{1}) = \inf_n \max_{t \in \{-1,1\}^n} \alpha_1^{(t_1)} \cdots \alpha_n^{(t_n)} \cdot \mathbf{1},$$

from which the result of the theorem is immediate. □

The above theorem shows that if we choose $\frac{1}{2} \leqslant \alpha_n \leqslant 1$ $(n = 1, 2, \ldots)$ the convolution operator T_μ is diffuse if and only if $\lim_{n\to\infty} \alpha_1 \cdots \alpha_n = 0$, i.e., $\Pi_{n=1}^{\infty} \alpha_n$ diverges to 0. Furthermore, if $0 < \delta \leqslant \alpha_n \leqslant 1 - \delta$ $(n = 1, 2, \ldots)$ and $\alpha_n \nrightarrow \frac{1}{2}$, then we obtain examples of diffuse convolution operators T_μ which are neither compact nor kernel operator.

It was proved in Proposition 3.2 that $T \in \mathscr{D}(E)$, $S \in \mathscr{L}_n(E)$ (with E a Dedekind complete vector lattice) implies $ST \in \mathscr{D}(E)$. We now present an example which illustrates that TS need not be diffuse. To this end, define the measure preserving transformation $\sigma: \Delta \to \Delta$ by

$$\sigma(t) = \sigma(t_1, t_2, \ldots, t_n, \ldots) = (t_1, t_4, \ldots, t_{n^2}, \ldots)$$

for all $t \in \Delta$ and let $S \in \mathrm{Hom}(E)$ be defined by $(Sf)(t) = f(\sigma(t))$, $f \in E$, $t \in \Delta$. A similar computation as in the proof of Theorem 4.5 shows that

$$p_{T_\mu S}(\mathbf{1}) = \inf_n \max_{t \in \{-1,1\}^n} \alpha_1^{(t_1)} \alpha_4^{(t_2)} \cdots \alpha_{n^2}^{(t_n)} \cdot \mathbf{1}.$$

Consequently, if $\frac{1}{2} \leqslant \alpha_n \leqslant 1$ $(n = 1, 2, \ldots)$, then $T_\mu S$ is diffuse if and only if $\Pi_{n=1}^{\infty} \alpha_{n^2}$ diverges to 0. Hence, for the particular choice $\alpha_n = 1 - (n+1)^{-1}$ $(n = 1, 2, \ldots)$ we obtain a diffuse operator T_μ for which $T_\mu S$ is not diffuse.

5. The projection onto the band generated by a disjointness preserving operator

Throughout this section E and F are Dedekind complete vector lattices. For any $T \in \mathscr{L}_b(E, F)$ we shall denote the principal band generated by T in $\mathscr{L}_b(E, F)$ with $\{T\}^{dd}$. If T is disjointness preserving then $|T| \in \mathrm{Hom}(E, F)$. It follows from $\{T\}^{dd} = \{|T|\}^{dd}$ that, when considering the band generated by an order bounded disjointness preserving operator T, we may assume from the beginning on that $T \in \mathrm{Hom}(E, F)$. Let $\mathscr{P}_T$ be the band projection of $\mathscr{L}_b(E, F)$ onto $\{T\}^{dd}$. In the present section we shall investigate properties of $\mathscr{P}_T$ in detail.

In our observations the 'Luxemburg t-map' will play an important role (see [L], section 3.3). We denote the Boolean algebra of all band projections in E by $\mathscr{B}(E)$ (see [LZ], Section 30). Fix $T \in \mathrm{Hom}(E, F)$. For any $P \in \mathscr{B}(E)$ the band projection in F onto $\{T(P(E))\}^{dd}$ is denoted by $t(P)$. The mapping $t: \mathscr{B}(E) \to \mathscr{B}(F)$ defines a Boolean homomorphism satisfying $t(P)T = TP$ for all $P \in \mathscr{B}(E)$ (for details, see [L], 3.3 and [H1], 3.15).

If $S \in \mathscr{L}_b(E, F)^+$, then, as already observed, $S \in \{T\}^{dd}$ implies $S \in \mathrm{Hom}(E, F)$. It was proved by W. A. J. Luxemburg and A. R. Schep in [LS], Theorem 4.2 that $S \in \{T\}^{dd}$ if and only if S is absolutely continuous with respect to T, i.e., $Su \in \{Tu\}^{dd}$ for all $u \in E^+$. For our purposes we need a slight refinement of their result.

LEMMA 5.1. *Let E, F be Dedekind complete vector lattices, $T \in \mathrm{Hom}(E, F)$ and $S \in \mathscr{L}_b(E, F)^+$. Then the following statements are equivalent.*

(i) $S \in \{T\}^{dd}$
(ii) $t(P)S = SP$ *for all* $P \in \mathscr{B}(E)$
(iii) $Su \in \{T\{u\}^{dd}\}^{dd}$ *for all* $u \in E^+$.

Proof. (i) $\Rightarrow$ (ii). For fixed P the mappings $S \mapsto SP$ and $S \mapsto t(P)S$ are band projections in $\mathscr{L}_b(E, F)$ which coincide on T and hence on $\{T\}^{dd}$.

(ii) $\Rightarrow$ (i). First we show that $S + T \in \mathrm{Hom}(E, F)$. Suppose $u \wedge v = 0$ in E and let P be the band projection of E onto $\{u\}^{dd}$. Then

$$\begin{aligned}(S+T)u \wedge (S+T)v &= (S+T)Pu \wedge (S+T)(I_E - P)v \\ &= t(P)(S+T)u \wedge t(I_E - P)(S+T)v = 0.\end{aligned}$$

Now decompose $S = S_1 + S_2$ with $0 \leqslant S_1 \in \{T\}^d$ and $0 \leqslant S_2 \in \{T\}^{dd}$. It follows from $S_1 + T \in \mathrm{Hom}(E, F)$, $0 \leqslant S_1 \leqslant S_1 + T$, $0 \leqslant T \leqslant S_1 + T$ that there exist π_1, $\pi_2 \in \mathrm{Orth}(F)$, $0 \leqslant \pi_1, \pi_2 \leqslant I_F$ such that $S_1 = \pi_1(S_1 + T)$ and $T = \pi_2(S_1 + T)$ (a result due to S. S. Kutateladze; see e.g. [AB], Theorem 8.16). Hence

$$(\pi_1 \wedge \pi_2)(S_1 + T) = \pi_1(S_1 + T) \wedge \pi_2(S_1 + T) = S_1 \wedge T = 0$$

(use the result of [AB], Exercise 8.11).

Put $\rho_1 = \pi_1 - \pi_1 \wedge \pi_2$ and $\rho_2 = \pi_2 - \pi_1 \wedge \pi_2$. Then $\rho_1 \wedge \rho_2 = 0$ in $\mathrm{Orth}(F)$, $S_1 = \rho_1(S_1 + T)$ and $T = \rho_2(S_1 + T)$. Since the infimum of two orthomorphisms is pointwise on positive elements (see [Z], Theorem 140.4) we infer $S_1 w \wedge Tw = 0$ for all $w \in E^+$. This implies easily that $S_1 u \wedge Tv = 0$ for all u, $v \in E^+$, so $S_1(E) \subset \{T(E)\}^d$. On the other hand, the hypothesis on S yields that $S(E) \subset \{T(E)\}^{dd}$, so it follows from $0 \leqslant S_1 u \leqslant Su$ that $S_1 u \in \{T(E)\}^{dd}$ for all $u \in E^+$, showing that $S_1(E) \subset \{T(E)\}^{dd}$. Hence, $S_1 = 0$ and consequently $S = S_2 \in \{T\}^{dd}$.

As to the equivalence of (ii) and (iii), condition (ii) is evidently equivalent to $S(B) \subset \{T(B)\}^{dd}$ for all bands B in E and hence equivalent to (iii). The proof is complete. □

Observing that $S \in \mathscr{L}_b(E, F)$ satisfies $t(P)S = SP$ for all $P \in \mathscr{B}(E)$ if and only if $t(P)|S| = |S|P$ for all $P \in \mathscr{B}(E)$ the following corollary is immediate.

COROLLARY 5.2. *E, F and T as in Lemma* 5.1. *Then*

$$\{T\}^{dd} = \{S \in \mathscr{L}_b(E, F) \colon t(P)S = SP \text{ for all } P \in \mathscr{B}(E)\}.$$

The above corollary is the main ingredient in the proof of the following result.

PROPOSITION 5.3. *Let E, F be Dedekind complete vector lattices, $T \in \mathrm{Hom}(E, F)$, $\mathscr{P}_T$ the band projection in $\mathscr{L}_b(E, F)$ onto $\{T\}^{dd}$ and $S \in \mathscr{L}_b(E, F)^+$. Then*

$$\mathscr{P}_T(S) = \inf\left\{\sum_{i=1}^{n} t(P_i)SP_i \colon I_E = \sum_{i=1}^{n} P_i \text{ in } \mathscr{B}(E)\right\}.$$

Proof. Denote the infimum on the right hand side of the above formula by R. Obviously, $R \leqslant S$. Since $\mathscr{P}_T(S) \in \{T\}^{dd}$ we have by Lemma 5.1 that $t(P)\mathscr{P}_T(S) = \mathscr{P}_T(S)P$ for all $P \in \mathscr{B}(E)$. Take disjoint band projections $P_1, \ldots, P_n \in \mathscr{B}(E)$ such that $\Sigma_{i=1}^n P_i = I_E$. Then

$$\mathscr{P}_T(S) = \sum_{i=1}^n \mathscr{P}_T(S)P_i = \sum_{i=1}^n t(P_i)\mathscr{P}_T(S)P_i$$
$$\leqslant \sum_{i=1}^n t(P_i)SP_i,$$

showing that $\mathscr{P}_T(S) \leqslant R$. Furthermore, for all $P \in \mathscr{B}(E)$ we have

$$0 \leqslant R \leqslant t(P)SP + t(I_E - P)S(I_E - P).$$

Put $U = t(P)SP + t(I_E - P)S(I_E - P)$. As observed earlier, left multiplication with $t(P)$ and right multiplication with P are band projections of $\mathscr{L}_b(E, F)$, which agree on U, as $t(P)U = UP = t(P)SP$. Therefore, they coincide on $\{U\}^{dd}$, so $t(P)R = RP$ for all $P \in \mathscr{B}(E)$. By Corollary 5.2, $R \in \{T\}^{dd}$, i.e., $\mathscr{P}_T(R) = R$. Consequently, $R \leqslant S$ yields $R \leqslant \mathscr{P}_T(S)$ and hence $R = \mathscr{P}_T(S)$. □

In the special case that $E = F$ and $T = I$ (so $t = \text{id.}$ and $\{T\}^{dd} = \text{Orth}(E)$) the above result is due to A. R. Schep ([S2], Theorem 1.1).

Let E be a Dedekind complete vector lattice, $T \in \text{Hom}(E)$ and S a linear operator on E. For any partition $\Pi = \{P_1, \ldots, P_n\}$ of I_E into (disjoint) band projections $P_i \in \mathscr{B}(E)$ $(i = 1, \ldots, n)$ (so $I_E = \Sigma_{i=1}^n P_i$) we will write

$$\mathscr{P}_\Pi(S) = \sum_{i=1}^n t(P_i)SP_i.$$

In the next lemma we shall mean by $\Sigma_{(\varepsilon)}$ the summation over all n-tuples $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ for all possible choices of $\varepsilon_i = \pm 1$ $(i = 1, \ldots, n)$.

LEMMA 5.4

$$|\mathscr{P}_\Pi(S)f| \leqslant 2^{-n} \sum_{(\varepsilon)} \left| S\left(\sum_{i=1}^n \varepsilon_i P_i f\right)\right|$$

for all $f \in E$.

Proof. It is sufficient to treat the case $n = 2$. The general situation follows then from a standard induction argument. To this end, take band projections P_1,

$P_2 \in \mathscr{B}(E)$ such that $I_E = P_1 + P_2$. Then

$$
\begin{aligned}
|\mathscr{P}_\Pi(S)f| &= |t(P_1)SP_1f + t(P_2)SP_2f| \\
&= \tfrac{1}{2}|t(I_E)(SP_1f + SP_2f) + (t(P_1) - t(P_2))(SP_1f - SP_2f)| \\
&\leqslant \tfrac{1}{2}t(I_E)|SP_1f + SP_2f| + \tfrac{1}{2}|t(P_1) - t(P_2)||SP_1f - SP_2f| \\
&= \tfrac{1}{2}t(I_E)\{|SP_1f + SP_2f| + |SP_1f - SP_2f|\} \\
&\leqslant \tfrac{1}{4}|SP_1f + SP_2f| + \tfrac{1}{4}|SP_1f + S(-P_2f)| \\
&\quad + \tfrac{1}{4}|S(-P_1f) + SP_2f| + \tfrac{1}{4}|S(-P_1f) + S(-P_2f)|.
\end{aligned}
$$

This shows the validity of the inequality for $n = 2$. □

Recall that the norm in the Banach lattice E is said to be *lower semicontinuous* whenever for every net $\{f_\alpha\}$ in E which converges in order to $f \in E$ (i.e., there exist $p_\alpha \in E^+$, $p_\alpha \downarrow 0$ such that $|f - f_\alpha| \leqslant p_\alpha$ for all α) we have $\|f\| \leqslant \liminf_\alpha \|f_\alpha\|$. It is easily seen that lower semicontinuity of the norm is equivalent to saying that the norm in E is a Fatou norm according to the definition in [Z], Section 107. Clearly, every order continuous norm is lower semicontinuous.

For a Dedekind complete Banach lattice E the space $\mathscr{L}_b(E)$ is a subalgebra of the Banach algebra $\mathscr{L}(E)$ of all norm bounded operators on E. In general, $\mathscr{L}_b(E)$ need not be closed in $\mathscr{L}(E)$ with respect to the operator norm. However, if we equip $\mathscr{L}_b(E)$ with the so-called regular norm (which is defined by $\|S\|_r = \||S|\|$ for all $S \in \mathscr{L}_b(E)$), then $\mathscr{L}_b(E)$ is a Banach lattice algebra (for details, see [AB], Section 15 and [Sch], Exercise IV.4). Obviously, any band projection in $\mathscr{L}_b(E)$ is contractive with respect to the regular norm, but in general not with respect to the operator norm. The following result shows that certain band projections in $\mathscr{L}_b(E)$ are contractive for the operator norm.

THEOREM 5.5. *Let E be a Dedekind complete Banach lattice with lower semicontinuous norm and $T \in \mathrm{Hom}(E)$. Then the band projection $\mathscr{P}_T$ in $\mathscr{L}_b(E)$ onto $\{T\}^{dd}$ is contractive with respect to the operator norm, so $\|\mathscr{P}_T(S)\| \leqslant \|S\|$ for all $S \in \mathscr{L}_b(E)$.*

Proof. Take $S \in \mathscr{L}_b(E)^+$. If Π runs over all partitions of I_E consisting of band projections in $\mathscr{B}(E)$, then the set $\{\mathscr{P}_\Pi(S): \Pi \text{ partition of } I_E\}$ is downwards directed under refinement of the partitions, so it follows from Proposition 5.3 that $\mathscr{P}_\Pi(S)u \downarrow_\Pi \mathscr{P}_T(S)u$ for all $u \in E^+$. Hence, if $S \in \mathscr{L}_b(E)$ is arbitrary, then $\mathscr{P}_\Pi(S)f \to \mathscr{P}_T(S)f$ in order for all $f \in E$. Since the norm in E is lower semicontinuous it follows that

$$\|\mathscr{P}_T(S)f\| \leqslant \liminf_\Pi \|\mathscr{P}_\Pi(S)f\|$$

for all $f \in E$. Lemma 5.4 implies that for all partitions Π

$$\|\mathscr{P}_\Pi(S)f\| \leqslant 2^{-n}\sum_{(\varepsilon)}\left\|S\left(\sum_{i=1}^{n} \varepsilon_i P_i f\right)\right\| \leqslant 2^{-n}\|S\| \sum_{(\varepsilon)}\left\|\sum_{i=1}^{n} \varepsilon_i P_i f\right\|$$
$$=2^{-n}\|S\| \sum_{(\varepsilon)}\left\|\sum_{i=1}^{n} P_i f\right\| = 2^{-n}\|S\| 2^n\|f\| = \|S\|\,\|f\|$$

where we have used that $|\sum_{i=1}^{n} \varepsilon_i P_i f| = |\sum_{i=1}^{n} P_i f|$, as the elements $P_1 f, \dots, P_n f$ are pairwise disjoint. Hence, we may conclude that $\|\mathscr{P}_T(S)f\| \leqslant \|S\|\,\|f\|$ for all $f \in E$. □

The arguments in the proofs of Lemma 5.4 and Theorem 5.5 are for the case $T = I_E$ (so $\{T\}^{dd} = \mathrm{Orth}(E) = Z(E)$, due to J. Voigt, who kindly placed his (unpublished) preprint [V2] at our disposal and permitted us to use his proof in this more general context. In [V1] J. Voigt showed that the band projection of $\mathscr{L}_b(E)$ onto $Z(E)$ is contractive with respect to the operator norm even without the extra assumption that the norm is lower semicontinuous. At present we do not know whether the result of Theorem 5.5 also holds in this more general context.

6. Applications to spectral theory

In this final secton we indicate some consequences of Theorem 5.5 and Theorem 3.4 for the spectral theory of disjointness preserving operators.

6.1. *The peripheral spectrum*

Let E be a Banach lattice and $T \in \mathscr{L}(E)$. Recall that the *peripheral spectrum* $\mathrm{Per}\sigma(T)$ of T is the set $\{\lambda \in \mathbb{C} : |\lambda| = r(T)\} \cap \sigma(T)$, where $r(T)$ denotes the spectral radius of T. Obviously, $\mathrm{Per}\sigma(T)$ is a non-void closed subset of $\sigma(T)$. For information on $\mathrm{Per}\,\sigma(T)$ we refer to [Sch], V.4 and V.5.

If, in addition, E is Dedekind complete and $T \in \mathscr{L}_b(E)$, then T is said to be *aperiodic* whenever $T^n \perp I$ $(n = 1, 2, \dots)$ (see [A1], Kapitel 4 or [H1], Chapter 4).

THEOREM 6.1. *Let E be a Dedekind complete Banach lattice with lower semicontinuous norm and $T \in \mathscr{L}(E)$ an order continuous aperiodic disjointness preserving operator on E. Then*

$$\mathrm{Per}\,\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| = r(T)\}.$$

Particularly, any norm bounded aperiodic disjointness preserving operator on a Banach lattice with order continuous norm has this property.

Proof. We may assume without loss of generality that $r(T) = 1$. Using that T is order continuous and disjointness preserving it is easily seen that T is aperiodic if and only if $T^n \perp T^m$ $(n, m = 0, 1, \dots; n \neq m)$. Observe that $T^n \in \mathscr{L}_b(E)$ is disjointness preserving as well. Denote by $\mathscr{P}_n$ the band projection of $\mathscr{L}_b(E)$ onto $\{T^n\}^{dd}$ $(n = 0, 1, \dots)$. By Theorem 5.5, $\mathscr{P}_n$ is contractive with respect to the operator norm. The operator

$$e^{nT} = \sum_{k=0}^{\infty} \frac{(nT)^k}{k!}$$

is order bounded, as the series converges in the regular norm. It follows from $T^m \perp T^n$ $(m \neq n)$ that

$$\mathscr{P}_n(e^{nT}) = \frac{n^n T^n}{n!},$$

so

$$\frac{n^n}{n!} \|T^n\| = \|\mathscr{P}_n(e^{nT})\| \leqslant \|e^{nT}\| = \|(e^T)^n\|.$$

Hence,

$$\frac{n}{\sqrt[n]{n!}} \|T^n\|^{1/n} \leqslant \|(e^T)^n\|^{1/n} \qquad (n = 1, 2, \dots).$$

Letting $n \to \infty$, we find $er(T) = e \leqslant r(e^T)$, so

$$e \leqslant \max\{|e^\lambda| : \lambda \in \sigma(T)\} = \max\{e^{\mathrm{Re}\,\lambda} : \lambda \in \sigma(T)\}.$$

It follows that $1 \in \sigma(T)$. Replacing T by $e^{i\varphi} T$ we get $\{\lambda \in \mathbb{C} : |\lambda| = 1\} \subset \sigma(T)$, which yields the desired result. □

In [AH], Corollary 7, it was shown by W. Arendt and D. Hart that any aperiodic *quasi-invertible* disjointness preserving operator on a Dedekind complete Banach lattice has a rotationally invariant spectrum.

6.2. *The essential spectrum*

Recall that the Banach lattice E is said to have the *weak Fatou property* if every norm bounded upwards directed set in E^+ has a supremum, i.e., $0 \leqslant u_\alpha \uparrow$ in E and $\sup \|u_\alpha\| < \infty$ implies that $\sup_\alpha u_\alpha$ exists in E^+. It is clear that E is

Dedekind complete in this case. Notice furthermore that every AL-space has the weak Fatou property.

The following application of Theorem 3.4 is analogous to the argument used by L. W. Weis in [W1], proof of Theorem 11.b.

THEOREM 6.2. *Let E be a Banach lattice with the weak Fatou property and F an* AL-*space such that* E^* *and* F^* *are non-atomic. If* $T \in \mathscr{H}_n(E, F)$ *is a Fredholm operator, then T is invertible and* $T^{-1} \in \mathscr{H}_n(F, E)$.

Proof. Since T is Fredholm, T has a pseudo-inverse $S \in \mathscr{L}(F, E)$, i.e.,

$$ST = I_E + P,\ TS = I_F + Q$$

for some finite rank projections $P \in \mathscr{L}(E)$ and $Q \in \mathscr{L}(F)$. By [Z], Theorem 113.5, $\mathscr{L}(F, E) = \mathscr{L}_b(F, E) = \mathscr{L}_n(F, E)$, so $S \in \mathscr{L}_n(F, E)$. Since Q is a finite rank operator, $|Q|$ is compact (e.g. by an appeal to [AB], Theorem 16.8). It follows from Corollary 4.3 that Q is diffuse, i.e., $\mathscr{P}_F(Q) = 0$. Using Theorem 3.4 we see that

$$T\mathscr{P}_{FE}(S) = \mathscr{P}_F(T\mathscr{P}_{FE}(S)) = \mathscr{P}_F(TS) = I_F$$

and hence $T(S - \mathscr{P}_{FE}(S)) = Q$. Since the kernel of T is finite dimensional, the latter equality yields that $S - \mathscr{P}_{FE}(S)$ is a finite rank operator, so

$$I_E + P - \mathscr{P}_{FE}(S)T = (S - \mathscr{P}_{FE}(S))T$$

is of finite rank as well and therefore diffuse, as E^* is non-atomic. Since P is diffuse this implies $\mathscr{P}_{FE}(S)T = I_E$, showing that $T^{-1} = \mathscr{P}_{FE}(S) \in \mathscr{H}_n(F, E)$. □

In the next corollary, which is an immediate consequence of the above theorem, the *essential spectrum* of an operator $T \in \mathscr{L}(E)$ is

$$\sigma_{\text{ess}}(T) = \{\lambda \in \mathbb{C} : \lambda I_E - T \text{ is not Fredholm}\}.$$

COROLLARY 6.3. *Let E be a non-atomic* AL-*space. Then* $\sigma(T) = \sigma_{\text{ess}}(T)$ *for all* $T \in \mathscr{H}(E)$. *In particular,* $\sigma(T) = \sigma_{\text{ess}}(T)$ *for any norm bounded disjointness preserving operator T on E.*

For the special case that E is an L_1-space on some standard measure space the above result is due to L. W. Weis ([W1], Theorem 11.b). Notice in this connection that A. R. Schep showed in [S2], Theorem 1.11 that $\sigma(T) = \sigma_{\text{ess}}(T)$ for all orthomorphisms T on a non-atomic Banach lattice (cf. [PS], Proposition 3.20).

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Note added in proof

(i) It was pointed out to us by W. A. J. Luxemburg that the results of H. Gordon [G] already appear in an earlier paper by A. Sobczyk and P. C. Hammer, "A decomposition of additive set functions", *Duke Math. J.* 11 (1944), 839–846.

(ii) Extensions of the results of L. W. Weis [W1, W2] can be found in an article of I. I. Shamaev "Expansion and representation of regular operators", *Siberian Math. J.* 30 (1989), 323–331 (English translation).

(iii) The problem of Remark 2.3 has been solved in the meantime by S. J. Bernau and the authors. They prove in a forthcoming paper, entitled "Sums of lattice homomorphisms", the following:

if E, F are vector lattices and F is Dedekind complete, then $0 \leqslant T\colon E \to F$ is n-disjoint ($n \in \mathbf{N}$) (i.e., $\bigwedge_{i=0}^{n} Tx_i = 0$ in F for all disjoint sets $\{x_i\}_{i=0}^{n}$ in E^+) if and only if T is the sum of n lattice homomorphisms.

(iv) It was observed by W. Arendt that our method of proof of Theorem 6.1 also gives the following result:

if E is an AL-space and $T \in \mathscr{L}(E)$ satisfies $T^n \perp T^m$ ($n, m \geqslant 0$ integers, $n \neq m$), then

$$\operatorname{Per} \sigma(T) = \{\lambda \in \mathbf{C}\colon |\lambda| = r(T)\}.$$