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On Tunnell's formula for characters of $GL(2)$

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1. Introduction

For a local field F of characteristic 0, let π an irreducible admissible representation of $GL_2(F)$ of infinite dimension, and ω_π its central character, χ_π its character respectively. For each quadratic extension L of F, we fix an embedding of $L^\times$ into $GL_2(F)$, and consider $L^\times$ as the set of F-rational points of a Cartan subgroup of $GL_2(F)$. In [T], Tunnell gave an expression for the restriction of χ_π to $L^\times$ as a sum of quasicharacters of $L^\times$, in which the coefficients are written in terms of ε-factors of the base change lifting of π to L. His proof is based on the case by case computation of ε-factors and characters and is not transparent. Furthermore the case of residual characteristic 2 was only partially treated. The purpose of this paper is to give a natural proof of the formula of Tunnell including the case of residual characteristic 2.

We state the theorem more exactly. Let ψ be a nontrivial additive character of F, and set $\psi_L = \psi \circ tr_{L/F}$, where $tr_{L/F}$ is the trace from L to F. Let Π be the base change lifting of π to L (cf. [L]). Then the central character ω_Π of Π is given by $\omega_\pi \circ n_{L/F}$, where $n_{L/F}$ is the norm from L to F. For each quasicharacter λ of $L^\times$, let $\varepsilon(\Pi \otimes \lambda, \psi_L)$ be the ε-factor of the representation $\Pi \otimes \lambda$ of $GL_2(L)$ with respect to ψ_L. For λ whose restriction to $F^\times$ is ω_π, it is shown in [T] that $\varepsilon(\Pi \otimes \lambda, \psi_L)$ is independent of ψ and is equal to 1 or -1. In this notation, we can state the result as follows.

THEOREM. *Let π be an infinite dimensional irreducible admissible representation of $GL_2(F)$, and χ_π its character. For a quadratic extension L of F, let Π be the base change lifting of π to L. Then one has*

$$\chi_\pi|_{L^\times} = \sum_\lambda \frac{1 + \varepsilon(\Pi \otimes \lambda, \psi_L)\omega_\pi(-1)}{2} \lambda,$$

where λ runs through all quasicharacters of $L^\times$ whose restriction to $F^\times$ coincide with ω_π.

Here the functions on both sides are considered as continuous functions on

$L^{\times}_{\text{reg}} = L^{\times} - F^{\times}$, and the sum on the right-hand side is computed by partial summation with respect to the conductors of quasicharacters.

The idea of the proof is simple. Let σ be the generator of the Galois group $\text{Gal}(L/F)$. For $g \in GL_2(L)$, we set

$$N(g) = g^{\sigma}g.$$

Here ${}^{\sigma}g$ denotes the componentwise action of σ on g. For $a \in L^{\times}$, to distinguish components of $GL_2(L)$ and elements of $GL_2(F)$ under the embedding fixed above, we write $\tilde{a}$ when considered as an element of $GL_2(F)$. Then we have

$$N\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w\right) = \begin{pmatrix} a & 0 \\ 0 & {}^{\sigma}a \end{pmatrix} \sim \tilde{a} \in GL_2(F), \tag{1.1}$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\sim$ denotes the conjugacy in $GL_2(L)$. The idea is to reduce the calculation of $\chi_{\pi}|_{L^{\times}}$ to that in $GL_2(L)$ by means of (1.1) and the theory of base change lifting.

2. Proof of the theorem

Besides the notations introduced above, we use the following ones. For a local field L, let o_L be the maximal compact ring of L, and p_L the maximal ideal of o_L. Let v_L be the additive valuation of L such that $v_L(\varpi_L) = 1$ for a prime element ϖ_L of L. Let $|o_L/p_L| = q_L$, and let $|\ |$ be the absolute value of L such that $|\varpi_L| = q_L^{-1}$. For a quasicharacter λ of $L^{\times}$, we denote by $f(\lambda)$ the exponent of the conductor of λ. For quasicharacters λ_1, λ_2 of $L^{\times}$, we define $\lambda_1 \sim \lambda_2$ if $\lambda_1\lambda_2^{-1}$ is unramified. Let L and F be as in Section 1, and let $n(\psi_L)$ the largest integer which satisfies $\psi_L(p_L^{-n(\psi_L)}) = \{1\}$. For a positive integer n, set

$$\Gamma_n = \begin{pmatrix} 1 + p_L^n & p_L^n \\ p_L^n & 1 + p_L^n \end{pmatrix} \cap GL_2(o_L).$$

For an irreducible admissible representation Π of $GL_2(L)$, let $p_L^{f(\Pi)}$ the conductor of Π. For Π, let $L(s, \Pi)$ and $\varepsilon(s, \Pi, \psi_L)$ be as in [J-L], and set $\varepsilon(\Pi, \psi_L) = \varepsilon(1/2, \Pi, \psi_L)$.

The proof of the theorem in the cases of principal series and special representations is easy and treated completely in [T], and in the following we assume π is supercuspidal. Let $\mathscr{W}(\Pi)$ be the Whittaker model of Π with respect to the additive character ψ_L. Then $\mathscr{W}(\Pi)$ consists of functions W on $GL_2(L)$

which satisfy

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi_L(x)W(g), \quad x \in L,$$

and the right translation ρ gives Π. Let ${}^\sigma\Pi$ be the representation defined by ${}^\sigma\Pi(g) = \Pi({}^\sigma g)$ for $g \in GL_2(L)$. We set

$${}^\sigma\mathscr{W}(\Pi) = \{W({}^\sigma g) \mid W \in \mathscr{W}(\Pi)\},$$

and $(I_\sigma W)(g) = W({}^{\sigma^{-1}}g) = W({}^\sigma g)$. Then I_σ gives an isomorphism from ${}^\sigma\mathscr{W}(\Pi)$ to $\mathscr{W}(\Pi)$, and for $W' \in {}^\sigma\mathscr{W}(\Pi)$, we see $W'\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi_L(x)W'(g)$ and

$$\rho(g)W' = I_\sigma^{-1}\Pi({}^\sigma g)I_\sigma W'.$$

By the uniqueness of Whittaker models, we have $\mathscr{W}({}^\sigma\Pi) = {}^\sigma\mathscr{W}(\Pi)$. Since Π is the base change lifting of π to L, Π is equivalent to ${}^\sigma\Pi$ and $\mathscr{W}(\Pi) = \mathscr{W}({}^\sigma\Pi) = {}^\sigma\mathscr{W}(\Pi)$. We see also that I_σ gives an intertwining operator of Π to ${}^\sigma\Pi$ which satisfies $I_\sigma^2 = 1$, and fixes the linear form $L(W) = W(1)$ on $\mathscr{W}(\Pi)$. Setting $\Pi((g, \sigma)) = \Pi(g)I_\sigma$, we can extend Π to the semidirect product of $GL_2(L)$ and Gal (L/F) by the action of $\mathrm{Gal}(L/F)$ on $GL_2(L)$. Let $\chi_{\Pi,\sigma}$ be the twisted character of this representation, namely, the distribution on $C_c^\infty(GL_2(L))$ defined by

$$\chi_{\Pi,\sigma}(\varphi) = \mathrm{trace}(\Pi(\varphi)I_\sigma),$$

for $\varphi \in C_c^\infty(GL_2(L))$. Then the twisted character is given by a function which is locally integrable and locally constant on the set of σ-regular elements of $GL_2(L)$ and satisfies

$$\chi_{\Pi,\sigma}(g) = \chi_\pi(Ng),$$

for σ-regular elements (cf. [L], [A-C]). By (1.1), to obtain $\chi_\pi|_{L^\times}$, it is enough to compute $\chi_{\Pi,\sigma}\left(\begin{pmatrix}-a & 0\\ 0 & 1\end{pmatrix} w\right)$. We carry out this calculation in the Kirillov model $\mathscr{K}(\Pi)$ of Π, that is, on $\mathscr{K}(\Pi) = \left\{W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) \middle| W \in \mathscr{W}(\Pi)\right\}$. An element f of $\mathscr{K}(\Pi)$ is a locally constant function on $L^\times$ which satisfies

$$f\left(\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} x\right) = \psi_L(bx)f(ax),$$

and the action of I_σ can be written as $(I_\sigma f)(x) = f({}^{\sigma^{-1}}x) = f({}^{\sigma}x)$.

First we treat the case where Π is supercuspidal. Then $\mathscr{K}(\Pi)$ coincides with the space $\mathscr{S}(L^\times)$ of Schwartz-Bruhat functions on $L^\times$ and a basis of this space is given by the set of the following functions:

$$\xi_\lambda^{(n)} = \begin{cases} \lambda(x) & \text{if } v_L(x) = -n, \\ 0 & \text{otherwise.} \end{cases}$$

Here n is extended over all integers and λ is extended over a complete system of representatives of all quasicharacters of $L^\times$ modulo $\sim$ defined above. Later in the proof we choose representatives suitably. On this basis, the action of w is described by means of ε-factor as follows.

LEMMA 2.1. *Let Π be a supercuspidal representation of $GL_2(L)$ and $\xi_\lambda^{(n)}$ as above. Then one has*

$$\Pi(w)\xi_\lambda^{(n)} = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\xi^{(m)}_{\omega_\Pi\lambda^{-1}},$$

where $m = f(\Pi \otimes \lambda^{-1}) + 2n(\psi_L) - n$.

This is the formula (9) of [Y] and can be deduced from local functional equations of $GL_2(L)$.

We determine the subspace $\mathscr{K}(\Pi)^n$ of $\mathscr{K}(\Pi)$ consisting of elements invariant under Γ_n. Since an element of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in o_L^\times$, normalizes Γ_n, $\mathscr{K}(\Pi)^n$ has a basis consisting of elements of the form

$$v = \sum_m a_m \xi_\lambda^{(m)},$$

with λ such that $f(\lambda) \leqslant n$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(L)$ with $d \neq 0$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (ad-bc)/d & b \\ 0 & d \end{pmatrix} w^{-1} \begin{pmatrix} 1 & -c/d \\ 0 & 1 \end{pmatrix} w.$$

Hence for n such that $f(\omega_\Pi) \leqslant n$, v is invariant under Γ_n if and only if v and $\Pi(w)v$ are invariant under $\begin{pmatrix} 1 & p_L^n \\ 0 & 1 \end{pmatrix}$. This condition is equivalent to that the supports of v and $\Pi(w)v$ are contained in $p_L^{-n(\psi_L)-n}$, and that $a_m = 0$ unless $f(\Pi \otimes \lambda^{-1}) + n(\psi_L) - n \leqslant m \leqslant n(\psi_L) + n$ by the above lemma. Let B_n be the set of $\xi_\lambda^{(m)}$ such that $f(\lambda) \leqslant n$ and $f(\Pi \otimes \lambda^{-1}) + n(\psi_L) - n \leqslant m \leqslant n(\psi_L) + n$. Then B_n

gives a basis of $\mathscr{K}(\Pi)^n$ for n sufficiently large and the union $\bigcup_n B_n$ gives a basis of $\mathscr{K}(\Pi)$. Let P_n be the projection of $\mathscr{K}(\Pi)$ onto $\mathscr{K}(\Pi)^n$ defined by

$$\int_{\Gamma_n} \Pi(g)dg \Big/ \int_{\Gamma_n} dg,$$

where dg is a Haar measure on $GL_2(L)$. Then we can calculate the value of $\chi_{\Pi,\sigma}(\tilde{a})$ as trace $\left(\Pi\left(\begin{pmatrix}-a & 0\\ 0 & 1\end{pmatrix} w\right) I_\sigma P_n\right)$ with respect to this basis for a sufficiently large n (cf. [T] Lemma 2.2).

By the above lemma and the relation $\varepsilon(\Pi \otimes {}^\sigma\lambda^{-1}, \psi_L) = \varepsilon({}^\sigma\Pi \otimes \lambda^{-1}, \psi_L) = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)$, which follows from ${}^\sigma\Pi \sim \Pi$, we have

LEMMA 2.2. *The notation being as above, for $a \in L^\times_{\mathrm{reg}}$ one has*

$$\Pi\left(\begin{pmatrix}-a & 0\\ 0 & 1\end{pmatrix} w\right) I_\sigma \xi_\lambda^{(n)} = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\omega_\Pi{}^\sigma\lambda^{-1}(-a)\xi^{(m+v_L(a))}_{\omega_\Pi{}^\sigma\lambda^{-1}}.$$

Here $m = f(\Pi \otimes \lambda^{-1}) + 2n(\psi_L) - n$.

Hence if $\xi_\lambda^{(n)}$ contributes to the $\chi_{\Pi,\sigma}(\tilde{a})$, then it holds

(a) $n = n(\psi_L) + \frac{1}{2}(f(\Pi \otimes \lambda^{-1}) + v_L(a))$,
(b) $\lambda|_{o_L^\times} = \omega_\Pi{}^\sigma\lambda^{-1}|_{o_L^\times}$.

First we assume L/F is unramified. Then (b) implies that $\lambda|_{o_F^\times} = \omega_\pi|_{o_F^\times}$. As a representative of the class of λ, we take λ such that $\lambda(\varpi_F) = \omega_\pi(\varpi_F)$. Then we have $\omega_\Pi{}^\sigma\lambda^{-1} = \lambda$. Let λ' be the quasicharacter of $L^\times$ defined by the condition $\lambda|_{o_L^\times} = \lambda'|_{o_L^\times}$ and $\lambda(\varpi_F) = -\lambda'(\varpi_F)$. Then we see the contribution to $\chi_{\Pi,\sigma}$ of $\xi_\lambda^{(n)}$ for the above λ is equal to

$$\begin{cases}\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-a) & \text{if } v_L(a) \equiv f(\Pi \otimes \lambda^{-1}) \pmod 2,\\ 0 & \text{otherwise,}\end{cases}$$

and noticing $\varepsilon(\Pi \otimes \lambda'^{-1}, \psi_L) = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)(-1)^{f(\Pi\otimes\lambda^{-1})}$, we see this is equal to

$$\frac{1}{2}(\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-a) + \varepsilon(\Pi \otimes \lambda'^{-1}, \psi_L)\lambda'(-a)).$$

Let $\chi_{L/F}$ be the unramified character of $F^\times$ corresponding to the quadratic extension L/F. Then we have

$$\chi_\pi(\tilde{a}) = \chi_{\Pi,\sigma}\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w\right) = \sum_{\lambda|_{F^\times} = \omega_\pi} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a)$$

$$+ \sum_{\lambda|_{F^\times} = \omega_\pi \chi_{L/F}} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a)$$

$$= \sum_{\lambda|_{F^\times} = \omega_\pi} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\omega_\pi(-1)}{2} \lambda(a)$$

$$+ \sum_{\lambda_{F^\times} = \omega_\pi \chi_{L/F}} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\omega_\pi(-1)}{2} \lambda(a),$$

since $\lambda(-1) = \omega_\pi(-1)$ and $\Sigma_{\lambda|_{F^\times} = \omega_\pi} \lambda = 0$, $\Sigma_{\lambda|_{F^\times} = \omega_\pi \chi_{L/F}} \lambda = 0$. We note $\chi_\pi(r\tilde{a}) = \omega_\pi(r)\chi_\pi(\tilde{a})$ for $r \in F^\times$. Therefore the second factor of the above sum vanishes and our assertion has been proved in this case.

Now assume L/F is ramified. Let ϖ_F be a prime element of F which is contained in the norm of L. Let $\chi_{L/F}$ the quadratic character of $F^\times$ corresponding to the extension L/F as above. In this case the condition (b) implies only that $\lambda|_{n_{L/F}(o_L^\times)} = \omega_\pi|_{n_{L/F}(o_L^\times)}$, hence that $\lambda|_{o_F^\times} = \omega_\pi|_{o_F^\times}$ or $\omega_\pi \chi_{L/F}|_{o_F^\times}$. In the class of λ satisfying (b), there exists exactly two characters satisfying $\lambda_i(\varpi_F) = \omega_\pi(\varpi_F)$, $i = 1, 2$, and they satisfy $\omega_\Pi{}^\sigma \lambda_i^{-1} = \lambda_i$ and $\lambda_1(\varpi_L) = -\lambda_2(\varpi_L)$. In the same way as in the unramified case, we obtain

$$\chi_\pi(a) = \sum_{\lambda|_{F^\times} = \omega_\pi} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-1)}{2} \lambda(a)$$

$$+ \sum_{\lambda|_{F^\times} = \omega_\pi \chi_{L/F}} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-1)}{2} \lambda(a).$$

Also as in the unramified case, the second sum vanishes and for λ in the first sum, one has $\lambda(-1) = \omega_\pi(-1)$. This completes the proof of the theorem in the case where Π is supercuspidal.

Next assume Π is a principal series representation $\pi(\mu_1, \mu_2)$. Since Π is a base change lifting and π is neither principal series nor special, we have $\mu_2 = {}^\sigma\mu_1$. In this case, we have to take care of the difference between $\mathscr{K}(\Pi)$ and $\mathscr{S}(L^\times)$. As a basis of $\mathscr{K}(\Pi)$, we employ the following: $\{\xi_\lambda^{(n)}\} \cup \{\eta_1^{(n)}, \eta_2^{(n)}\}$. Here $n \in \mathbf{Z}$ and λ is extended over all classes of quasicharacters with respect to $\sim$ which do not contain μ_1 nor μ_2. The element $\eta_i^{(n)}$, for $i = 1, 2$, is defined by

$$\eta_i^{(n)}(x) = \begin{cases} \mu_i(x)|x|^{1/2} & \text{if } v_L(x) \geqslant -n \\ 0 & \text{otherwise.} \end{cases}$$

We note $\omega_\Pi=\omega_\pi\circ n_{L/F}=\mu_1\mu_2=\mu_1\circ n_{L/F}=\mu_2\circ n_{L/F}$. Hence we have $\mu_i=\omega_\pi$ or $\omega_\pi\chi_{L/F}$ on $F^\times$. The action of w on this basis can be described by

LEMMA 2.3. *The notation being as above, let* $\mu=\mu_1\mu_2^{-1}$. *For* $\lambda\not\sim\mu_1,\mu_2$, *one has*

$$\Pi(w)\xi_\lambda^{(n)}=\varepsilon(\Pi\otimes\lambda^{-1},\psi_L)\xi_{\omega_\Pi\lambda^{-1}}^{(f(\Pi\otimes\lambda^{-1})+2n(\psi_L)-n)}.$$

If μ *is ramified, one has*

$$\Pi(w)\eta_1^{(n)}=\varepsilon(\Pi\otimes\mu_1^{-1},\psi_L)|\varpi_L|^{f(\Pi\otimes\mu_1^{-1})/2+n(\psi_L)-n}\eta_2^{(f(\Pi\otimes\mu_1^{-1})+2n(\psi_L)-n)},$$
$$\Pi(w)\eta_2^{(n)}=\varepsilon(\Pi\otimes\mu_2^{-1},\psi_L)|\varpi_L|^{f(\Pi\otimes\mu_2^{-1})/2+n(\psi_L)-n}\eta_1^{(f(\Pi\otimes\mu_2^{-1})+2n(\psi_L)-n)},$$

and if μ *is unramified, one has*

$$\Pi(w)\eta_1^{(n)}=\tfrac{1}{2}|\varpi_L|^{n(\psi_L)-n}\{\varepsilon(\Pi\otimes\mu_1^{-1},\psi_L)(\eta_2^{(2n(\psi_L)-n)}+|\varpi_L|\eta_2^{(2n(\psi_L)-n+1)})$$
$$+(-1)^n(\eta_1^{(2n(\psi_L)-n)}-|\varpi_L|\eta_1^{(2n(\psi_L)-n+1)})\},$$
$$\Pi(w)\eta_2^{(n)}=\tfrac{1}{2}|\varpi_L|^{n(\psi_L)-n}\{\varepsilon(\Pi\otimes\mu_2^{-1},\psi_L)(\eta_1^{(2n(\psi_L)-n)}+|\varpi_L|\eta_1^{(2n(\psi_L)-n+1)})$$
$$+(-1)^n(\eta_2^{(2n(\psi_L)-n)}-|\varpi_L|\eta_2^{(2n(\psi_L)-n+1)})\}.$$

Proof. The case of $\xi_\lambda^{(n)}$ can be proved in the same way as in the case of cuspidal representations, since the support of $\Pi(w)\xi_\lambda^{(n)}$ is also compact. To treat the case of $\eta_i^{(n)}$, we recall the construction of Kirillov models in the case of principal series representations ([G]). Let $\mathscr{F}_\mu$ be the space of locally constant functions ϕ on L such that $\phi(x)\mu(x)|x|$ is constant for large $|x|$. For $\phi\in\mathscr{F}_\mu$, we set

$$\hat{\phi}(x)=\sum_{m\in\mathbf{Z}}\int_{v_L(y)=m}\phi(y)\bar{\psi}_L(xy)\,dy,$$

where dy is the Haar measure of L such that $\int_{o_L}dy=|\varpi_L|^{n(\psi_L)/2}$. Then the map $\phi\mapsto\mu_2(x)|x|^{1/2}\hat{\phi}$ gives an isomorphism from $\mathscr{F}_\mu$ to $\mathscr{K}(\Pi)$. We denote this isomorphism by $\mathscr{F}$. The action of w in $\mathscr{F}_\mu$ is given by $\Pi(w)\phi(x)=\mu(-x^{-1})|x|^{-1}\phi(-x^{-1})$. Let

$$\phi_1^{(n)}(x)=\begin{cases}1 & v_L(x)\geqslant n,\\ 0 & \text{otherwise,}\end{cases}$$

$$\phi_\mu^{(n)}(x)=\begin{cases}\mu^{-1}(x)|x|^{-1} & v_L(x)\leqslant -n,\\ 0 & \text{otherwise.}\end{cases}$$

Then these functions belong to $\mathscr{F}_\mu$ and satisfy $\Pi(w)\phi_1^{(n)}=\mu(-1)\phi_\mu^{(n)}$.

First assume μ is ramified. Then by some calculations, we obtain

$$\mathscr{F}(\phi_1^{(n)}) = |\varpi_L|^{n(\psi_L)/2+n}\eta_2^{(n+n(\psi_L))},$$

$$\mathscr{F}(\phi_\mu^{(n)}) = a\eta_1^{(f(\mu)+n(\psi_L)-n)}, \quad a = \int_{v_L(y)=-n(\psi_L)-f(\mu)} \mu^{-1}(y)\bar{\psi}_L(y)d^\times y,$$

where $d^\times y = |y|^{-1}dy$. Hence we have

$$\Pi(w)\eta_2^{(n)} = a\mu(-1)|\varpi_L|^{n(\psi_L)/2-n}\eta_1^{(f(\mu)+2n(\psi_L)-n)}.$$

To determine the relation between a and ε-factors, we use local functional equations. By [J-L], we have

$$\int_{L^\times} (\Pi(w)\xi)(x)\omega_\Pi^{-1}(x)\chi^{-1}(x)|x|^{1/2-s}d^\times x$$
$$= \varepsilon(s, \Pi\otimes\chi, \psi_L)\frac{L(1-s, \widetilde{\Pi\otimes\chi})}{L(s, \Pi\otimes\chi)}\int_{L^\times} \xi(x)\chi(x)|x|^{s-1/2}d^\times x.$$

for $\xi\in\mathscr{K}(\Pi)$ and a quasicharacter χ of $L^\times$. Here dx is the Haar measure of $L^\times$ such that $\int_{O_L^\times} d^\times x = 1$. We take $\xi = \eta_2^{(0)}$, $\chi = \mu_2^{-1}$. Then we see the second integral is equal to $(1-|\varpi_L|^s)^{-1} = L(s, \Pi\otimes\mu_2^{-1})$ and the first one is equal to

$$a\mu(-1)|\varpi_L|^{n(\psi_L)/2-(f(\mu)+2n(\psi_L))(1-s)}(1-|\varpi_L|^{1-s})^{-1}$$
$$= a\mu(-1)|\varpi_L|^{n(\psi_L)/2-(f(\mu)+2n(\psi_L))(1-s)}L(1-s, \widetilde{\Pi\otimes\mu_2^{-1}}).$$

Since $f(\mu) = f(\Pi\otimes\mu_i^{-1})$ for $i = 1, 2$, we obtain

$$\varepsilon(\Pi\otimes\mu_2^{-1}, \psi_L) = a\mu(-1)|\varpi_L|^{-(f(\Pi\otimes\mu_2^{-1})+n(\psi_L))/2}.$$

This shows our result on $\eta_2^{(n)}$. Interchanging μ_1 and μ_2, we obtain the equation for $\eta_1^{(n)}$.

Next assume μ is unramified and $\mu = |\ |^{s_0}$. If L/F is unramified, then $\mu(\varpi_F)=1$, and $\mu_1=\mu_2={}^\sigma\mu_1$. Hence Π is a lifting of a principal series representation of $GL_2(F)$, which is contrary to our assumption. Therefore L/F is ramified, and by the relation $\mu(\varpi_L)=|\varpi_L|^{s_0}$, we see $|\varpi_L|^{2s_0} = 1$. If $|\varpi_L|^{s_0} = 1$, then again Π is a lifting of a principal series representation. Hence we may assume $\mu(\varpi_L) = -1$. For $\phi_1^{(n)}$, in the same way as above, we have

$$\mathscr{F}(\phi_1^{(n)}) = |\varpi_L|^{n(\psi_L)/2+n}\eta_2^{(n+n(\psi_L))}.$$

By (131) of [G], we have

$$\mathscr{F}(\phi_\mu^{(n)}) = |\varpi_L|^{n(\psi_L)/2}|x|^{1/2}\mu_2(x)(F_{so}^{(n)}(x) - |\varpi_L| F_{so}^{(n)}(\varpi_L x)),$$

where

$$\begin{aligned} F_{so}^{(n)}(x) &= \sum_{-n(\psi_L)-v_L(x) \leqslant m \leqslant -n} |\varpi_L|^{-mso} \\ &= \tfrac{1}{2}((-1)^{n(\psi_L)}\mu(x) + (-1)^n)\phi_1^{(n-n(\psi_L))}(x). \end{aligned}$$

Since $\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) = \varepsilon(1, \psi_L)\varepsilon(\mu, \psi_L) = \mu(\varpi_L^{n(\psi_L)}) = (-1)^{n(\psi_L)}$, we have

$$|x|^{1/2}\mu_2(x) F_{so}^{(n)}(x) = \tfrac{1}{2}(\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\eta_1^{(n(\psi_L)-n)} + (-1)^n \eta_2^{(n(\psi_L)-n)}).$$

From these formulas, we obtain our formula for $\Pi(w)\eta_2^{(n)}$. The case of $\eta_1^{(n)}$ can be proved in the same way. This completes the proof of Lemma 2.3.

By this lemma, we can proceed in the same way as in the case where Π is supercuspidal. We give a proof for the case where μ is unramified. The contribution from $\xi_\lambda^{(n)}$ for $\lambda \not\sim \mu_1, \mu_2$ is equal to

$$\sum_{\substack{\lambda|_{F^\times} = \omega_\pi \text{ or } \omega_\pi \chi_{L/F} \\ \lambda \not\sim \mu_1, \mu_2}} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a). \tag{2.1}$$

For $\eta_1^{(n)}$, we have

$$\begin{aligned} &\Pi\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w\right) I_\sigma \eta_1^{(n)} \\ &= \frac{|\varpi_L|^{n(\psi_L)-n}}{2} \{\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_1(-a)|a|^{1/2} \\ &\quad \times (\eta_1^{(2n(\psi_L)-n+v_L(a))} + |\varpi_L|\eta_1^{(2n(\psi_L)-n+1+v_L(a))}) \\ &\quad + (-1)^n \mu_2(-a)|a|^{1/2}(\eta_2^{(2n(\psi_L)-n+v_L(a))} - |\varpi_L|\eta_2^{(2n(\psi_L)-n+1+v_L(a))})\}, \end{aligned}$$

Hence if $\eta_i^{(n)}$ contributes to the trace, n is equal to $n(\psi_L) + v_L(a)/2$ or $n(\psi_L) + (v_L(a)+1)/2$, and the contribution is equal to

$$\begin{cases} \tfrac{1}{2}\varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)\mu_1(-1)\mu_1(a) & \text{if } v_L(a) \text{ is even,} \\ \tfrac{1}{2}\varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)\mu_1(-1)|\varpi_L|^{1/2}\mu_1(a) & \text{if } v_L(a) \text{ is odd.} \end{cases}$$

In the same way the contribution from the elements of the form $\eta_2^{(n)}$ is equal to

$$\begin{cases} \frac{1}{2}\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)\mu_2(a) & \text{if } v_L(a) \text{ is even,} \\ \frac{1}{2}\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)|\varpi_L|^{1/2}\mu_2(a) & \text{if } v_L(a) \text{ is odd.} \end{cases}$$

The sum of these contributions is equal to

$$\tfrac{1}{2}(\varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)\mu_1(-1)\mu_1(a) + \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)\mu_2(a)). \tag{2.2}$$

Here we used the fact that when $v_L(a)$ is odd, $\mu_1(a) + \mu_2(a) = 0$ and that $\varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L) = \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)$. The sum of (2.1) and (2.2) can be transformed into the form in the theorem as in the supercuspidal case. The case Π special does not occur, since π is supercuspidal. This completes the whole proof of the theorem.

As a corollary of the proof, we see

COROLLARY 2.4. *Let π be a supercuspidal representation of $GL_2(F)$ with the central character ω_π, and let Π be the base change lifting of π to $GL_2(L)$. Then for quasicharacters λ of $L^\times$ which satisfy $\lambda|_{F^\times} = \omega_\pi \chi_{L/F}$, $\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-1)$ is independent of λ.*

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