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K_2 of elliptic curves with sufficient torsion over \mathbf{Q}

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1. Introduction

The conjectures of Beilinson and Bloch ([1]–[3]) relate the conjectural behavior at $s=0$ of the Hasse–Weil L -function $L(s, E)$ of an elliptic curve E defined over \mathbf{Q} to K_2E via a regulator which generalizes that of Dirichlet. Part of what the conjectures assert is that K_2E is a finitely generated abelian group of rank $1 + |\text{Spl}(E)|$, where $\text{Spl}(E)$ denotes the set of primes where E has split multiplicative reduction [3].

In case E has complex multiplication, there is partial evidence in support of the part of the conjecture concerning the rank of K_2E : in this case the conjectural rank of K_2E is 1, and Bloch has constructed a rank 1 subgroup ([2], also see [8]). But in case E does not have CM, there were only finitely many examples for which one knew that K_2E had positive rank. In this paper, we show that for all but finitely many elliptic curves E/\mathbf{Q} possessing a rational torsion point of order at least 3, K_2E has positive rank. Our method is as follows. In the case of an elliptic curve E defined over \mathbf{C} , one may view the regulator as a homomorphism $K_2E \rightarrow \mathbf{C}$. Parametrize elliptic curves in the usual manner by points in the complex upper half-plane \mathcal{H} ; denote by E_λ the elliptic curve corresponding to $\lambda \in \mathcal{H}$. For each λ , we construct an element $\alpha_\lambda \in K_2E_\lambda$ using torsion points on E_λ , and show that the map $\lambda \mapsto \text{reg}_{E_\lambda}(\alpha_\lambda)$ is real analytic on \mathcal{H} and behaves well near the cusps. (Here, we are denoting by reg_{E_λ} the regulator homomorphism on K_2E_λ .) This allows us to conclude our result with \mathbf{Q} replaced by \mathbf{R} ; using the twisting theory of elliptic curves allows us to descend to \mathbf{Q} .

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2. Analytic behavior of the regulator

Let E be an elliptic curve defined over \mathbf{C} . In this section, we only care about the \mathbf{C} -isomorphism class of E , and thus identify $E(\mathbf{C})$ with a complex torus \mathbf{C}/Λ ,

where Λ is a lattice in \mathbf{C} . Let ω be a nonzero holomorphic 1-form on $E(\mathbf{C})$. In [1], Beilinson defines a regulator

$$\text{reg}_E: K_2\mathbf{C}(E) \rightarrow \mathbf{C}$$

by

$$\text{reg}_E(\{f, g\}) = \frac{1}{2\pi i} \int_{E(\mathbf{C})} \log |f| \overline{\log g} \wedge \omega.$$

Note that this depends on the choice of ω . To eliminate this dependence, we normalize the regulator as follows. The period lattice of ω is homothetic to $\Lambda = \mathbf{Z} + \mathbf{Z}\lambda$ for some $\lambda \in \mathcal{H}$. Let Γ_E denote the element of $H_1(E(\mathbf{C}), \mathbf{Z})$ determined by the segment of the real axis connecting 0 to 1. Put

$$\Omega_E = \int_{\Gamma_E} \omega.$$

Then define $\rho_E: K_2\mathbf{C}(E) \rightarrow \mathbf{C}$ by $\rho_E(\{f, g\}) = \Omega_E^{-1} \text{reg}_E(\{f, g\})$. We want to express $\rho_E(\{f, g\})$ in terms of the homothety class of Λ , $\text{div}(f)$, and $\text{div}(g)$.

Let $r, s \in \mathbf{R}$, $\lambda = x + iy \in \mathcal{H}$, and define $\mathcal{E}(r, s; \lambda)$ by:

$$\mathcal{E}(r, s; \lambda) = \sum'_{(m,n)} (m\lambda + n) |m\lambda + n|^{-4} e^{2\pi i(mr + ns)}.$$

Here, the prime indicates that the sum is over all pairs of integers $(m, n) \neq (0, 0)$. Note that \mathcal{E} depends on r and s only mod \mathbf{Z} . \mathcal{E} has the following modular behavior: If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, then

$$\mathcal{E}(r, s; \gamma\lambda) = \frac{|c\lambda + d|^4}{c\lambda + d} \mathcal{E}(dr - bs, as - cr; \lambda). \tag{1}$$

Beilinson, in [1], gives a formula for $\rho_E(\{f, g\})$, which we state in the following lemma.

LEMMA 2.1. *Let E be an elliptic curve defined over \mathbf{C} , and let $\lambda \in \mathcal{H}$ be such that the period lattice of E is homothetic to $\Lambda = \mathbf{Z} + \mathbf{Z}\lambda$. For $z \in \mathbf{C}/\Lambda$, write $z = u(z)\lambda + v(z) \bmod \Lambda$ with $u(z)$ and $v(z)$ in $[0, 1)$. Let $f, g \in \mathbf{C}(E)^*$, and identify f and g with functions on \mathbf{C}/Λ . Then*

$$\rho_E(\{f, g\}) = \frac{(\text{Im } \lambda)^2}{\pi^2} \sum_{z, w \in \mathbf{C}/\Lambda} (\text{ord}_z f)(\text{ord}_w g) \mathcal{E}(v(z - w), -u(z - w); \lambda).$$

Proof. [4], Lemma (3.2). □

We now examine the analytic properties of this expression for ρ_E . We begin with the following lemma, which gives a Fourier expansion for $\mathcal{E}(r, s; \lambda)$. Let $\lambda = x + iy$.

LEMMA 2.2. *Suppose that $s \in \mathbf{Q}$ and $N \in \mathbf{N}$ satisfy $Ns \in \mathbf{Z}$. Then*

$$\operatorname{Re} \mathcal{E}(r, s; \lambda) = y^{-2} \sum_{k=0}^{\infty} a_k e^{-2\pi ky} + y^{-1} \sum_{k=0}^{\infty} b_k e^{-2\pi ky/N}$$

and

$$\operatorname{Im} \mathcal{E}(r, s; \lambda) = 4\pi^3 B(s) + y^{-1} \sum_{k=0}^{\infty} c_k e^{-2\pi ky/N}$$

where $a_k, b_k, c_k \in \mathbf{R}$ depend only on r, s, N , and x , and $B(s) = \frac{1}{3}s^3 - \frac{1}{2}s^2 + \frac{1}{6}s$ for $s \in [0, 1]$, and for general s , $B(s) = B(s - [s])$, where $[s]$ denotes the greatest integer less than or equal to s .

Proof. For $z \in \mathbf{C}$, $\operatorname{Re} z > \frac{3}{4}$, define $\mathcal{E}(r, s; \lambda, z)$ by

$$\mathcal{E}(r, s; \lambda, z) = \sum'_{(m,n)} (m\lambda + n) |m\lambda + n|^{-4z} e^{2\pi i(mr + ns)}.$$

For z in this half-plane, the sum converges absolutely and uniformly on compact sets. Assume now that $\operatorname{Re} z > 1$. Letting

$$S(\lambda, z) = \sum_{m \neq 0} \sum_n \frac{1}{m} |m\lambda + n|^{2-4z} e^{2\pi i(mr + ns)},$$

we may write

$$\begin{aligned} \mathcal{E}(r, s; \lambda, z) &= \sum_{n \neq 0} \frac{n}{|n|^{4z}} e^{2\pi i ns} - \frac{1}{2z-1} \frac{\partial}{\partial \bar{\lambda}} S(\lambda, z) \\ &= 4\pi i B(s) - \frac{1}{2z-1} \frac{\partial}{\partial \bar{\lambda}} S(\lambda, z). \end{aligned}$$

We have

$$\begin{aligned} S(\lambda, z) &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m \neq 0} \sum_n \frac{1}{m} e^{2\pi i(mr + ns)} \int_0^{\infty} e^{-\pi t |m\lambda + n|^2} t^{2z-1} \frac{dt}{t} \\ &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m \neq 0} \frac{1}{m} e^{2\pi i m(r - sx)} \int_0^{\infty} \left(\sum_n e^{-\pi t(n + mx - is/t)^2} \right) e^{-\pi(ty^2 m^2 + s^2/t)} t^{2z-1} \frac{dt}{t}. \end{aligned}$$

By Poisson summation,

$$\sum_n e^{-\pi t(n+mx-is/t)^2} = \frac{1}{\sqrt{t}} \sum_n e^{-\pi n^2/t} e^{2\pi i n(mx-is/t)}.$$

Substituting the right-hand side into the expression for S and simplifying, we obtain

$$\begin{aligned} S(\lambda, z) &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m \neq 0} \frac{1}{m} e^{2\pi i m(r-sx)} \sum_n e^{2\pi i mnx} \int_0^\infty e^{-\pi(tm^2y^2 + (n-s)^2/t)} t^{2z-3/2} \frac{dt}{t} \\ &= \frac{-\pi^{2z-1}}{\Gamma(2z)} \sum_{m \neq 0} \frac{1}{m} e^{2\pi i m(r-sx)} \sum_n e^{2\pi i mnx} K_{2z-3/2}(\sqrt{\pi} |m|y, \sqrt{\pi} |n-s|), \end{aligned}$$

where, following [5],

$$K_\nu(a, b) = \int_0^\infty e^{-(a^2t+b^2/t)} t^\nu \frac{dt}{t}.$$

By analytic continuation, the expression above for $S(\lambda, z)$ holds for all z . In particular, it holds for $z = 1$.

We have (see [5], pp. 270–271)

$$K_{1/2}(\sqrt{\pi} |m|y, \sqrt{\pi} |n-s|) = |m|y^{-1} e^{-2\pi |m| |n-s|y}.$$

Hence, we have the following expression for $S(\lambda, 1)$:

$$S(\lambda, 1) = -\pi y^{-1} \sum_{m \neq 0} \sum_n \frac{1}{m|m|} e^{2\pi i(m(r-sx) + nm x + iy|n-s||m|)}, \tag{2}$$

and therefore obtain the following expression for \mathcal{E} :

$$\mathcal{E}(r, s; \lambda) = 4\pi^3 i B(s) + \frac{\pi i}{2} y^{-2} S(\lambda, 1) + \pi y^{-1} \frac{\partial S(\lambda, 1)}{\partial \bar{\lambda}}.$$

Noting that $S(\lambda, 1)$ is totally imaginary, we find that

$$\operatorname{Re} \mathcal{E}(r, s; \lambda) = \frac{\pi i}{2} y^{-2} S(\lambda, 1) + \pi y^{-1} \operatorname{Re} \frac{\partial S(\lambda, 1)}{\partial \bar{\lambda}}$$

and

$$\operatorname{Im} \mathcal{E}(r, s; \lambda) = 4\pi^3 B(s) + \pi y^{-1} \operatorname{Im} \frac{\partial S(\lambda, 1)}{\partial \bar{\lambda}}.$$

In view of (2), the lemma now follows. □

We now turn our attention to functions of the form

$$\phi(\lambda) = \sum_{j=1}^r m_j \mathcal{E}(r_j, s_j; \lambda)$$

where $m_j \in \mathbf{Z}$, and $r_j, s_j \in [0, 1)$ with $s_j \in \mathbf{Q}$. For such a ϕ , we will choose a natural number N such that for all j , $Ns_j \in \mathbf{Z}$. It is clear that ϕ is a complex-valued real analytic function on \mathcal{H} . We now proceed to examine the behavior of ϕ near the cusps.

We will need the following simple lemma.

LEMMA 2.3. For $y > 0$ consider the function

$$\Phi(y) = y^{-1} \sum_{k=0}^{\infty} a_k e^{-2\pi ky/N} + \sum_{k=0}^{\infty} b_k e^{-2\pi ky/N},$$

where $a_k, b_k \in \mathbf{R}$ and $N \in \mathbf{N}$. Suppose that Φ is not identically zero. Then for all y sufficiently large, $\Phi(y) \neq 0$.

Proof. Let $w = e^{-2\pi y/N}$. It suffices to show that for all $w > 0$ sufficiently small, the function

$$f(w) = -\frac{2\pi}{N} (\log w)^{-1} \sum_{k=0}^{\infty} a_k w^k + \sum_{k=0}^{\infty} b_k w^k$$

has no zeros. This is straightforward. □

We now return to ϕ . For $x \in \mathbf{R}$, we let L_x denote the vertical ray in \mathcal{H} defined by $L_x = \{x + iy : y > 0\}$.

LEMMA 2.4. Let $x \in \mathbf{Q}$. Suppose that $\text{Re } \phi$ (resp. $\text{Im } \phi$) is not identically zero on L_x . Then $\text{Re } \phi$ (resp. $\text{Im } \phi$) has at most finitely many zeros on L_x .

Proof. We prove this only for $\text{Re } \phi$, the proof for $\text{Im } \phi$ being similar.

By Lemma 2.2 we have

$$\text{Re } \phi(x + iy) = y^{-1} \left(y^{-1} \sum_{k=0}^{\infty} A_k e^{-2\pi ky/N} + \sum_{k=0}^{\infty} B_k e^{-2\pi ky/N} \right)$$

which, by Lemma 2.3, has no zeros for y sufficiently large.

If $x \neq 0$, write $x = A/C$ with $A, C \in \mathbf{Z}$, $C > 0$, and $(A, C) = 1$. Let B and D be integers such that $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}_2(\mathbf{Z})$. Put $x' = -D/C$. If $x = 0$, let $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $x' = 0$. Give each L_x the orientation induced by the usual

ordering on y . Note that γ gives an orientation-reversing map of $L_{x'}$ onto L_x . Thus, by Equation (1), we are led to examine

$$\Phi(y) = C^3 y^3 \operatorname{Im} \tilde{\phi}(x' + iy)$$

for large values of y , where

$$\tilde{\phi}(\lambda) = \sum_{j=1}^r m_j \mathcal{E}(Dr_j - Bs_j, As_j - Cr_j; \lambda).$$

$\operatorname{Im} \tilde{\phi}$ is not identically zero on $L_{x'}$ because $\operatorname{Re} \phi$ is not identically zero on L_x . Then, by Lemma 2.3, we conclude that $\operatorname{Im} \tilde{\phi}$ has no zeros on $L_{x'}$ for y large enough.

Therefore the zeros of $\operatorname{Re} \phi$ on L_x are contained in a compact subset of L_x . Since $\operatorname{Re} \phi$ is real analytic, it follows that it has only finitely many zeros on L_x . \square

3. The main theorem

We now construct elements in the K_2 groups of elliptic curves defined over \mathbf{Q} with a rational torsion point of order at least three, and study the relevant regulator expression.

We begin by standardizing our choice of period lattice for E . Let O denote the identity element for the group law on E .

LEMMA 3.1. *Let E be an elliptic curve defined over \mathbf{R} . Fix an orientation on $E(\mathbf{R})^\circ$, the connected component of the identity in $E(\mathbf{R})$. Then there exists a unique pair (Λ, θ) where $\Lambda \subset \mathbf{C}$ is a lattice and $\theta: \mathbf{C}/\Lambda \rightarrow E(\mathbf{C})$ is a complex analytic isomorphism such that:*

- (a) θ is defined over \mathbf{R} .
- (b) $\Lambda \cap \mathbf{R} = \mathbf{Z}$ and $\theta|_{\mathbf{R}/\mathbf{Z}}$ maps \mathbf{R}/\mathbf{Z} isomorphically onto $E(\mathbf{R})^\circ$ in an orientation-preserving manner, where \mathbf{R}/\mathbf{Z} is given the orientation induced by the usual order on \mathbf{R} . Hence $\Gamma_E = E(\mathbf{R})^\circ$ with the specified orientation.
- (c) $\Lambda = \mathbf{Z} + \mathbf{Z}\lambda$ with $\operatorname{Re} \lambda = 0$ or $1/2$ and $\operatorname{Im} \lambda > 0$. Furthermore, $\operatorname{Re} \lambda = 0$ (resp. $1/2$) if $[E(\mathbf{R}): E(\mathbf{R})^\circ] = 2$ (resp. 1).

Proof. Let ω be a non-zero holomorphic 1-form on $E(\mathbf{C})$ defined over \mathbf{R} . Let Λ be the period lattice of ω . Then Λ is invariant under complex conjugation, whence $\Lambda \cap \mathbf{R} \neq \emptyset$. By suitably renormalizing ω , we may assume that $\Lambda \cap \mathbf{R} = \mathbf{Z}$. Let ψ denote the Abel–Jacobi map:

$$\psi: E(\mathbf{C}) \rightarrow \mathbf{C}/\Lambda \quad \psi: P \mapsto \int_O^P \omega \bmod \Lambda.$$

Then ψ is defined over \mathbf{R} . Let $\theta = \psi^{-1}$. By replacing θ with $-\theta$ if necessary, we may assume that $\theta|_{\mathbf{R}/\mathbf{Z}}$ preserves orientations. This shows (a) and (b).

Now let $\Lambda = \mathbf{Z}\lambda_1 + \mathbf{Z}\lambda_2$. Then there exist integers a and b such that $1 = a\lambda_1 + b\lambda_2$. Because $\Lambda \cap \mathbf{R} = \mathbf{Z}$, a and b must be relatively prime. Choose integers c and d such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, and let $\lambda = c\lambda_1 + d\lambda_2$. Then $\Lambda = \mathbf{Z} + \mathbf{Z}\lambda$. By replacing λ with $-\lambda$ if necessary, we may assume that $\lambda \in \mathcal{H}$. Since $\bar{\lambda} \in \Lambda$, we find that $\operatorname{Re} \lambda \in \frac{1}{2}\mathbf{Z}$. Adding a suitable integer to λ allows us to assume that $\operatorname{Re} \lambda = 0$ or $1/2$.

Suppose that $\operatorname{Re} \lambda = 0$, and put $\lambda = iy$, $y > 0$. Let $X = \{x + \frac{1}{2}iy : 0 \leq x < 1\}$. Then $\bar{X} \equiv X \pmod{\Lambda}$, where the bar denotes complex conjugation, and $\bar{X} \not\equiv \{x : 0 \leq x < 1\} \pmod{\Lambda}$. So $E(\mathbf{R})$ has two components.

Suppose that $\operatorname{Re} \lambda = \frac{1}{2}$. Note then that $\Lambda = \mathbf{Z}\lambda + \mathbf{Z}\bar{\lambda}$, and that the fundamental parallelogram \mathcal{P} defined by λ and $\bar{\lambda}$ is invariant under complex conjugation. So if $z \in \mathcal{P}$ satisfies $z \equiv \bar{z} \pmod{\Lambda}$, then $z = \bar{z}$, whence $z \in \mathbf{R}$. So in this case $E(\mathbf{R})$ has only one component.

To verify the uniqueness of (Λ, θ) , assume that we have another pair (Λ', θ') satisfying (a), (b), and (c) above. Then $\phi = \theta'^{-1} \circ \theta : \mathbf{C}/\Lambda \rightarrow \mathbf{C}/\Lambda'$ is a complex analytic isomorphism defined over \mathbf{R} . Therefore, $\Lambda = c\Lambda'$ for some $c \in \mathbf{C}^*$, (a) implies that $c \in \mathbf{R}$ and then (b) implies that $c = 1$. □

Now let E be defined over \mathbf{Q} , and let $N \in \{3, 4, 5, 6, 7, 8, 9, 10, 12\}$. We assume that E has a rational torsion point of exact order N . For each of these values of N , there are infinitely many such E/\mathbf{Q} , because the modular curve $X_1(N)$ has genus zero in these cases. A well-known theorem of Mazur implies that these values of N , together with 1 and 2, are the only ones possible.

Let $P \in E(\mathbf{Q})$ be a point of exact order N , and write $P = \theta(u\lambda + a/N)$ where θ and λ are as in Lemma 3.1, and a is unique modulo N . Since $2P \in E(\mathbf{R})^\circ$, we may assume that $u = 0$ or $\frac{1}{2}$. If $\operatorname{Re} \lambda = \frac{1}{2}$, so that $E(\mathbf{R})$ has only one component, we necessarily have $u = 0$.

LEMMA 3.2. *For each N , let $P \in E(\mathbf{Q})$ be a point of exact order N . Then there exist functions f and g in $\mathbf{Q}(E)$ such that $\operatorname{div}(f) = N(P) - N(O)$, $\operatorname{div}(g) = N(-P) - N(O)$, and $\{f, g\} \in \ker \tau$, where τ is the global tame symbol on $K_2\mathbf{Q}(E)$ [7].*

Proof. Since P is of order N and defined over \mathbf{Q} , there exist functions f and g defined over \mathbf{Q} having the indicated divisors. By multiplying these functions by suitable rational numbers, we may assume that $f(-P) = g(P) = 1$. Weil Reciprocity implies that the symbol $\{f, g\} \in \ker \tau$. □

Let f and g be as in Lemma 3.2. An easy calculation gives:

$$\rho_E(\{f, g\}) = \frac{N^2(\operatorname{Im} \lambda)^2}{\pi^2} \left(\mathcal{E} \left(\frac{2a}{N}, 0; \lambda \right) - 2\mathcal{E} \left(\frac{a}{N}, u; \lambda \right) \right),$$

where $E = E_\lambda$ and λ is given by Lemma 3.1. Let $\phi(u, a, N; \lambda) = \mathcal{E}(2a/N, 0; \lambda) - 2\mathcal{E}(a/N, u; \lambda)$. Note that $\phi(u, a, N; \lambda) \in \mathbf{R}$ for $\text{Re } \lambda = 0$ or $\frac{1}{2}$.

LEMMA 3.3. *Let $u, a,$ and N be as above. Then $\phi(u, a, N; \lambda)$ has only finitely many zeros on L_0 and $L_{1/2}$.*

Proof. Let $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Note that $\sigma(L_0) = L_0$ and $\gamma(L_{-1/2}) = L_{1/2}$. Note also that if E/\mathbf{R} has period lattice $\mathbf{Z} + \mathbf{Z}\lambda$ with $\text{Re } \lambda = \frac{1}{2}$, then $E(\mathbf{R}) = E(\mathbf{R})^\circ$; hence in this case $u = 0$.

By Lemma 2.4, it suffices to show that $\text{Re } \phi(u, a, N; \sigma\lambda)$ is not identically zero on L_0 and that $\text{Re } \phi(0, a, N; \gamma\lambda)$ is not identically zero on $L_{-1/2}$ for each of the values of $u, a,$ and N which can occur. Computing using equation (1) and discarding an automorphy factor which never vanishes, it suffices to show that

$$\text{Im} \left(\mathcal{E} \left(0, \frac{2a}{N}; \lambda \right) - 2\mathcal{E} \left(-u, \frac{a}{N}; \lambda \right) \right)$$

is not identically zero on L_0 , and that

$$\text{Im} \left(\mathcal{E} \left(\frac{2a}{N}, \frac{-4a}{N}; \lambda \right) - 2\mathcal{E} \left(\frac{a}{N}, \frac{-2a}{N}; \lambda \right) \right)$$

is not identically zero on $L_{-1/2}$. We do this by examining the Fourier coefficients of these expressions, using Lemma 2.2.

Note that the leading term of the first expression is $4\pi^3 \left(B\left(\frac{2a}{N}\right) - 2B\left(\frac{a}{N}\right) \right)$. Since $B(2t) - 2B(t) = 2t^3 - t^2$ for t between 0 and 1, we see that this term is nonzero for all admissible values of a and N .

As for the second expression, note that its leading term is $4\pi^3 \left(B\left(-\frac{4a}{N}\right) - 2B\left(-\frac{2a}{N}\right) \right)$, which is nonzero for all admissible values of a and N except $N = 4$ and $a = \pm 1$.

To take care of this case, we return to

$$\phi(0, \pm 1, 4; \lambda) = \pm(\mathcal{E}(\frac{1}{2}, 0; \lambda) - 2\mathcal{E}(\frac{1}{4}, 0; \lambda)),$$

where we have used the fact that $\mathcal{E}(-r, -s; \lambda) = -\mathcal{E}(r, s; \lambda)$. This fact also implies in particular that $\mathcal{E}(\frac{1}{2}, 0; \lambda) = 0$. Returning to the proof of Lemma 2.2, we find that

$$\mathcal{E}(\frac{1}{4}, 0; \frac{1}{2} + iy) = -\pi \frac{\partial}{\partial \lambda} y^{-1} \sum_{m \neq 0} \sum_n \frac{1}{m|m|} e^{2\pi i(m/4 + mnx + iy|mn|)} \Big|_{\lambda = 1/2 + iy}.$$

Break this into two sums, one for which $n=0$ and one for which $n \neq 0$. Denote this latter sum by $S(x, y)$. We thus obtain

$$\operatorname{Re} \mathcal{E}(\frac{1}{4}, 0; \frac{1}{2} + iy) = -\pi y^{-2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} + S(\frac{1}{2}, y).$$

The term $S(\frac{1}{2}, y)$ decays like $y^{-1} e^{-2\pi y}$ as $y \rightarrow \infty$. Hence,

$$\lim_{y \rightarrow \infty} y^2 \mathcal{E}(\frac{1}{4}, 0; \frac{1}{2} + iy) = -\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \neq 0. \quad \square$$

PROPOSITION 3.1. *Let K be a perfect field of characteristic $\neq 2, 3$. Let $j \in K$, $j \neq 0$, and let $N \geq 3$ be an integer. Then there are only finitely many K -isomorphism classes of elliptic curves E/K such that $j(E) = j$ and $E(K)$ has a point of exact order N .*

Proof. Suppose that $j \neq 1728$. Let E/K have invariant j . Choose a Weierstrass equation for E :

$$E: y^2 = x^3 + Ax + B$$

with $A, B \in K$. The set of K -isomorphism classes of elliptic curves E'/K such that $j(E') = j$ is in one-to-one correspondence with K^*/K^{*2} ; this correspondence is given explicitly by

$$D \bmod K^{*2} \leftrightarrow E_D: y^2 = x^3 + D^2 Ax + D^3 B$$

and an isomorphism $\phi_D: E \rightarrow E_D$, defined over \bar{K} , is given by

$$\phi_D: (x, y) \mapsto (Dx, D^{3/2}y),$$

where $D^{3/2}$ is some fixed square root of D^3 [10].

Let $(x, y) \in E(\bar{K})$ be of exact order N ; since $N \geq 3$, we know that $y \neq 0$. We claim that there is at most one $D \bmod K^{*2}$ such that $\phi_D(x, y) \in E_D(K)$. For suppose that D' were also such that $\phi_{D'}(x, y) \in E_{D'}(K)$. Then both $\sqrt{D}y$ and $\sqrt{D'}y$ belong to K . Since $y \neq 0$, we conclude that $D \equiv D' \bmod K^{*2}$. Hence we obtain the proposition in case $j \neq 1728$.

If $j = 1728$, consider the following elliptic curve

$$E: y^2 = x^3 + x.$$

The set of K -isomorphism classes of elliptic curves E'/K with $j(E') = 1728$ is in

one-to-one correspondence with K^*/K^{*4} ; this correspondence is given explicitly by

$$D \bmod K^{*4} \leftrightarrow E_D: y^2 = x^3 + Dx,$$

and an isomorphism $\psi_D: E \rightarrow E_D$, defined over \bar{K} , is given by

$$\psi_D: (x, y) \mapsto (\delta^2 x, \delta^3 y)$$

where δ is any fourth-root of D [10].

Let $(x, y) \in E(\bar{K})$ be of exact order N ; since $N \geq 3$, we know that $xy \neq 0$. Again there is at most one $D \bmod K^{*4}$ such that $\psi_D(x, y) \in E_D(K)$. For if $D' \bmod K^{*4}$ were also such that $\psi_{D'}(x, y) \in E_{D'}(K)$, then, letting δ' be a fourth-root of D' , we have $\delta'^2 x$ and $\delta'^3 y$ belonging to K . Since $xy \neq 0$, we have $(\delta/\delta')^2 \in K^*$ and $(\delta/\delta')^3 \in K^*$. So $\delta/\delta' \in K^*$, that is, $D \equiv D' \bmod K^{*4}$. \square

REMARKS. (1) In the case $K = \mathbf{Q}$, this is a weak version of the main result of [6].

(2) As stated, the proposition is false for curves of j invariant 0. As a counterexample, consider the family E_d of curves defined over \mathbf{Q} by

$$E_d: y^2 = x^3 + d^2$$

where $d \in \mathbf{Q}^{*2}$. Then the 3-torsion in $E_d(\mathbf{Q})$ consists of $(0, d)$, $(0, -d)$, and ∞ .

We may now state our main result:

THEOREM 3.1. *Let N be an integer greater than or equal to 3. Then for all but finitely many \mathbf{Q} -isomorphism classes of elliptic curves E/\mathbf{Q} such that $E(\mathbf{Q})$ possesses a torsion point of order N , there exists $\alpha \in K_2 E$ such that $\rho_E(\alpha) \neq 0$.*

Proof. If $j(E) = 0$, then the statement follows from Bloch's theorem [2]. Hence, we may assume that $j(E) \neq 0$. For each such curve, choose a point P of exact order N defined over \mathbf{Q} and construct $\{f, g\}$ as in Lemma 3.2. Since $\{f, g\}$ is in the kernel of the tame symbol, it follows from the localization sequence in K -theory that $\{f, g\}$ represents an element $\alpha \in K_2 E$. Let λ be the point in \mathcal{H} corresponding to E , as determined in Lemma 3.1. Then $\rho_E(\alpha) = \phi(u, a, N; \lambda)$ for some admissible choice of u, a, N .

By Lemma 3.3, there are at most finitely many values λ_0 for λ such that the corresponding value $\rho_E(\alpha)$ is zero. By Proposition 3.1, to each of these values λ_0 there are associated only finitely many elliptic curves of the type we are considering. The theorem follows. \square

Using the functoriality of the regulator, we immediately obtain the following:

THEOREM 3.2. *For all but finitely many elliptic curves E/\mathbf{Q} which are isogenous*

over \mathbf{Q} to an elliptic curve defined over \mathbf{Q} containing a rational torsion point of order at least three, K_2E contains an element of infinite order.

We remark that this generalization is non-vacuous, since any elliptic curve defined over \mathbf{Q} is isogenous over \mathbf{Q} to an elliptic curve E'/\mathbf{Q} such that $|E'(\mathbf{Q})_{\text{tors}}| = 1$ or 2 ([9]).

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