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On metric theorems in the theory of uniform distribution

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1. Introduction

As is apparent from the work of Cigler [2] and [3], Helmberg and Paalman-De Miranda [4], Hlawka [5], Kemperman [6], Niederreiter [8] and many others, the methods and results of probability theory play an important role in the theory of uniform distribution. For example, it follows from the strong law of large numbers that almost all sequences in a second countable compact Hausdorff space are uniformly distributed with respect to a given probability measure on the space (see p. 182, [7]). In this note, we shall obtain a general probabilistic result corresponding to another metric theorem "almost no sequence is well distributed" (see [8]), namely, for a sequence $\{f_n\}_{n=1}^{\infty}$ of independent, identically distributed, real-valued random variables on a probability space $(\Omega, \mathscr{B}, P)$, if the common distribution of f_n's is not a point measure and if $\int_\Omega |f_1| \, \mathrm{d}P < +\infty$, then the set U of all the ω's such that

$$\operatorname*{Lim}_{n \to \infty} \frac{f_{k+1}(\omega) + \cdots + f_{k+n}(\omega)}{n} = \int_\Omega f_1 \, \mathrm{d}P$$

uniformly in $k = 0, 1, 2, \ldots$ is a null set. Moreover, for the case that Ω is the sample space of an infinite coin-tossing and f_n is the random variable corresponding to the output of the n-th toss, we characterize the set U as the image of all well distributed sequences mod 1 under a natural map. Note that by the metric theorem on well distributed sequences, we know that well distributed sequences are rather scarce. However, in the next section we shall show that if there exists a well distributed sequence in a space with at least two points, then there are at least continuum many well distributed sequences in that space. The same result also holds for uniformly distributed sequences.

2. The Results

Let μ be a regular Borel probability measure on a compact Hausdorff space X. A sequence $(x_n)_{n=1}^{\infty}$ in X is said to be μ-well distributed in X, if for any given continuous function f on X,

$$\operatorname*{Lim}_{N\to\infty} \frac{f(x_{k+1}) + \cdots + f(x_{k+N})}{N} = \int_X f\,d\mu$$

uniformly in $k = 0, 1, 2, \ldots$. Now let $\mathscr{W}_X$ be the set of all μ-well distributed sequences in X. Let X^∞ be the set of all sequences in X and μ_∞ be the completion of the product measure on X^∞ induced by μ. It was shown by Niederreiter [8] that if μ is not a point measure, then $\mu_\infty(\mathscr{W}_X) = 0$. Note that the result for the case that X is second countable can be found in [4]. In the following proposition we shall show that if X is not a singleton and if there is a μ-well distributed sequence in X, then there are at least continuum many μ-well distributed sequences in X. In fact, the set $\mathscr{W}_X$ of μ-well distributed sequences and the set $\mathscr{S}_X$ of μ-u.d. sequences have the same cardinality as X^∞.

PROPOSITION 1. *If $\mathscr{W}_X \neq \phi$, then the cardinality of $\mathscr{W}_X$ is the same as that of X^∞.*

Proof. Let $\omega_0 = (a_n)_{n=1}^{\infty}$ be a μ-well distributed sequence in X. For any sequence $\omega = (x_n)_{n=1}^{\infty}$ in X^∞, define a new sequence $F(\omega) = (y_n)_{n=1}^{\infty}$ by letting

$$y_n = \begin{cases} a_n & \text{if } n \neq m^2 \quad \text{for any } m \geqslant 1, \\ x_m & \text{if } n = m^2 \quad \text{for some } m \geqslant 1. \end{cases}$$

Since a good distribution is not afflicted by change on square indices (see p. 140 or p. 202, [7]), the sequence $F(\omega)$ is still μ-well distributed. Therefore, the cardinalities of $\mathscr{W}_X$ and X^∞ are the same. □

REMARKS. (1) It is shown in [1] that there always exists a μ-well distributed sequence in X if X is second countable. Hence, in this case, we have continuum many μ-well distributed sequences and continuum many μ-u.d. sequences in X if X is not a singleton.

(2) For a regular probability measure μ on a compact Hausdorff space X, if there is a μ-u.d. sequence in X, then the set $\mathscr{S}_X$ of all μ-u.d. sequences has the same cardinality as X^∞, no matter whether μ-well distributed sequences exist at all in X.

(3) For any nonatomic Borel probability measure μ on a second countable compact Hausdorff space X, it is shown in [9] that there is a Borel measurable, measure preserving mapping F from the unit interval I together with the Lebesgue measure λ on I to X such that the sets of discontinuity of F and F^{-1} are null sets. Thus for any sequence $\omega = (x_n)_{n=1}^{\infty}$ in I, ω is u.d. (well distributed) mod 1 if and only if $(F(x_n))_{n=1}^{\infty}$ is μ-u.d. (μ-well distributed) in X. Thus, u.d.

sequences and well distributed sequences in X behave in the same way as those in I.

Now, let $X = \{0, 1\}$, $\mu(\{0\}) = \mu(\{1\}) = 1/2$. The space X^∞ could be interpreted as the sample space of an infinite coin-tossing. For each $n \geqslant 1$, let π_n be the projection such that for any $\omega = (x_n)_{n=1}^\infty$ in X^∞, $\pi_n(\omega) = x_n$. It is clear that $\{\pi_n\}_{n=1}^\infty$ is a sequence of independent, identically distributed random variables. It is easily verified that ω is μ-well distributed in X if and only if

$$\operatorname*{Lim}_{N\to\infty} \frac{\pi_{k+1}(\omega) + \cdots + \pi_{k+N}(\omega)}{N} = \operatorname*{Lim}_{N\to\infty} \frac{x_{k+1} + \cdots + x_{k+N}}{N} = \frac{1}{2}$$

uniformly in $k = 0, 1, 2, \ldots$. Since $\mu_\infty(\mathscr{W}_X) = 0$, we know that for almost no ω,

$$\operatorname*{Lim}_{N\to\infty} \frac{\pi_{k+1}(\omega) + \cdots + \pi_{k+N}(\omega)}{N} = \int_{X^\infty} \pi_1 \, \mathrm{d}\mu_\infty$$

uniformly in $k = 0, 1, 2, \ldots$. Thus it is natural to consider whether a similar result still holds for a general sequence of independent, identically distributed, real-valued random variables. The following proposition answers this question, whose proof is similar to the proof of the metric theorem for well distributed sequences in [8].

PROPOSITION 2. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of independent, identically distributed, real-valued, random variables on a probability space $(\Omega, \mathscr{B}, P)$. Assume that $\int_\Omega |f_1| \, \mathrm{d}P < +\infty$. Let U be the set all ω such that*

$$\operatorname*{Lim}_{N\to\infty} \frac{f_{k+1}(\omega) + \cdots + f_{k+N}(\omega)}{N} = \int_\Omega f_1 \, \mathrm{d}P$$

uniformly in $k = 0, 1, 2, \ldots$. If the distribution of f_1 is not a point measure, then $P(U) = 0$.

Proof. Let μ be the distribution of f_1 on $\mathbb{R}$. Since μ is not a point measure, there is a real number b such that $b < \int_\Omega f_1 \, \mathrm{d}P = \int_{\mathbb{R}} x \, \mathrm{d}\mu$ and $0 < \mu((-\infty, b]) < 1$. Let $\alpha = \mu((-\infty, b])$ and $a = \int_\Omega f_1 \, \mathrm{d}P$. For any given $N \geqslant 1$, $k \geqslant 0$, let

$$A_k^N = \left\{\omega \in \Omega \colon \frac{f_{kN+1}(\omega) + \cdots + f_{kN+N}(\omega)}{N} > b\right\}.$$

It is clear that

$$A_k^N \subseteq \Omega - \{\omega \in \Omega \colon f_{kN+j}(\omega) \leqslant b \quad \text{for all } 1 \leqslant j \leqslant N\}.$$

Hence $P(A_k^N) \leqslant (1 - \alpha^N)$. Now let

$$A = \bigcup_{N=1}^{\infty} \bigcap_{k=0}^{\infty} A_k^N.$$

For any given $N \geqslant 1$, the sequence

$$\left\{\frac{f_{kN+1} + \cdots + f_{kN+N}}{N}\right\}_{k=0}^{\infty}$$

of random variables is independent, and hence

$$P\left(\bigcap_{k=0}^{\infty} A_k^N\right) = \operatorname*{Lim}_{m\to\infty} P\left(\bigcap_{k=0}^{m} A_k^N\right) = \operatorname*{Lim}_{m\to\infty} \prod_{k=0}^{m} P(A_k^N)$$

$$\leqslant \operatorname*{Lim}_{m\to\infty} (1 - \alpha^N)^m = 0.$$

Thus we have $P(A) = 0$. Next, we show that U is a subset of A. To this end, pick any ω in U. Then there is a positive integer N_0 such that

$$\left|\frac{f_{k+1}(\omega) + \cdots + f_{k+N_0}(\omega)}{N_0} - a\right| < (a - b)$$

for all $k = 0, 1, 2, \ldots$. Therefore, for each $k = 0, 1, 2, \ldots$, we have

$$\frac{f_{kN_0+1}(\omega) + \cdots + f_{kN_0+N_0}(\omega)}{N_0} > b.$$

Thus, $\omega \in A_k^{N_0}$ for all $k \geqslant 0$ and hence $\omega \in A$. Therefore, we have $P(U) = 0$. □

REMARKS. (1) By the strong law of large numbers, we know that for almost all ω in Ω.

$$\operatorname*{Lim}_{N\to\infty} \frac{f_{k+1}(\omega) + \cdots + f_{k+N}(\omega)}{N} = \int_\Omega f_1 \, \mathrm{d}P$$

for each $k = 0, 1, 2, \ldots$. However, the above proposition shows that for almost no ω in Ω, the convergence is uniform in $k = 0, 1, 2, \ldots$.

(2) Let X, μ and $\mathscr{W}_X$ be the same as in the beginning of this section. Then the metric result $\mu_\infty(\mathscr{W}_X) = 0$ clearly follows from the above proposition. In fact, if μ is not a point measure, then there exists a μ-continuity subset M of X with

$0 < \mu(M) < 1$ (see [8]). To obtain the metric result, we can apply the proposition to the sequence $\{f_n\}_{n=1}^{\infty}$ of random variables on $(X^\infty, \mathscr{B}_{X^\infty}, \mu_\infty)$ defined by $f_n(\omega) = \chi_M(x_n)$ for any $\omega = (x_1, \ldots, x_n, \ldots)$ in X^∞, where χ_M is the characteristic function of M in X.

In the rest of this paper, we shall give a characterization of the set U of all the sequences $\omega' = (i_n)_{n=1}^{\infty}$ of an infinite coin-tossing such that

$$\operatorname*{Lim}_{N\to\infty} \frac{i_{k+1} + \cdots + i_{k+N}}{N} = \frac{1}{2}$$

uniformly in $k = 0, 1, 2, \ldots$. As noted earlier, $U = \mathscr{W}_{\{0,1\}^\infty}$. Define a function F from $[0, 1)$ to $\{0, 1\}$ by

$$F(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}), \\ 1 & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

Then we have the following

PROPOSITION 3. *An infinite coin-tossing* $\omega' = (i_n)_{n=1}^{\infty}$ *is in* U *if and only if there is a well distributed sequence* $\omega = (x_n)_{n=1}^{\infty}$ mod 1 *such that* $\omega' = F(\omega)$, *i.e.*, $i_n = F(x_n)$ *for each* n.

Proof. To prove the sufficiency, let $\omega = (x_n)_{n=1}^{\infty}$ be well distributed mod 1 and $\omega' = F(\omega)$. Since F is Riemann integrable, we have

$$\operatorname*{Lim}_{N\to\infty} \frac{F(x_{k+1}) + \cdots + F(x_{k+N})}{N} = \int_0^1 F(x)\,dx = \frac{1}{2}$$

uniformly in $k = 0, 1, 2, \ldots$. Therefore ω' is an element of U.

To prove the necessity, let $\omega' = (i_n)_{n=1}^{\infty}$ be a member of U. For each $n \geqslant 1$, let $a_n = \{n\sqrt{2}\}$ – the fractional part of $n\sqrt{2}$. Then the sequence $(a_n)_{n=1}^{\infty}$ is well distributed mod 1. Define a new sequence $\omega = (x_n)_{n=1}^{\infty}$ by letting

$$x_n = \begin{cases} \dfrac{1}{2} a_{n-(i_1 + \cdots + i_n)} & \text{if} \quad i_n = 0, \\[2ex] \dfrac{1}{2} + \dfrac{1}{2} a_{i_1 + \cdots + i_n} & \text{if} \quad i_n = 1. \end{cases}$$

Then it is clear that $F(\omega) = \omega'$. All that remains to be shown is that ω is well distributed mod 1. Let $I^0(N, k)$ and $I^1(N, k)$ be the number of 0's and 1's in the finite sequence $\{i_{k+1}, i_{k+2}, \ldots, i_{k+N}\}$ respectively. Let $Z(N, k)$ be the first n such that $k \leqslant n \leqslant k + N - 1$ and $i_{n+1} = 0$. Then it is easy to see that

$$x_{Z(N,k)+1} = \frac{1}{2} a_{I^0(Z(N,k),0)+1}.$$

Now, for a subset E of $[0,1]$, let $A(E; N, k)$ be the number of terms $a_{k+1}, a_{k+2}, \ldots, a_{k+N}$ that are lying in E and let $X(E; N, k)$ be the number of terms $x_{k+1}, x_{k+2}, \ldots, x_{k+N}$ that are in E. For any pair of real numbers a, b with $0 \leqslant a < b < \frac{1}{2}$, it is easily verified that

$$X((a, b); N, k) = A((2a, 2b); I^0(N, k), I^0(Z(N, k), 0)).$$

Since $\omega' \in U$,

$$\operatorname*{Lim}_{N\to\infty} \frac{I^0(N, k)}{N} = \frac{1}{2}$$

uniformly in $k = 0, 1, 2, \ldots$, and since

$$\operatorname*{Lim}_{M\to\infty} \frac{A((2a, 2b); M, p)}{M} = 2(b - a)$$

uniformly in $p = 0, 1, 2, \ldots$, it is easy to see that

$$\operatorname*{Lim}_{N\to\infty} \frac{X((a, b); N, k)}{N} = b - a$$

uniformly in $k = 0, 1, 2, \ldots$. The proof for the case that $\frac{1}{2} \leqslant a < b < 1$ is the same. Hence ω is well distributed mod 1 and the proof is now complete. □

REMARK. It is clear that the sequence $\omega_0 = (0, 1, 0, 1, 0, 1, \ldots)$ is an element of U. The proof of Proposition 1 yields that we can obtain continuum many members of U by revising ω_0. Now let α, β be any two real numbers such that 1, α, β are independent over the rational field Q. Since the sequence $((n\alpha, n\beta))_{n=1}^{\infty}$ is u.d. mod 1 in $\mathbb{R}^2$ (see p. 48, [7]), we know that the sequences $(F(\{n\alpha\}))_{n=1}^{\infty}$ and $(F(\{n\beta\}))_{n=1}^{\infty}$ are different elements of $\{0, 1\}^{\infty}$. Thus, if we let $\mathscr{H}$ be a Hamel basis of $\mathbb{R}$ including 1, then the set

$$\{(F(a_n))_{n=1}^{\infty}: a_n = \{n\alpha) \text{ for some } \alpha \in \mathscr{H} - \{1\}\}$$

is a subset of U with continuum many elements, which are in a certain sense more "random" than ω_0.

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