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Nil-manifolds as links of isolated singularities

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By a nil-manifold we shall mean a manifold having the homotopy type of the classifying space of a finitely generated (necessarily torsion free) nilpotent group. Nil-manifolds do occur as links of isolated singularities: if $\mathscr{L} \to A$ is a line bundle over an abelian variety whose inverse is ample, then the zero section of $\mathscr{L}$ can be collapsed to form an isolated singularity whose link L is the unit circle bundle of $\mathscr{L}$. This is easily seen to be a nil-manifold whose fundamental group Γ can be expressed as a central extension

$$0 \to \mathbb{Z} \to \Gamma \to H_1(A) \to 0.$$

This may be seen by considering the long exact sequence of homotopy groups associated to the circle bundle $L \to A$.

It is natural to ask which nil-manifolds occur as links of isolated singularities. Our main result asserts that, up to homotopy, these are the only examples. I would like to thank Bill Goldman and John Millson for bringing this question to my attention. I would also like to thank Vincente Navarro-Aznar for carefully reading an earlier version and saving from some embarassing slips.

To describe our result in more detail, we need to introduce the notion of a group of *Heisenberg type*. First recall that a central extension

$$0 \to \mathbb{Z} \to \Gamma \to G \to 1$$

of a group G is classified by a cohomology class $e \in H^2(G, \mathbb{Z})$, where $\mathbb{Z}$ has the trivial G module structure. If G is a torsion free abelian group, then $H^2(G, \mathbb{Z})$ is isomorphic to the set of skew symmetric bilinear forms $G \times G \to \mathbb{Z}$. In this case, the skew form $q\colon G \times G \to \mathbb{Z}$ associated to the extension is defined by

$$q(a, b) = \{\tilde{a}, \tilde{b}\},$$

where $\tilde{a}, \tilde{b}$ denote lifts of a, b to Γ, and $\{x, y\}$ denotes the commutator $xyx^{-1}y^{-1}$.

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In the examples above, the class of the extension is the Chern class of the line bundle $\mathscr{L} \to A$.

We will say that a group Γ is of *Heisenberg type* if Γ is finitely generated and can be expressed as a central extension

$$0 \to \mathbb{Z} \to \Gamma \to H \to 0$$

where H is a torsion free abelian group, and the class of the extension is non-degenerate as a bilinear form. This last statement is equivalent to the statement that the kernel of $\Gamma \to H$ is the center of Γ. The fundamental groups of the singularities constructed above are all of Heisenberg type as Chern classes of ample line bundles over abelian varieties are non-degenerate (see, for example [4, p. 317]).

The following is our main result. It is a partial generalization of the well-known fact* that the only compact Kähler manifolds that are nil-manifolds are the compact complex tori. It is also a partial generalization of a result of Wagreigh who proves a stronger theorem for surface singularities [11]. By an n-fold singularity, we shall mean a singularity of a complex algebraic variety of dimension n.

THEOREM A. *Suppose that L is the link of an isolated n-fold singularity. If L has the homotopy type of a nil-manifold with fundamental group* Γ, *then* Γ *is a group of Heisenberg type. Moreover, the canonical mixed Hodge structure on* $H^k(L)$ *is pure of weight* k *when* $k < n$ *and pure of weight* $k+1$ *when* $k \geq n$. *Finally, the class of the extension* $e \in \Lambda^2 H^1(L)$ *is a Hodge class; that is, it is integral and of type* (1, 1).

It would be interesting to know whether the negative of e satisfies the Riemann bilinear relations, for then the *Albanese* of L

$$\mathrm{Alb}(L) := H_1(L, \mathbb{C})/(H_1(L, \mathbb{Z}) + F^0 H_1(L))$$

would be an abelian variety polarized by the inverse of the line bundle with Chern class e. As in the introduction, one can then construct an isolated singularity whose link is homotopy equivalent to L and whose cohomology has a mixed Hodge structure isomorphic to that of L. One could then hope for a Torelli theorem, which would assert that the singularity which gave rise to L is isomorphic to the one obtained from the ample line bundle over $\mathrm{Alb}(L)$. This would imply that the only isolated singularities whose links are nil-manifolds are those obtained from ample line bundles over abelian varieties.

PRELIMINARIES. Assume that the nil-manifold L is the link of an isolated n-fold singularity. Morgan's work [6] combined with results from either [3] or [8]

*A proof follows easily using the methods of the first part of the proof of our main theorem.

imply that the Sullivan minimal model [10] $\mathcal{M}^\bullet$ of L has a (not necessarily canonical) mixed Hodge structure such that the natural isomorphism

$$H^\bullet(\mathcal{M}^\bullet) \approx H^\bullet(L)$$

is an isomorphism of mixed Hodge structures.

Let $\mathfrak{g}$ be the nilpotent Lie algebra associated to the fundamental group of L. Denote the Chevalley–Eilenberg complex associated to $\mathfrak{g}$ by $\mathscr{G}^\bullet(\mathfrak{g})$. This is the exterior algebra generated by the dual of $\mathfrak{g}$ with differential dual to the bracket of $\mathfrak{g}$. It computes $H^\bullet(\mathfrak{g})$. (See, for example, [5].) This is a minimal algebra. Since L is a nil-manifold, it follows from [9] and the uniqueness of minimal models that $\mathcal{M}^\bullet$ is isomorphic to $\mathscr{C}^\bullet(\mathfrak{g})$. Since $\mathfrak{g}$ is the dual of $\mathcal{M}^1$, it follows that $\mathfrak{g}$ has a mixed Hodge structure. Since the differential $d\colon \mathcal{M}^1 \to \mathcal{M}^2$ is a morphism of mixed Hodge structures, it follows that the bracket of $\mathfrak{g}$ is also a morphism of mixed Hodge structures. This all implies that $\mathscr{G}^\bullet(\mathfrak{g})$ has a mixed Hodge structure and that the d.g.a. isomorphism $\mathscr{C}^\bullet(\mathfrak{g}) \to \mathcal{M}^\bullet$ is an isomorphism of mixed Hodge structures. That is, there is a mixed Hodge structure on $\mathfrak{g}$ which induces one on $H^\bullet(\mathfrak{g})$, and the natural isomorphism $H^\bullet(\mathfrak{g}) \approx H^\bullet(L)$ is an isomorphism of mixed Hodge structures.

Let $d = \dim \mathfrak{g}$. Since L is a compact orientable manifold, and since $H^i(\mathfrak{g})$ is isomorphic to $\mathbb{Q}$ when $i = d$, and 0 when $i > d$, it follows that $d = \dim_{\mathbb{R}} L = 2n-1$.

Let $\mathfrak{n}$ be a finite dimensional Lie algebra over $\mathbb{R}$, and N the corresponding simply connected Lie group. It follows by induction on the length of the lower central series of the Lie algebra that there is a discrete, cocompact subgroup Γ of N. The quotient $\Gamma\backslash N$ is a compact manifold. By [9] the real cohomology of this nil-manifold is isomorphic to the cohomology of $\mathfrak{n}$. It follows that the cohomology of every nilpotent Lie algebra satisfies Poincaré duality.

To prove that the fundamental group of L is of Heisenberg type, it will suffice to show that the Lie algebra $\mathfrak{g}$ is also of Heisenberg type. That is, it can be expressed as a central extension

$$0 \to \mathbb{Q} \to \mathfrak{g} \to H_1(\mathfrak{g}) \to 0$$

whose class $q \in \Lambda^2 H^1(\mathfrak{g})$ is non-degenerate as a bilinear form.

We will say that an integer l is a *weight* of a $\mathbb{Q}$-mixed Hodge structure H if

$$\mathrm{Gr}_l^W H := W_l H / W_{l-1} H$$

is non-trivial. Recall from [2] that the functors Gr_l^W are exact functors from the category of mixed Hodge structures to the category of rational vector spaces, and that the complex part of every mixed Hodge structure is canonically isomorphic to the associated graded module of the weight filtration. In particular, the complex form of a Lie algebra with a mixed Hodge structure is a graded Lie algebra.

Finally, we recall a consequence of Gabber's Purity Theorem [1]: The weights l on $H^k(L)$ satisfy $0 \leqslant l \leqslant k$ when $k < n$, and satisfy $k < l \leqslant 2n$ when $k \geqslant n$. (This applies to the cohomology of links of all isolated n-fold singularities, not just those that are nil-manifolds.) A relatively elementary proof of this fact is given by Navarro in [7].

FIRST STEPS. In this section we prove Theorem A under the additional hypothesis that the natural mixed Hodge structure on $H^1(L)$ is pure of weight 1. As already noted, the mixed Hodge structure on $H^\bullet(\mathfrak{g})$ is isomorphic to that of $H^\bullet(L)$. Since $H^1(L)$ is pure of weight 1, $\mathfrak{g} = W_{-1}\mathfrak{g}$. Set

$$w_l = \dim \mathrm{Gr}^W_{-l}\mathfrak{g}.$$

Then

$$2n - 1 = \dim \mathfrak{g} = \sum_{l>0} w_l.$$

The fundamental class of L is of type (n, n), and therefore of weight $2n$. The fundamental class of $\mathfrak{g}$ is the dual of a generator of $\Lambda^{\dim \mathfrak{g}}\mathfrak{g}$, which has weight $\Sigma\, l w_l$. It follows that

$$2n = \sum_{l>0} l w_l.$$

Combining this with the previous equality, we have

$$1 + \sum_{l>0} w_l = \sum_{l>0} l w_l.$$

That is,

$$\sum_{l>0} (l - 1) w_l = 1.$$

Since each w_l is non-negative, this implies that $w_l = 0$ when $l > 2$, and that $w_2 = 1$. Since $H^1(\mathfrak{g})$ is pure of weight 1, the weight filtration of $\mathfrak{g}$ is its lower central series. This implies that $W_{-2}\mathfrak{g}$ is one dimensional and equal to $[\mathfrak{g}, \mathfrak{g}]$, so that $\mathfrak{g}$ is a central extension

$$0 \to \mathbb{Q} \to \mathfrak{g} \to H_1(\mathfrak{g}) \to 0.$$

It also follows that $W_{-2}\mathfrak{g}$ is the Hodge structure $\mathbb{Q}(1)$ of type $(-1, -1)$. Since the bracket

$$H_1(\mathfrak{g}) \otimes H_1(\mathfrak{g}) \to W_{-2}\mathfrak{g}$$

is a morphism of mixed Hodge structures, it follows that the class $q \in \Lambda^2 H^1(\mathfrak{g})$ is a Hodge class. It remains to show that the quadratic form q is non-degenerate.

If q is degenerate, we can write $H_1(\mathfrak{g})$ as a direct sum

$$H_1(\mathfrak{g}) = A \oplus B$$

with $B \neq 0$, where the restriction of q to A is non-degenerate, and the restriction of q to B vanishes. This implies that $\mathfrak{g}$ is the Lie algebra sum of the abelian Lie algebra B, and the nilpotent Lie algebra $\mathfrak{h}$ which is the extension of A by $\mathbb{Q}$ determined by the restriction of q to A. By the Künneth Theorem, there is a ring isomorphism

$$H^\bullet(\mathfrak{g}) \approx H^\bullet(\mathfrak{h}) \otimes H^\bullet(B).$$

PROPOSITION 1. *Suppose that* $\mathfrak{g}$ *is a Lie algebra of dimension* $2a+1$ *over a field* F *of characteristic* 0. *If* $\mathfrak{g}$ *is a central extension*

$$0 \to F \to \mathfrak{g} \to A \to 0$$

where A *is abelian and the class* $q: \Lambda^2 A \to F$ *of the extension is non-degenerate, then the homomorphism* $H^i(A) \to H^i(\mathfrak{g})$ *induces an isomorphism*

$$H^i(\mathfrak{g}) \approx PH^i(A) := \operatorname{coker}\{q \wedge \_ : H^{i-2}(A) \to H^i(A)\}$$

when $i \leqslant a$. □

This is easily proved using the algebraic analogue of the Gysin sequence.

Presently, the important point is that cohomology of $\mathfrak{h}$ is non-trivial in every dimension. It follows that if q is degenerate, there exist integers a, b satisfying $a, b < n$ and $a+b > n$ such that the cup product

$$H^a(\mathfrak{g}) \otimes H^b(\mathfrak{g}) \to H^{a+b}(\mathfrak{g})$$

does not vanish. By [3] the cup product

$$H^a(L, \mathbb{Q}) \otimes H^b(L, \mathbb{Q}) \to H^{a+b}(L, \mathbb{Q})$$

vanishes when $a + b > n$. It follows that q must be non-degenerate.

Finally, the assertion about the cohomology of L follows from Proposition 1 and the fact that the cup product

$$H^i(\mathfrak{g}) \otimes H^{2n-i-1}(\mathfrak{g}) \to H^{2n-1}(\mathfrak{g}) \approx \mathbb{Q}(-n)$$

is a non-degenerate pairing of mixed Hodge structures.

PURITY. In this section we complete the proof of Theorem A by establishing the purity of $H^1(L)$. Specifically, we shall prove:

THEOREM B. *Suppose that L is the link of an isolated n-fold singularity. If $n > 1$ and L is a nil-manifold, then the natural mixed Hodge structure on $H^k(L)$ is pure of weight k when $k < n$.*

The following fact is needed in the proof. It is possible that it is well known, but I could not find a proof in the literature. The proof here, due to Thierry Levasseur, is considerably simpler than my original proof and gives a slightly stronger result.

LEMMA. *If $\mathfrak{g}$ is a nilpotent Lie algebra of dimension d over a field F of characteristic zero, then*

$$H^i(\mathfrak{g}, F) \neq 0 \quad \textit{if and only if} \quad 0 \leqslant i \leqslant d.$$

Proof. The "only if" part is easy. Since $\mathfrak{g}$ is nilpotent, $H^1(\mathfrak{g})$ is non-trivial. The kernel $\mathfrak{h}$ of a non-zero linear functional on $H^1(\mathfrak{g})$ is an ideal, so we have a short exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \to F \to 0.$$

The associated Hochschild–Serre spectral sequence

$$E_2^{s,t} = H^s(F, H^t(\mathfrak{h})) \Rightarrow H^{s+t}(\mathfrak{g})$$

has only 2 non-zero columns, and therefore degenerates at E_2. Since F has euler characteristic 0, $h^0(F, V) = h^1(F, V)$ for all coefficient modules V. Since each $H^k(\mathfrak{h})$ is a nilpotent F-module, each $H^0(F, H^k(\mathfrak{h}))$, and thus each $H^1(F, H^k(\mathfrak{h}))$, is non-zero. The result now follows by induction on the dimension of $\mathfrak{g}$. □

The following fact will be needed in the proof of Theorem B.

PROPOSITION 2. *Suppose that $\mathfrak{g}$ is a nilpotent Lie algebra over a field F of* *characteristic zero. If V is a non-zero, finite dimensional nilpotent* $\mathfrak{g}$ *module, then* $H^0(\mathfrak{g}, V) \neq 0$.

Proof. The result follows as, each finite dimensional $\mathfrak{g}$ module V has a filtration

$$V = V^1 \supseteq V^2 \supseteq \cdots \supseteq V^m \supseteq V^{m+1} = 0$$

such that $\mathfrak{g}V^i \subseteq V^{i+1}$. □

We now prove Theorem B. Since the weights on $H_1(L)$ are 0 and -1, the nilpotent Lie algebra $\mathfrak{g}$ associated to $\pi_1(L)$ has weights $\leqslant 0$. It can thus be written as an extension

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{z} \to 0,$$

where

$$\mathfrak{z} = \mathrm{Gr}_0^W \mathfrak{g} \quad \text{and} \quad \mathfrak{h} = W_{-1}\mathfrak{g}.$$

Set $z = \dim \mathfrak{z}$ and $h = \dim \mathfrak{h}$. Then $z + h = 2n - 1$, where n is the complex dimension of the singularity of which L is the link.

PROPOSITION 3. *If* $i < n$, *then the mixed Hodge structure on* $H^i(\mathfrak{h})$ *is pure of weight* i.

First observe, since $H^{2n-1}(L)$ has weight $2n$, that $\mathfrak{h}$ cannot be zero. Consider the spectral sequence

$$E_2^{s,t} = H^s(\mathfrak{z}, H^t(\mathfrak{h})) \Rightarrow H^{s+t}(\mathfrak{g})$$

associated to the extension

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{z} \to 0.$$

This is easily seen to be a spectral sequence of mixed Hodge structures as it can be constructed in the category of mixed Hodge structures, an abelian category.

Since $\mathfrak{h}$ has negative weights, the weights on $H^t(\mathfrak{h})$ are $\geqslant t$. Since $\mathfrak{z}$ is of weight 0, the weights on $E_2^{s,t}$ are $\geqslant t$.

Since $E_2^{s,0}$ is pure of weight 0, and since the differential preserves the splittings of the weight filtration, it follows that $d_2\colon E_2^{0,1} \to E_2^{2,0}$ is zero. Consequently,

$$\mathrm{Gr}_l^W H^1(L) \approx H^0(\mathfrak{z}, \mathrm{Gr}_l^W H^1(\mathfrak{h}))$$

for each $l > 0$. Since each $\mathrm{Gr}_l^W H^1(\mathfrak{h})$ is a nilpotent $\mathfrak{z}$ module, and $H^1(L)$ has weights $\leqslant 1$, it follows from Proposition 2 that $H^1(\mathfrak{h})$ must be pure of weight one.

This now implies that each $E_2^{s,1}$ is pure of weight 1. As above, it follows that the differentials

$$d_2\colon E_2^{0,2} \to E_2^{2,1} \quad \text{and} \quad d_3\colon E_3^{0,2} \to E_3^{3,0}$$

must both be zero. Provided that $2 < n$, it follows, by an argument similar to the one above and the fact that $H^2(L)$ has weights $\leqslant 2$, that $H^2(\mathfrak{h})$ is pure of weight 2.

One continues similarly to prove that $H^i(\mathfrak{h})$ is pure of weight i when $i < n$. □

To complete the argument, note that the mixed Hodge structure on the top cohomology group $H^h(\mathfrak{h})$ of $\mathfrak{h}$ is pure of weight

$$\sum_{l \geqslant 1} l \dim \mathrm{Gr}_{-l}^W \mathfrak{h} = \sum_{l \geqslant 0} l \dim \mathrm{Gr}_{-l}^W \mathfrak{g} = 2n.$$

Since the cohomology of every finite dimensional nilpotent Lie algebra satisfies Poincaré duality, and since the cup product

$$H^i(\mathfrak{h}) \otimes H^{h-i}(\mathfrak{h}) \to H^h(\mathfrak{h})$$

is a morphism of mixed Hodge structures, Proposition 3 implies that $H^i(\mathfrak{h})$ is pure of weight

$$2n - (h - i) = z + i + 1$$

when $i > h - n$. This and Proposition 3 imply that $H^i(\mathfrak{h})$ vanishes when $h - n < i < n$. That is, when $n - z \leqslant i \leqslant n - 1$. But, by the Lemma, $\mathfrak{h}$ cannot have any vanishing cohomology. It follows that $z = 0$, which implies that $\mathfrak{g} = \mathfrak{h}$. But we have already proved that $H^1(\mathfrak{h})$ is pure of weight 1. This completes the proof of the Theorem B, and with it, Theorem A. □

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