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Metaplectic correspondences and unitary representations

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1. Introduction

It is observed in [H] that there is a correspondence from the irreducible unitary almost spherical representations (cf. Definition 5.1 of [H]) of the universal covering group of $GL(n, \mathbb{R})$ to the irreducible unitary spherical representations of $GL(n, \mathbb{R})$. When n is greater than or equal to 3, the universal covering group of $GL(n, \mathbb{R})$ is just a double cover. We write $\tilde{G} = \widetilde{GL}(n, \mathbb{R})$ for the double cover of $G = GL(n, \mathbb{R})$. We let $B = MAN$ be a Borel subgroup of $GL(n, \mathbb{R})$ and let $\tilde{B} = \tilde{M}AN$ be the corresponding Borel subgroup of $\widetilde{GL}(n, \mathbb{R})$. We let δ be the pin representation of $\tilde{M}$. Then we observe that the Langlands quotient of the principal series $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\delta \otimes \nu \otimes 1)$ is unitary if and only if the Langlands quotient of the principal series $\mathrm{Ind}_B^G(1 \otimes 2\nu \otimes 1)$ is unitary. The correspondence is given by

$$\text{Langlands quotient of } \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\delta \otimes \nu \otimes 1) \leftrightarrow$$
$$\leftrightarrow \text{Langlands quotient of } \mathrm{Ind}_B^G(1 \otimes 2\nu \otimes 1).$$

In this paper we will investigate the same kind of phenomenon for p-adic groups. We restrict our attention to simple split groups first.

Let $\mathbf{G}$ be a simply connected simple Chevalley group. Let F be a p-adic field of residual characteristic q, R its ring of integers with maximal ideal P. Fix a prime element $\pi \in P$. Write $G = G(F)$ for the group of F-rational points of $\mathbf{G}$. As an algebraic group G is simply connected, but as a topological group or abstract group the fundamental group $\pi_1(G)$ of G is equal to the group of all roots of unity in F (cf. [Ma] and [Mo]). Let n be a positive integer. We define

$$\mu_n(F) = \{x \in F \mid x^n = 1\}. \tag{1.1}$$

We assume that $|\mu_n(F)| = n$ and that n divides the order of $\pi_1(G)$. We let $\tilde{G}$ be the

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central extension of G with a preferred section s:

$$1 \to \mu_n(F) \to \tilde{G} \underset{s}{\overset{p}{\rightleftarrows}} G \to 1. \tag{1.2}$$

We also assume that $(q, n) = 1$ so that the cover splits over the maximal compact subgroup $K = G(R)$. We write $K^* = s(K)$, the lift of K. We let $I \subset K$ be the Iwahori subgroup of G. Since the cover is also split over I, we write I^* for the lift of I under the section s. The group I^* will be called the Iwahori subgroup of $\tilde{G}$.

Fix a faithful character ε of $\mu_n(F)$. Let $\mathrm{R}^\varepsilon(\tilde{G})$ be the category of (equivalence classes of) admissible representations of $\tilde{G}$ which are generated by its I^*-fixed vectors and such that $\mu_n(F)$ acts via ε. Let $\mathscr{H}^\varepsilon(\tilde{G})$ be the set of I^*-bi-invariant, compactly supported functions on $\tilde{G}$ satisfying the condition $f(g\xi) = f(g)\varepsilon(\xi)$ for $g \in \tilde{G}$ and $\xi \in \mu_n(F)$. Fix a Haar measure on $\tilde{G}$. The set $\mathscr{H}^\varepsilon(\tilde{G})$ is an algebra with respect to the convolution product

$$f * g(x) = \int_{\tilde{G}} f(xy^{-1})g(y)dy = \int_{\tilde{G}} f(y)g(y^{-1}x)dy. \tag{1.3}$$

The algebra $\mathscr{H}^\varepsilon(\tilde{G})$ is usually called the Hecke algebra. It is also a $*$-algebra. For $f \in \mathscr{H}^\varepsilon(\tilde{G})$ we define

$$f^*(g) = \overline{f(g^{-1})}. \tag{1.4}$$

It is easy to see that $f^* \in \mathscr{H}^\varepsilon(\tilde{G})$ if $f \in \mathscr{H}^\varepsilon(\tilde{G})$. We have $(f^*)^* = f$ for $f \in \mathscr{H}^\varepsilon(\tilde{G})$. For a representation $(\tilde{\pi}, \tilde{V}) \in \mathrm{R}^\varepsilon(\tilde{G})$, we write $\tilde{V}^{I^*}$ for the space of its Iwahori fixed vectors. Then the linear map $f \to \pi(f)|\tilde{V}^{I^*}$ is a representation of $\mathscr{H}^\varepsilon(\tilde{G})$. It is well known that the map

$$\tilde{V} \mapsto \tilde{V}^{I^*} \tag{1.5a}$$

gives an equivalence of categories from $\mathrm{R}^\varepsilon(\tilde{G})$ to the category $\mathrm{R}(\mathscr{H}^\varepsilon(\tilde{G}))$ of finite dimensional representations of $\mathscr{H}^\varepsilon(\tilde{G})$. Let $C^\infty_{c,\varepsilon}(\tilde{G}/I^*)$ denote the set of compactly supported I^* right invariant smooth functions on $\tilde{G}$ satisfying the condition $f(g\xi) = f(g)\varepsilon(\xi)$ for $g \in \tilde{G}$ and $\xi \in \mu_n(F)$. Then the inverse functor is

$$\tilde{V}^{I^*} \mapsto C^\infty_{c,\varepsilon}(\tilde{G}/I^*) \underset{\mathscr{H}^\varepsilon(\tilde{G})}{\otimes} \tilde{V}^{I^*}. \tag{1.6a}$$

In particular, when $n = 1$, $\tilde{G} = G$. In this case the Hecke algebra will be denoted by $\mathscr{H}(G)$ and the category of representations with I-fixed vectors will be denoted by $\mathrm{R}(G)$. For $(\pi, V) \in \mathrm{R}(G)$ the map

$$V \mapsto V^I \tag{1.5b}$$

gives the equivalence of categories from R(G) to the category R($\mathscr{H}(G)$) of finite dimensional representations of $\mathscr{H}(G)$. The inverse functor is

$$V^I \mapsto C_c^\infty(G/I) \underset{\mathscr{H}(G)}{\otimes} V^I. \tag{1.6b}$$

We will make use of the computations in [Sa] about the generators and relations of $\mathscr{H}^\varepsilon(\tilde{G})$ which imply that we have an isomorphism $\mathscr{H}^\varepsilon(\tilde{G}) \cong \mathscr{H}(G')$, where G' is an algebraic group. We will see that G' is isomorphic to either G modulo a central cyclic subgroup (possibly trivial) or the dual group ${}^LG^0$ of G. For any given group G and a fixed positive integer n we can easily determine G'. In Section 5 we will define the unramified principal series $I(\tilde{\chi})$ for $\tilde{G}$. Moreover we will construct a bijection from the set of unramified principal series of $\tilde{G}$ to the set of unramified principal series of G'. Fix an unramified principal series $I(\tilde{\chi})$ of $\tilde{G}$ and let $I(\chi)$ be the unramified principal series of G' corresponding to it. We will see that χ is the "nth power" of $\tilde{\chi}$. Write $I(\tilde{\chi})^{I^*}$ (resp. $I(\chi)^{I'}$) as the space of Iwahori fixed vectors of $I(\tilde{\chi})$ (resp. $I(\chi)$). We will prove

THEOREM 5.4. *The spaces $I(\tilde{\chi})^{I^*}$ and $I(\chi)^{I'}$ yield equivalent representations of the two isomorphic Hecke algebras $\mathscr{H}^\varepsilon(\tilde{G})$ and $\mathscr{H}(G')$ respectively.*

We write $\mathrm{R}_0^\varepsilon(\tilde{G})$ (resp. $\mathrm{R}_0(G')$) for the subcategory of $\mathrm{R}^\varepsilon(\tilde{G})$ (resp. R(G)) consisting of irreducible representations. We will construct a bijection between $\mathrm{R}_0^\varepsilon(\tilde{G})$ and $\mathrm{R}_0(G')$. Fix a hermitian representation $(\tilde{\pi}, \tilde{V}) \in \mathrm{R}_0^\varepsilon(\tilde{G})$, let $(\pi, V) \in \mathrm{R}_0(G')$ be the representation corresponding to it. Suppose V is real hermitian, then we will prove in Section 6

THEOREM 6.2. *$\tilde{V}$ is unitary if and only if V is unitary.*

As a consequence we extend the result of D. Barbasch and A. Moy in [BM] to the nonlinear group $\tilde{G}$. That is, the unitarity of $\tilde{V}$ can be detected on the space of its Iwahori fixed vectors $\tilde{V}^{I^*}$.

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2. Notations and preliminaries

We retain the notion in the previous section. Fix a maximal split torus T of G. Let $X = \mathrm{Hom}_{\mathrm{alg}}(T, F^\times)$ be the group of algebraic characters of T and let $\check{X} = \mathrm{Hom}_{\mathrm{alg}}(F^\times, T)$ be the group of algebraic one-parameter subgroups in T. Then the sets X and $\check{X}$ are in duality via the natural pairing $\langle\, , \rangle$. Let $\Phi \subset X$ be the root system of G with respect to T and let $\check{\Phi} \subset \check{X}$ be the set of coroots in bijection with Φ. For a root $\alpha \in \Phi$ let $\check{\alpha} \in \check{\Phi}$ be the corresponding coroot. The quaternity $(X, \Phi, \check{X}, \check{\Phi})$ is the root datum associated with the pair (G, T). The quaternity $(\check{X}, \check{\Phi}, X, \Phi)$ is the root datum for the dual group ${}^LG^0$. Since G is simply connected as an algebraic group, ${}^LG^0$ is of adjoint type. (For more detail about dual groups, cf. [B2], [Sp] and [T].) Therefore $\check{X}$ is the $\mathbb{Z}$-span of $\check{\Phi}$. Let T_α be the identity component of Ker α and Z_α be the centralizer of T_α in G. Then there is an isomorphism

$$\phi_\alpha: SL(2, F) \to Z_\alpha. \tag{2.1}$$

Let N_T be the normalizer of T and $W = N_T/T$ be the finite Weyl group of G. Let $s_\alpha = \phi_\alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the s_α generate W which acts on T as well as on X and $\check{X}$. We write w_l as the longest element of W.

Now fix a choice of the set of positive roots Φ^+. This uniquely determines the Iwahori subgroup I and the Borel subgroup $B = TN$. Let $W_{\text{aff}} = N_T/T_0$ be the affine Weyl group, where T_0 is the maximal compact subgroup of T. We have the standard decomposition $W_{\text{aff}} = \check{X} \times W$. Note that this is only a semi-direct product. We may identify each element in $\check{X}$ as an element of a coset of T/T_0 by evaluation at π, i.e.

$$\lambda \mapsto \lambda(\pi) \in T. \tag{2.2}$$

Let $\Delta \subset \Phi^+$ be the set of simple roots. Define

$$T^- = \{t \in T \mid |\alpha(t)| \leqslant 1, \forall \alpha \in \Delta\}. \tag{2.3}$$

Let $\mathfrak{n}$ denote the Lie algebra of N and let $Ad_\mathfrak{n}$ be the adjoint action of T on $\mathfrak{n}$. Let δ be the modulus character of $B = TN: tn \mapsto |\det Ad_\mathfrak{n}(t)|$, where $t \in T$ and $n \in N$. For $\lambda \in \check{X}$ and $w \in W$, define $l(\lambda)$ and $l(w)$ such that $q^{l(\lambda)} = [I\lambda I : I]$ and $q^{l(w)} = [IwI : I]$. Assign G a Haar measure. We have

$$[I\lambda I : I] = \text{meas}(I\lambda I)/\text{meas}(I) = \delta^{-1}(\lambda), \tag{2.4}$$

if we use the identification of (2.2) (cf. [C1] Lemma 1.5.1).

In the rest of this section we assume that G is a reductive p-adic group (which may be nonlinear, i.e. a covering group of a linear group). An admissible representation (π, V) of G is called hermitian if there is a hermitian form $\langle\,,\rangle$ on V such that

$$\langle \pi(g)v, w\rangle = \langle v, \pi(g^{-1})w\rangle (v, w \in V, g \in G). \tag{2.5}$$

If (π^h, V^h) denotes the hermitian dual of (π, V), then (π, V) is hermitian if and only if $(\pi, V) \cong (\pi^h, V^h)$.

We can also define the notions of hermitian and unitary $\mathscr{H}(G)$ modules. A $\mathscr{H}(G)$-module E is called hermitian if there is a hermitian form $\langle\,,\rangle$ on E such that

$$\langle\pi(f)v, w\rangle = \langle v, \pi(f^*)w\rangle (v, w \in E, f \in \mathscr{H}(G)). \tag{2.6}$$

In the equivalence of categories between R(G) and R($\mathscr{H}(G)$), it follows from (1.6a) and (1.6b) that the property of being hermitian is preserved. In one direction, we need only to restrict the hermitian form on V to V^I. In the other direction the hermitian form on V is given by

$$\langle f \otimes v, g \otimes w\rangle_V = \langle\pi(g * f)v, w\rangle_{V^I} (v, w \in E, f, g \in C_c^\infty(G)). \tag{2.7}$$

3. The *W*-invariant forms associated with metaplectic covers

In order to construct the central extension of G, it suffices to construct a W-invariant cover $\tilde{T}$ of T by the method of Matsumoto. Assume $\tilde{x}, \tilde{y} \in \tilde{T}$ such that $p(\tilde{x}) = x, p(\tilde{y}) = y \in T$. The commutator $[\tilde{x}, \tilde{y}]$ lies in $\mu_n(F)$ and depends only on x and y. Hence a central extension of T determines a map

$$c: T \times T \to \mu_n(F)$$

given by $c(x, y) = [\tilde{x}, \tilde{y}]$. The map c is called the associated commutator map and it satisfies following conditions:

$$\begin{aligned} &c(xy, z) = c(x, z)c(y, z);\\ &c(x, yz) = c(x, y)c(x, z);\\ &c(x, x) = 1. \end{aligned} \tag{3.1}$$

Conversely, a commutator map c determines a cover of T. For $\lambda \otimes s$, $\mu \otimes t \in T = \check{X} \otimes_{\mathbb{Z}} F^\times$ we define c by

$$c(\lambda \otimes s, \mu \otimes t) = (s, t)_H^{(\lambda,\mu)_W}, \tag{3.2}$$

where $(,)_H$ is the nth-order Hilbert symbol and $(,)_W$ is the W-invariant even symmetric form on $\check{X}$. The W-invariant bilinear form $(\,,)_W$ is uniquely determined up to a scalar. We always take $(\,,)_W$ as the minimal $\mathbb{Z}$-valued form and therefore it is unique.

Now we have an imbedding of $\check{X}$ into X given by

$$\imath: \check{\lambda} \mapsto (\check{\lambda}, \cdot)_W. \tag{3.3}$$

Let d be the smallest positive integer such that $X^d \subset \imath(\check{X})$. The integer d is important for understanding the isomorphism $H^\varepsilon(\tilde{G}) \cong H(G')$. We will now determine d for each case. For simplicity we will write $(\,,)$ for $(\,,)_W$ in the following table. Notice that $(\check{\alpha}_i, \check{\alpha}_j) = 0$ if $|i-j| > 1$ except for the case where G is of type D_l and $\{i, j\} = \{l-2, l\}$:

$$G = A_l (l \geqslant 1), \quad (\check{\alpha}_i, \check{\alpha}_i) = 2, \quad (\check{\alpha}_i, \check{\alpha}_{i+1}) = -1;$$
$$X/\imath(\check{X}) = \mathbb{Z}_{l+1}, \quad d = l+1.$$

$$G = B_l \ (l \geqslant 2), \quad (\check{\alpha}_i, \check{\alpha}_i) = \begin{cases} 4 \ (i < l) \\ 2 \ (i = l) \end{cases}, \quad (\check{\alpha}_i, \check{\alpha}_{i+1}) = -2;$$

$$X/\imath(\check{X}) = \overbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}^{l \text{ times}}, \quad d = 2.$$

$$G = C_l \ (l \geqslant 3), \quad (\check{\alpha}_i, \check{\alpha}_i) = \begin{cases} 2 \ (i < l) \\ 4 \ (i = l) \end{cases}, \quad (\check{\alpha}_i, \check{\alpha}_{i+1}) = \begin{cases} -1 \ (i < l-1) \\ -2 \ (i = l-1) \end{cases};$$
$$X/\imath(\check{X}) = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad d = 2.$$

$$G = D_l \ (l \geqslant 4), \quad (\check{\alpha}_i, \check{\alpha}_i) = 2; \quad (\check{\alpha}_i, \check{\alpha}_{i+1}) = \begin{cases} -1 \ (i < l-1) \\ \ \ 0 \ (i = l-1) \end{cases}, \quad (\check{\alpha}_{l-2}, \check{\alpha}_l) = -1;$$

$$X/\imath(\check{X}) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \ (2 \mid l) \\ \mathbb{Z}_4 \qquad (2 \nmid l) \end{cases}, \quad d = \begin{cases} 2 \ (2 \mid l) \\ 4 \ (2 \nmid l) \end{cases}.$$

$$G = G_2, \quad (\check{\alpha}_1, \check{\alpha}_1) = 6, \quad (\check{\alpha}_2, \check{\alpha}_2) = 2, \quad (\check{\alpha}_1, \check{\alpha}_2) = -3;$$
$$X/\imath(\check{X}) = \mathbb{Z}_3, \quad d = 3.$$

$$G = F_4, \quad (\check{\alpha}_i, \check{\alpha}_i) = \begin{cases} 2 \ (i \leqslant 2) \\ 4 \ (i \geqslant 3) \end{cases}, \quad (\check{\alpha}_i, \check{\alpha}_{i+1}) = \begin{cases} -2 \ (i \leqslant 2) \\ -1 \ (i = 3) \end{cases};$$
$$X/\imath(\check{X}) = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad d = 2.$$

$$G = E_l \ (l = 6, 7, 8), \quad (\check{\alpha}_i, \check{\alpha}_i) = 2, \quad (\check{\alpha}_i, \check{\alpha}_{i+1}) = -1;$$
$$X/\imath(\check{X}) = \begin{cases} \mathbb{Z}_3 (l = 6) \\ \mathbb{Z}_2 (l = 7) \\ 1 \ \ (l = 8) \end{cases}, \quad d = \begin{cases} 3 (l = 6) \\ 2 (l = 7) \\ 1 (l = 8) \end{cases}.$$

4. The isomorphism between two Hecke algebras

In this section we describe $H^\varepsilon(\tilde{G})$ and construct an isomorphism between $H^\varepsilon(\tilde{G})$ and $H(G')$, where G' is a linear group to be determined. Take a nonzero I^*-bi-invariant function $f \in H^\varepsilon(\tilde{G})$ and $\tilde{x} \in \tilde{T}$ such that $p(\tilde{x}) = x = \lambda \otimes s \in T$. Then for any $\tilde{y} \in s(T_0) \subset I^*$, we have

$$f(\tilde{x}) = f(\tilde{x}\tilde{y}) = f(\tilde{y}\tilde{x}\xi) = f(\tilde{x})\varepsilon(\xi), \tag{4.1}$$

where $\xi = [\tilde{x}, \tilde{y}]$. Since the function f is nonzero and the character ε is faithful, we must have $\xi = [\tilde{x}, \tilde{y}] = 1$. That implies $(\lambda, \check{X})_W = 0 \pmod n$. Hence, we have $\iota(\lambda) \in X^n$ or $\lambda \in \tilde{X} = \iota^{-1}(X^n)$.

The isomorphism between $H^\varepsilon(\tilde{G})$ and $H(G')$ depends on the greatest common divisor (d, n) of d and n (d is defined and determined for each case in Section 3):

(i) If $(d, n) = 1$, then $\tilde{X} \cong \check{X}^n$, we take $G' \cong G$.
(ii) If $d \mid n$, then $\tilde{X} \cong X^n$, we take $G' \cong {}^LG^0$.
(iii) If $1 < (d, n) = m < d$, which only happens when $G = SL(l+1)$ or $G = \mathrm{Spin}(2l+1, 2l+1)$, we take G' such that there is a central isogony $\phi: G \to G'$ with $\mathrm{Ker}\,\phi = \mu_m(F)$.

Let $(X', \Phi', \check{X}', \check{\Phi}')$ be the root datum of G'. Then $\check{X}'^n \cong \tilde{X}$. Write $\psi: \check{X}'^n \to \tilde{X}$ for the isomorphism. In any case we can identify the Weyl group W of G with the Weyl group of G'. Fix the set of positive roots of G' which corresponds to the set of fixed positive roots of G. Let I' be the Iwahori subgroup and B' be the Borel subgroup of G'. Let K' be the maximal compact subgroup of G'. Observe that the isomorphism of lattices ψ induces a homomorphism from $T' = \check{X}' \otimes_{\mathbb{Z}} F^\times$ to $T = \check{X} \otimes_{\mathbb{Z}} F^\times$ by

$$\lambda \otimes s \mapsto \lambda^n \otimes s \mapsto \psi(\lambda^n) \otimes s.$$

We compose this homomorphism with the section s to get a homomorphism $\eta: T' \to \tilde{T}$. We need this homomorphism η in the next section.

For $\lambda \in \tilde{X}$ let S_λ be the function on $\tilde{G}$ with value $\varepsilon(\xi)$ on $I^*\lambda I^* \times \{\xi\}$ and 0 elsewhere. For $w \in W$ let S_w be the function on $\tilde{G}$ with value $\varepsilon(\xi)$ on $I^*wI^* \times \{\xi\}$ and 0 elsewhere. Then the $\{S_\lambda S_w \mid \tilde{X}, w \in W\}$ forms a basis of $H^\varepsilon(\tilde{G})$.

The isomorphism of the two lattices $\check{X}'^n \cong \tilde{X}$ also suggests an isomorphism between $H(G')$ and $H^\varepsilon(\tilde{G})$. For $\lambda \in \check{X}'$ let T_λ be the characteristic function on $I'\lambda I'$. For $w \in W$ let T_w be the characteristic function on $I'wI'$. Then the set $\{T_\lambda T_w \mid \lambda \in \check{X}', w \in W\}$ forms a basis of $H(\tilde{G}')$.

Write Λ for $\check{X}'$ and $\tilde{\Lambda}$ for $\tilde{X}$. Define

$$\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha \rangle \geqslant 0, \alpha \in \Delta\}; \tag{4.2a}$$

$$\tilde{\Lambda}^+ = \{\lambda \in \tilde{\Lambda} \mid \langle \lambda, \alpha \rangle \geqslant 0, \alpha \in \Delta\}. \tag{4.2b}$$

By definition $\lambda \in \Lambda^+$ if and only if $\lambda(\pi) \in T^-$ (cf. (2.3)). For any $\lambda \in \tilde{\Lambda}$ there exist $\lambda_1, \lambda_2 \in \tilde{\Lambda}^+$ such that $\lambda = \lambda_1 - \lambda_2$. Define

$$\hat{S}_\lambda = q^{1/2(-l(\lambda_1)+(\lambda_2))} S_{\lambda_1} S_{\lambda_2}^{-1}.$$

For $\lambda \in \Lambda$ we can find $\lambda_1, \lambda_2 \in \Lambda^+$ such that $\lambda = \lambda_1 - \lambda_2$. We define

$$\hat{T}_\lambda = q^{1/2(-l(\lambda_1)+l(\lambda_2))} T_{\lambda_1} T_{\lambda_2}^{-1}.$$

For $w \in W$ we define $\hat{S}_w = q^{-\frac{1}{2}l(w)} S_w$ and $\hat{T}_w = q^{-\frac{1}{2}l(w)} T_w$. Now we define $\mathscr{I}: H(G') \to H^\varepsilon(\tilde{G})$ to be the isomorphism of vector spaces induced by

$$\hat{T}_\lambda \mapsto \hat{S}_{\psi(\lambda^n)}, \ \hat{T}_w \mapsto \hat{S}_w, \quad \text{for } \lambda \in \Lambda^+,\ w \in W. \tag{4.3}$$

We will often omit ψ to write $\hat{S}_{\lambda^n}$ for $\hat{S}_{\psi(\lambda^n)}$. Using the relations computed in [Sa] and the fact that $(\mathscr{I}(f))^* = \mathscr{I}(f^*)$, we have

THEOREM 4.1. *The map $\mathscr{I}: H^\varepsilon(\tilde{G}) \to H(G')$ is actually an isomorphism of $*$-algebras.*

Let $\mathscr{H}(K^*)$ be the Hecke algebra of I^*-bi-invariant functions on $\tilde{K} = p^{-1}(K) \cong K^* \times \mu_n(F)$ satisfying the condition $f(g\xi) = f(g)\varepsilon(\xi)$, for $g \in \tilde{K}$ and $\xi \in \mu_n(F)$. A better notation for it might be $\mathscr{H}^\varepsilon(\tilde{K})$ but we use $\mathscr{H}(K^*)$ for simplicity. The algebra $\mathscr{H}(K^*)$ is a subalgebra of the Hecke algebra $\mathscr{H}^\varepsilon(\tilde{G})$. Let $\mathscr{H}(K')$ be the Hecke algebra of I'-bi-invariant functions on K'. The algebra $\mathscr{H}(K')$ is a subalgebra of the Hecke algebra $\mathscr{H}(G')$.

Since we have the decompositions $K^* = \bigcup_{w\in W} I^*wI^*$ and $K' = \bigcup_{w\in W} I'wI'$. It follows that $\mathscr{H}(K^*)$ is generated by $\{\hat{S}_w \mid w \in W\}$ and $\mathscr{H}(K')$ is generated by $\{\hat{T}_w \mid w \in W\}$. It is easy to see from (4.3) that the $*$-algebra isomorphism $\mathscr{I}: \mathscr{H}^\varepsilon(\tilde{G}) \to \mathscr{H}(G')$ restricted to $\mathscr{H}(K^*)$ is an $*$-algebra isomorphism from $\mathscr{H}(K^*)$ to $\mathscr{H}(K')$.

5. The unramified principal series and its Iwahori-fixed vectors

Let G' be the group defined in Section 4. Let T', I' and B' be its maximal torus, Iwahori subgroup and maximal Borel subgroup respectively. Let K' be the maximal compact subgroup of G'. Write $T_0' = T' \cap K'$; that is the maximal compact subgroup of T'. Let $\tilde{G}$, $\tilde{T}$ be as before. Write $N^* \subset \tilde{G}$ for the lift of $\underset{\cdot}{N}$. Let $T^n = \{t^n \mid t \in T\}$. Observe that $\tilde{T}^n = p^{-1}(T^n) \subset \eta(T')$. Let $\tilde{T}_*$ be the maximal abelian subgroup of $\tilde{T}$ containing $\eta(T')$.

Let $\tilde{\chi}$ be a character of $\eta(T')$. We can extend it to $\tilde{T}_*$ so that $\tilde{\chi}|_{\mu_n(F)} = \varepsilon$. We still write it as $\tilde{\chi}$. We extend $\tilde{\chi}$ further to $\tilde{B}_* = \tilde{T}_* N^*$ by $\tilde{\chi}(tn) = \tilde{\chi}(t)$ for $t \in \tilde{T}_*$ and $n \in N^*$.

Let $I(\tilde{\chi})$ be the space of locally constant functions $f: \tilde{G} \to \mathbb{C}$ such that

$$f(bg) = (\tilde{\chi}\delta^{1/2})(b) f(g) \quad \text{for } b \in \tilde{B}_* \quad \text{and} \quad g \in \tilde{G}. \tag{5.1}$$

Let $\tilde{G}$ act on $I(\tilde{\chi})$ by right translation. The representation $I(\tilde{\chi})$ is a principal series of $\tilde{G}$ and is admissible.

The character $\tilde{\chi}$ also induces a character χ of T' by $\chi(t) = \tilde{\chi}(\eta(t))$. The character χ is the "n-th power" of $\tilde{\chi}$ because of the nature of the homomorphism η. We can also extend χ to B' and define the principal series representation $I(\chi)$ for G' similarly. The character $\tilde{\chi}$ is called unramified if χ is unramified, i.e. $\chi|_{T'_0} = 1$. From now on we assume that all characters $\tilde{\chi}$ of $\tilde{T}_*$ or χ of T' are unramified.

PROPOSITION 5.1. *The correspondence between the unramified characters, $\tilde{\chi} \leftrightarrow \chi$, is a bijection. Therefore this correspondence induces a bijection between the set of unramified principal series of $\tilde{G}$ and the set of unramified principal series of G'.*

We only need to observe that any unramified character χ on T' can factor through T'^n. Hence there is a character $\tilde{\chi}$ on $\eta(T')$ such that $\chi(t) = \tilde{\chi}(\eta(t))$ for all $t \in T'$. We will see that this bijection has nice properties in Theorem 5.4.

PROPOSITION 5.2 (cf. [C2] Prop. 2.7).

(1) *The principal series $I(\chi)$ is generated by its Iwahori-fixed vectors $I(\chi)^{I'}$;*
(2) *The principal series $I(\tilde{\chi})$ is generated by its Iwahori-fixed vectors $I(\tilde{\chi})^{I^*}$.*

Define the G'-projection $\mathscr{P}$ from $C_c^\infty(G')$ onto $I(\chi)$:

$$\mathscr{P}(f)(g) = \int_{B'} \chi^{-1}\delta^{1/2}(b)f(bg)db. \tag{5.2a}$$

Here the measure is the left invariant-Haar measure for which $\text{meas}(B' \cap K') = 1$. For each $w \in W$ let $\phi_w = \mathscr{P}(T_w)$, where T_w is the function defined in Section 4.

Similarly, we define the $\tilde{G}$-projection $\tilde{\mathscr{P}}$ from $C_c^\infty(\tilde{G})$ onto $I(\tilde{\chi})$:

$$\tilde{\mathscr{P}}(f)(g) = \int_{\tilde{B}_*} \tilde{\chi}^{-1}\delta^{1/2}(b)f(bg)db. \tag{5.2b}$$

Here the measure is the left-invariant Haar measure for which $\text{meas}(\tilde{B}_* \cap K^*) = 1$. For each $w \in W$ let $\tilde{\phi}_w = \tilde{\mathscr{P}}(S_w)$, where S_w is the function defined in Section 4.

PROPOSITION 5.3

(1) *The functions ϕ_w $(w \in W)$ form a basis of $I(\chi)^{I'}$;*
(2) *The functions $\tilde{\phi}_w$ $(w \in W)$ form a basis of $I(\tilde{\chi})^{I^*}$.*

Proof. (1) This is because G' is the disjoint union of the open subsets $B'wI'$.
(2) Let $\{\gamma\}$ be a set of representatives of cosets $\tilde{T}/\tilde{T}_*$. Then $\tilde{G} = \bigcup \tilde{B}_* \gamma w I^*$. Suppose $f \in I(\tilde{\chi})^{I^*}$ and $h \in s(T_0) \subset I^*$. For any $w \in W$ we have

$$f(\gamma w) = f(\gamma wh) = f(\gamma whw^{-1}\gamma^{-1}\gamma w) = \tilde{\chi}\delta^{1/2}(\gamma whw^{-1}\gamma^{-1})f(\gamma w). \tag{5.3}$$

If $f(w\gamma)$ is non-zero, we must have $\tilde{\chi}\delta^{1/2}(\gamma w h w^{-1}\gamma^{-1})=1$. This is only true when $\gamma \in \tilde{T}_*$ because $\tilde{T}_*$ is maximal abelian in $\tilde{T}$. □

Proposition 5.3 suggests a linear isomorphism $\mathscr{M}$ between two vector spaces $I(\tilde{\chi})^{I^*}$ and $I(\chi)^{I'}$ given by mapping the base elements to the base elements. Actually we can prove

THEOREM 5.4. *Take $I(\tilde{\chi})^{I^*}$ and $I(\chi)^{I'}$ as representations of $\mathscr{H}^{\varepsilon}(\tilde{G})$ and $\mathscr{H}(G')$ respectively, then $\mathscr{M}: I(\tilde{\chi})^{I^*} \to I(\chi)^{I'}$ given by $\mathscr{M}(\tilde{\phi}_w) = \phi_w$ defines an equivalence of representations of the two isomorphic Hecke algebras.*

Proof. It is enough to show that for any $w, w_1 \in W$ and $\lambda \in \Lambda$ we have

$$\mathscr{M}(\hat{S}_{w_1}\tilde{\phi}_w) = \hat{T}_{w_1}\phi_w. \tag{5.4a}$$

$$\mathscr{M}(\hat{S}_{\lambda^n}\tilde{\phi}_w) = \hat{T}_{\lambda}\phi_w. \tag{5.4b}$$

Observe that $T_w\phi_1 = \phi_w$ and $S_w\tilde{\phi}_1 = \tilde{\phi}_w$. So (5.4a) is obvious. To prove (5.4b) we need Lemma 3.9 in [C2] which says:

$$T_\lambda \phi_{w_l} = [I'\lambda I' : I']w_l\chi\delta^{1/2}(\lambda)\phi_{w_l} \quad \text{for } \lambda \in \Lambda^+. \tag{5.5a}$$

By the same method we have

$$S_{\lambda^n}\tilde{\phi}_{w_l} = [I\lambda^n I : I]w_l\chi\delta^{1/2}(\lambda^n)\tilde{\phi}_{w_l} \quad \text{for } \lambda \in \Lambda^+. \tag{5.5b}$$

Therefore by (2.4) we have (5.4b) for $w = w_l$ and λ in the positive chamber. This is also true for arbitrary λ by the definition of $\hat{T}_\lambda$ and $\hat{S}_\lambda$. Since ϕ_{w_l} (resp. $\tilde{\phi}_{w_l}$) is a cyclic vector of $I(\chi)^{I'}$ under the action of $\{\hat{T}_w \mid w \in W\}$ (resp. $I(\tilde{\chi})^{I^*}$ under the action of $\{\hat{S}_w \mid w \in W\}$), we only need to show

$$\mathscr{M}(\hat{S}_{\lambda^n}S_w\tilde{\phi}_{w_l}) = \hat{T}_\lambda T_w\phi_{w_l} \quad \text{for any } w \in W,\ \lambda \in \Lambda. \tag{5.6}$$

By (5.4a) we can apply a simple reflection on both sides of (5.4b) (for $w = w_l$ case) to get

$$\mathscr{M}(S_i\hat{S}_{\lambda^n}\tilde{\phi}_{w_l}) = T_i\hat{T}_\lambda\phi_{w_l}. \tag{5.7}$$

By the multiplication relations in the Hecke algebra (but beware of that $T_iT_\lambda \neq T_\lambda T_i$ unless $\langle \lambda, \alpha_i \rangle = 0$), we have

$$\mathscr{M}(\hat{S}_{\lambda^n}S_i\tilde{\phi}_{w_l}) = \hat{T}_\lambda T_i\phi_{w_l}. \tag{5.8}$$

Then we can apply another simple reflection on both sides of (5.8). Therefore an induction will do the job of proving (5.6). □

6. Metaplectic correspondences and unitary representations

In the previous section we had a bijection between the set of unramified principal series of $\tilde{G}$ and the set of unramified principal series of G'. We proved that if $I(\tilde{\chi}) \in \mathrm{R}^{\varepsilon}(\tilde{G})$ corresponds to $I(\chi) \in \mathrm{R}(G')$ then the spaces of their Iwahori-fixed vectors $I(\tilde{\chi})^{I^*}$ and $I(\chi)^{I'}$ are equivalent as modules of the isomorphic Hecke algebras $\mathscr{H}^{\varepsilon}(\tilde{G})$ and $\mathscr{H}(G')$ respectively. In this section we will construct a bijection between $\mathrm{R}_0^{\varepsilon}(\tilde{G})$ and $\mathrm{R}_0(G')$. We need the following well-known theorem.

THEOREM 6.1. *An admissible irreducible representation* $(\tilde{\pi}, \tilde{V})$ *(resp.* (π, V)*) of* $\tilde{G}$ *(resp.* G'*) has a non-trivial Iwahori-fixed vector, if and only if it is a subquotient of an unramified principal series* $I(\tilde{\chi})$ *(resp.* $I(\chi)$*).*

For the proof of this theorem, cf. [FK] and [B1].

Fix a $\tilde{V} \in \mathrm{R}_0^{\varepsilon}(\tilde{G})$. We can imbed it into a unramified principal series $I(\tilde{\chi})$. Write $\tilde{V}^{I^*} \subset I(\tilde{\chi})$ as its I^*-fixed vector. Let $I(\chi)$ be the unramified principal series of G' corresponding to $I(\tilde{\chi})$. Write $I(\chi)^{I'}$ for its Iwahori-fixed vectors. By Theorem 6.4 $I(\tilde{\chi})^{I^*}$ and $I(\chi)^{I'}$ are equivalent as modules of the isomorphic Hecke algebras $\mathscr{H}^{\varepsilon}(\tilde{G})$ and $\mathscr{H}(G')$ respectively. Therefore their composition factors correspond to each other. Let $V^{I'} = \mathscr{M}(\tilde{V}^{I^*})$ be the irreducible module of $\mathscr{H}(G')$ corresponding to $\tilde{V}^{I^*}$. Then we get a $V \in \mathrm{R}_0(G')$ corresponding to $V^{I'}$ (cf 1.6b). We call the map $\tilde{V} \mapsto V$ the metaplectic correspondence. We still write this map as $\mathscr{M}$. This correspondence is actually a composition of three correspondences

$$(\tilde{\pi}, \tilde{V}) \leftrightarrow (\tilde{\pi}|_{\tilde{V}^{I^*}}, \tilde{V}^{I^*}) \leftrightarrow (\pi|_{V^{I'}}, V^{I'}) \leftrightarrow (\pi, V). \tag{6.1}$$

It follows from the fact that $\mathscr{I}: \mathscr{H}^{\varepsilon}(\tilde{G}) \to \mathscr{H}(G')$ is an isomorphism of $*$-algebras that the correspondence in the middle preserves the property of being hermitian. Because the other two correspondences also preserve the property of being hermitian, the metaplectic correspondence $\mathscr{M}$ takes hermitian representations to hermitian representations. We can use Harish-Chandra's criterion ([Si] Theorem 4.4.4) to prove that $\mathscr{M}$ also takes discrete series to discrete series and tempered representations to tempered representations (cf. [FK] §17). We call a representation $(\tilde{\pi}, \tilde{V}) \in \mathrm{R}_0^{\varepsilon}(\tilde{G})$ real hermitian if $\mathscr{M}(\tilde{V}) = V \in \mathrm{R}_0(G')$ is real hermitian. The main theorem we want to prove here is

THEOREM 6.2. *If* $(\tilde{\pi}, \tilde{V}) \in \mathrm{R}_0^{\varepsilon}(\tilde{G})$ *and* $(\pi, V) \in \mathrm{R}_0(G')$ *are real hermitian such that* $\mathscr{M}(\tilde{V}) = V$*, then* $\tilde{V}$ *is unitary if and only if* V *is unitary.*

In [BM] D. Barbasch and A. Moy show that V is unitary if and only if $V^{I'}$ is unitary. Since the isomorphism $\mathscr{I}: \mathscr{H}^{\varepsilon}(\tilde{G}) \to \mathscr{H}(G')$ is an isomorphism of $*$-algebras, $\tilde{V}^{I^*}$ is unitary if and only if $V^{I'} = \mathscr{M}(\tilde{V}^{I^*})$ is unitary. It is obvious that the unitarity of $\tilde{V}$ implies the unitarity of $\tilde{V}^{I^*}$. We need only to prove that the unitarity of $\tilde{V}^{I^*}$ implies the unitarity of $\tilde{V}$. We need to introduce the K-character and signature character of a hermitian representation V of a reductive group and their analogous characters of its Iwahori-fixed vectors V^I. These notions are first introduced by D. Vogan in [V] for real groups and by D. Barbasch and A. Moy in [BM] for p-adic groups.

Let $(\tilde{\pi}, \tilde{V}) \in R_0^\varepsilon(\tilde{G})$ be an irreducible representation with a hermitian form $\langle , \rangle$. For each irreducible representation δ of K^* fix a positive definite hermitian form. Let $\tilde{V}(\delta)$ be the δ-isotropical component of $\tilde{V}$. The finite dimensional vector space $F(\delta) = \mathrm{Hom}(\delta, \tilde{V}(\delta))$ acquires a nondegenerate form $\langle , \rangle$. Denote the dimension and signature of $F(\delta)$ by $m(\delta)$ and $(p(\delta), q(\delta))$ respectively. Define the formal K^*-character of $\tilde{V}$ to be

$$\theta_{K^*}(\tilde{V}) = \sum_\delta m(\delta)\delta. \tag{6.2}$$

Define the signature character of $\langle , \rangle$ to be the pair of formal sums

$$\left(\sum_\delta p(\delta)\delta,\ \sum_\delta q(\delta)\delta\right). \tag{6.3}$$

Let $(\pi, V) \in R_0(G')$ be an irreducible representation with a hermitian form $\langle , \rangle$. The K'-character and signature character of V are defined in a similar way.

Let $\tilde{E}$ be a hermitian $\mathscr{H}^\varepsilon(\tilde{G})$-module. If δ is a simple module of $\mathscr{H}(K^*)$, let $(p(\delta), q(\delta))$ be the signature of $\mathrm{Hom}_{\mathscr{H}(K^*)}(\delta, \tilde{E})$ and set $m(\delta) = p(\delta) + q(\delta)$. In analogy with (6.2) and (6.3), the formal $\mathscr{H}(K^*)$-character is

$$\theta_{\mathscr{H}(K^*)}(\tilde{E}) = \sum_\delta m(\delta)\delta; \tag{6.4}$$

and the signature character is

$$\left(\sum_\delta p(\delta)\delta,\ \sum_\delta q(\delta)\delta\right). \tag{6.5}$$

Let $\mathscr{S}$ be the set of all irreducible representations of K^* occurring in the induced representation $\mathrm{Ind}_{I^*}^{K^*} 1$. For $\delta \in \mathscr{S}$ the map

$$\delta \mapsto \delta^{I^*} \tag{6.6}$$

is a bijection from $\mathscr{S}$ to the set of simple $\mathscr{H}(K^*)$-modules. Suppose $(\tilde{\pi}, \tilde{V}) \in R_0^\varepsilon(\tilde{G})$ and $\tilde{E} = \tilde{V}^{I^*}$. Write the K^*-character of $\tilde{V}$ as the sum

$$\theta_{K^*}(\tilde{V}) = \sum_{\delta \in \mathscr{S}} m(\delta)\delta + \sum_{\delta \notin \mathscr{S}} m(\delta)\delta. \tag{6.7}$$

Then the $\mathscr{H}(K^*)$-character of $\tilde{E}$ is

$$\theta_{K^*}(\tilde{E}) = \sum_{\delta \in \mathscr{S}} m(\delta)\delta. \tag{6.8}$$

The signature character of $\tilde{E}$ can be gotten in a same fashion.

Let E be a hermitian $\mathscr{H}(G')$-module. The $\mathscr{H}(K')$-character and signature character are defined in a similar way. We also have a formula analogous to (6.8) for E.

It follows from the remark after the statement of Theorem 6.2 that we only have to prove the following proposition in order to prove Theorem 6.2.

PROPOSITION 6.3. *In the setting of Theorem* 6.2, *suppose $\tilde{V}^{I^*}$ is unitary, then $\tilde{V}$ is also unitary.*

Proof. Denote by $\langle\, , \rangle$ the hermitian form on $\tilde{V}$ and $\langle\, , \rangle^{I^*}$ its restriction on $\tilde{V}^{I^*}$. Denote by $\langle\, , \rangle^{I'}$ the hermitian form of $V^{I'} = \mathscr{M}(\tilde{V}^{I^*})$. Now we need a signature theorem which is suitable for our use. Following the same line of D. Vogan in [V] or D. Barbasch and A. Moy in [BM], we can find finitely many real tempered irreducible representations $\tilde{V}_1, \ldots, \tilde{V}_m \in \mathrm{R}_0^{\varepsilon}(\tilde{G})$ (i.e. all $\tilde{V}_i$'s are I^*-spherical and $\mu_n(F)$ acts on them via ε) and non-negative integers $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ (some of them may be zero) such that

$$\text{signature of } \langle\, , \rangle = \left(\sum_{i=1}^{m} a_i \theta_{K^*}(\tilde{V}_i), \sum_{j=1}^{m} b_j \theta_{K^*}(\tilde{V}_j) \right).$$

This is just a modified version of Theorem 5.2 in [BM], which can be proved through the same procedure as in Section 5 of [BM]. Then the signature character of $\tilde{V}^{I^*}$ is

$$\text{signature of } \langle\, , \rangle^{I^*} = \left(\sum_{i=1}^{m} a_i \theta_{\mathscr{H}(K^*)}(\tilde{V}_i^{I^*}), \sum_{j=1}^{m} b_j \theta_{\mathscr{H}(K^*)}(\tilde{V}_j^{I^*}) \right).$$

Applying the map $\mathscr{M}$, we have the signature character of $V^{I'}$ is

$$\text{signature of } \langle\, , \rangle^{I'} = \left(\sum_{i=1}^{m} a_i \theta_{\mathscr{H}(K')}(V_i^{I'}), \sum_{j=1}^{m} b_j \theta_{\mathscr{H}(K')}(V_j^{I'}) \right),$$

where $V^{I'} = \mathscr{M}(\tilde{V}^{I^*})$. By the linear independence of $\theta_{\mathscr{H}(K')}(V_j^{I'})$'s (cf. [BM] Corollary 4.8), the positive definiteness of $\langle\, , \rangle^{I'}$ implies $b_j = 0$ for all j. Hence $\langle\, , \rangle$ is also positive definite. □

COROLLARY 6.4. *Suppose $(\tilde{\pi}, \tilde{V}) \in \mathrm{R}_0^{\varepsilon}(\tilde{G})$ is real hermitian, then the unitarity of $\tilde{V}$ can be detected on the space of its Iwahori-fixed vectors $\tilde{V}^{I^*}$.*