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Localization of epimorphisms and monomorphisms in homotopy theory

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1. Introduction

Recall that $f: X \to Y \in \mathrm{HCW}^*$, the homotopy category of pointed path-connected CW-spaces, is a homotopy epimorphism (monomorphism) if given u, $v: Y \to Z \in \mathrm{HCW}^*$ (u, $v: Z \to X \in \mathrm{HCW}^*$), $u \circ f = v \circ f$ implies $u = v$ ($f \circ u = f \circ v$ implies $u = v$) [3].

The purpose of this note is to study the effect of p-localizing homotopy epimorphisms and homotopy monomorphisms. The following problems are due to Hilton and Roitberg [4].

PROBLEM A. If $f: X \to Y$ is a homotopy epimorphism (monomorphism) of nilpotent spaces, then is any p-localized map $f_p: X_p \to Y_p$ a homotopy epimorphism (monomorphism)?

PROBLEM B. If each p-localized map $f_p: X_p \to Y_p$ is a homotopy epimorphism (monomorphism), then is $f: X \to Y$ a homotopy epimorphism (monomorphism)?

In [4], Hilton and Roitberg obtained some partial information [4, Theorem 4.4, 4.4′, 4.5 and 4.5′] for these problems. In this note we shall prove the following theorems.

THEOREM 1. *If $f: X \to Y$ is a homotopy epimorphism of nilpotent spaces, then the p-localized map f_p: $X_p \to Y_p$ is a homotopy epimorphism. Conversely, let Y be quasifinite, if each p-localized map $f_p: X_p \to Y_p$ is a homotopy epimorphism, then $f: X \to Y$ is a homotopy epimorphism.*

THEOREM 2. *If $f: X \to Y$ is a homotopy monomorphism of nilpotent spaces, then the p-localized map $f_p: X_p \to Y_p$ is a homotopy monomorphism. Conversely, let each homotopy group of X be finite, if each p-localized map $f_p: X_p \to Y_p$ is a homotopy monomorphism, then $f: X \to Y$ is a homotopy monomorphism.*

This answers Problem A affirmatively.

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2. Proofs

At first, we characterize homotopy epimorphisms and homotopy monomorphisms in terms of homotopy pushouts and homotopy pullbacks.

THEOREM 3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j_1} \\ Y & \xrightarrow{j_2} & C \end{array}$$

be a homotopy pushout in HCW^*. Then f is a homotopy epimorphism if and only if $j_1 = j_2$.

Proof. Suppose f is a homotopy epimorphism. It follows from $j_1 \circ f = j_2 \circ f$ that $j_1 = j_2$. Conversely, given two maps $u, v: Y \to Z$ such that $u \circ f = v \circ f$. Since the square is a homotopy pushout, then there is a map $\varphi: C \to Z$ such that $u = \varphi \circ j_1$ and $v = \varphi \circ j_2$. If $j_1 = j_2$, then $u = v$, and so f is a homotopy epimorphism.

THEOREM 4. *Let*

$$\begin{array}{ccc} E & \xrightarrow{i_1} & X \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

be a homotopy pullback in HCW^*. Assume that E is path-connected (if not, replacing E by the path-component E^* of its base point). Then f is a homotopy monomorphism if and only if $i_1 = i_2$.

Proof. Suppose f is a homotopy monomorphism. It follows from $f \circ i_1 = f \circ i_2$ that $i_1 = i_2$. Conversely, given two maps $u, v: Z \to X$ such that $f \circ u = f \circ v$. Since the square is a homotopy pullback, then there is a map $\varphi: Z \to E$ such that $i_1 \circ \varphi = u$ and $i_2 \circ \varphi = v$. If $i_1 = i_2$, then $u = v$, and so f is a homotopy monomorphism.

Secondly, we must show the question of when we may infer that C and E in Theorem 3 and 4 are nilpotent if X and Y are nilpotent, since we want to localize them.

LEMMA 1. *If $f: X \to Y$ is a homotopy epimorphism of nilpotent spaces, then C in Theorem 3 is nilpotent.*

Proof. Note that the homotopy epimorphism $f: X \to Y$ induces an epimorphism $f_*: \pi_1 X \to \pi_1 Y$ [3, Proposition 1]. By [6, Theorem 2.1], C in Theorem 3 is nilpotent.

LEMMA 2. *If X and Y are nilpotent, then E in Theorem 4 is nilpotent.*
Proof. See [2, Corollary II.7.6].

Finally, we show p-localization of the square in Theorem 3 (4) is also a homotopy pushout (pullback).

Let X and Y be nilpotent, and the following square (*) be a homotopy pushout, and the following square (**) be a homotopy pullback

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j_2} \\ Y & \xrightarrow{j_1} & C \end{array} \quad \ldots(*) \qquad \begin{array}{ccc} E & \xrightarrow{i_1} & X \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array} \quad \ldots(**)$$

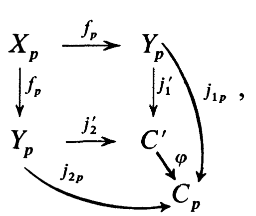
If C and E are nilpotent, then we can localize squares at prime p. Hence we obtain the following commutative squares:

$$\begin{array}{ccc} X_p & \xrightarrow{f_p} & Y_p \\ \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle j_{2p}} \\ Y_p & \xrightarrow{j_{1p}} & C_p \end{array} \quad \ldots(*)_p \qquad \begin{array}{ccc} E_p & \xrightarrow{i_{1p}} & X_p \\ \downarrow{\scriptstyle i_{2p}} & & \downarrow{\scriptstyle f_p} \\ X_p & \xrightarrow{f_p} & Y_p \end{array} \quad \ldots(**)_p$$

LEMMA 3. *If $f: X \to Y$ is a homotopy epimorphism of nilpotent spaces, then the square $(*)_p$ is a homotopy pushout.*
Proof. Let

$$\begin{array}{ccc} X_p & \xrightarrow{f_p} & Y_p \\ \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle j_1'} \\ Y_p & \xrightarrow{j_2'} & C' \end{array} \quad \ldots(*)'_p$$

be a homotopy pushout. Then there is a map $\varphi: C' \to C_p$ yielding a commutative diagram in HCW*



and hence a map of the Mayer–Vietoris sequence of the square $(*)'_p$ to the p-localization of the Mayer-Vietoris sequence of the square (*). In this map of Mayer-Vietoris sequences all groups except $H_n(C')$ are mapped by the identity.

Thus φ induces an isomorphism of homology groups. Since f is a homotopy epimorphism, $f_*: \pi_1 X \to \pi_1 Y$ is an epimorphism by [3, Proposition 1], and so is $f_{p*}: \pi_1 X_p \to \pi_1 Y_p$. Hence C (so C_p) and C' are nilpotent by [6, Theorem 2.1]. Therefore $\varphi: C' \to C_p$ is a homotopy equivalence by [1].

LEMMA 4. *The square* $(**)_p$ *is a homotopy pullback.*
Proof. See [2, Proposition II.7.9].

Now we can prove Theorem 1 and 2.

Proof of Theorem 1. Let $f: X \to Y$ be a homotopy epimorphism. Then $j_1 = j_2$ in the square (*) by Theorem 3, and C is nilpotent by Lemma 1. So $j_{1p} = j_{2p}$ in the square $(*)_p$. It follows from Lemma 3 and Theorem 3 that $f_p: X_p \to Y_p$ is a homotopy epimorphism. Conversely, let each p-localized map $f_p: X_p \to Y_p$ be a homotopy epimorphism. Then $f_{p*}: \pi_1 X_p \to \pi_1 Y_p$ is an epimorphism [3, Proposition 1]. It follows from [2, Theorem I.3.12] that $f_*: \pi_1 X \to \pi_1 Y$ is an epimorphism, and so C is nilpotent. This implies $j_{1p} = j_{2p}$ in the square $(*)_p$ by Theorem 3. By [2, Theorem II.5.14], we obtain $j_1 = j_2$ in the square (*), and so f is a homotopy epimorphism by Theorem 3.

Proof of Theorem 2. Let $f: X \to Y$ be a homotopy monomorphism. Then $i_1 = i_2$ in the square (**) by Theorem 4, and E is nilpotent by Lemma 2. So $i_{1p} = i_{2p}$ in the square $(**)_p$. It follows from Lemma 4 and Theorem 4 that $f_p: X_p \to Y_p$ is a homotopy monomorphism. Conversely, let each p-localized map $f_p: X_p \to Y_p$ is a homotopy monomorphism. By [4, Theorem 4.5′], $f: X \to Y$ satisfies that $f \circ u' = f \circ v'$ implies $u' = v'$ if given $u', v': W \to X$ and W finite complex. Given $u, v: Z \to X$ such that $f \circ u = f \circ v$. Let $\{Z_\alpha\}$ be the set of finite subcomplex of Z directed by inclusion $i_\alpha: Z_\alpha \to Z$. Then $u \circ i_\alpha = v \circ i_\alpha$ for all α. By [5, Theorem 1], the natural map

$$[Z, X] \to \varprojlim [Z_\alpha, X]$$

is bijective if each homotopy group of X is finite. It follows from $u \circ i_\alpha = v \circ i_\alpha$ that $u = v$, and f is a homotopy monomorphism.

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