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## On the conductor of the Jacobi sum Hecke character\*

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Recently, Coleman-McCallum [5] determined completely the precise conductor of the Jacobi sum Hecke character, using the stable reduction of Fermat curves.

In this paper, we will give a purely number theoretic proof of their results (see Theorem 3 and its Corollary in the present paper), not using the geometry of Fermat curves. Our proof is much simpler than theirs.

First, we give the definition of the Jacobi sum.

**DEFINITION.** For arbitrary positive integers  $m, r$  and any  $a = (a_1, \dots, a_r) \in \mathbb{Z}^r$  and for any prime ideal  $\mathfrak{p}$  of  $\mathbb{Q}(\zeta_m)$  which is prime to  $m$ , put

$$J_m^{(a)}(\mathfrak{p}) = (-1)^{r+1} \sum_{\substack{x_1 + \dots + x_r = -1 \\ x_1, \dots, x_r \in \mathbb{Z}[\zeta_m]/\mathfrak{p}}} \chi_{\mathfrak{p}}^{a_1}(x_1) \chi_{\mathfrak{p}}^{a_2}(x_2) \cdots \chi_{\mathfrak{p}}^{a_r}(x_r) \in \mathbb{Z}[\zeta_m],$$

where  $\mathbb{Q}$  is the field of rational numbers,  $\mathbb{Z}$  is the ring of rational integers,  $\zeta_m \in \mathbb{C}$  (the field of complex numbers) is a primitive  $m$ th root of unity, and  $\chi_{\mathfrak{p}}(x) = (x/\mathfrak{p})_m$  is the  $m$ th power residue symbol in  $\mathbb{Q}(\zeta_m)$ , i.e.  $\chi_{\mathfrak{p}}(x \bmod \mathfrak{p})$  is a unique  $m$ th root of unity in  $\mathbb{C}$  such that

$$\chi_{\mathfrak{p}}(x \bmod \mathfrak{p}) \equiv x^{(N_{\mathfrak{p}}-1)/m} \pmod{\mathfrak{p}}$$

for  $x \in \mathbb{Z}[\zeta_m], \notin \mathfrak{p}$ . Here  $N_{\mathfrak{p}}$  is the number of elements in  $\mathbb{Z}[\zeta_m]/\mathfrak{p}$ . Put  $\chi_{\mathfrak{p}}(0) = 0$ . For any fractional ideal  $\mathfrak{a}$  of  $\mathbb{Q}(\zeta_m)$  which is prime to  $m$ , put

$$J_m^{(a)}(\mathfrak{a}) = \prod_{\mathfrak{p}} J_m^{(a)}(\mathfrak{p})^{e_{\mathfrak{p}}},$$

where  $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$  is the prime ideal decomposition of  $\mathfrak{a}$ .  $J_m^{(a)}(\mathfrak{a})$  is called the *Jacobi sum*.

\*This paper is the details of a part of my talk in *Number Theory Seminar* (Goldfeld), Columbia University, March 21, 1988 (see [16]).

By Weil [23],  $J_m^{(a)}(\mathfrak{a})$  is a Hecke character of  $\mathbb{Q}(\zeta_m)$  as a function in  $\mathfrak{a}$  with conductor  $C_m^{(a)}$  dividing  $m^2$ . He raised the problem of giving the precise value of the conductor  $C_m^{(a)}$ . The Jacobi sum is an interesting Hecke character and it is a natural problem to give the precise conductor for a given Hecke character. Hasse [6] determined the precise  $C_m^{(a)}$  when  $r = 2$  and  $m = l$  is any odd prime number. Iwasawa [11] determined (essentially) the precise  $C_m^{(a)}$  when  $r \geq 2$  and  $m = l$  is any odd prime number. Jensen [12] and Schmidt [20] gave certain estimates for  $C_m^{(a)}$ . Rohrlich [19] proved that  $C_m^{(a)} | (\zeta_l - 1)^2$  when  $r = 2$  and  $m = l^n$  with any integer  $n \geq 1$  and any odd prime  $l$ , by using Artin-Hasse's and Iwasawa's explicit formulas for the Hilbert norm residue symbol [2], [8]. Miki [15] gave the precise  $C_m^{(a)}$  when  $r \geq 2$  and  $m = l^2$  with any odd prime  $l$ , by using a congruence for the Jacobi sum [14] which generalizes Hasse-Iwasawa-Ihara's [6], [7], [11]. The method of [15] can be regarded as a generalization of Hasse's [6] and Iwasawa's [11]. Coleman-McCallum [5] gave a complete solution of the problem by using the stable reduction of Fermat curves and Shimura-Taniyama's complex multiplication of abelian varieties [21]. We should also note that Coleman ([4], Section VI) (with G. Anderson) gave another proof (at least under the assumption  $(l, a_0 a_1 \cdots a_r) = 1$ ) as an application of Ihara [7] and Anderson [1], and that Kato [13] gave another proof as an application of his theory.

The present paper can be regarded as a generalization of Rohrlich [19] and Miki [15], and the main idea is to use the homomorphism  $\delta^{(n)}$  of  $U_n^{(1)}$  (the group of principal units) to  $\mathcal{O}_K/l^n \mathcal{O}_K$  which is related to Artin-Hasse's and Iwasawa's explicit formulas for the Hilbert norm residue symbol (see Lemma 1 in Section 1), instead of using the congruence for the Jacobi sum.

Our number theoretic proof involves the calculation of the Hilbert symbol  $(1 + l, J_m^{(a)}(\mathfrak{a}))_n$  and that of certain sums  $W_n(a)$  and  $S_m^{(a)}$  (see Theorems 1 and 2, and corollary to Theorem 2), which are new results not contained in Coleman-McCallum [5]. The determination of the conductor follows directly from those calculations (see Theorem 3 and its corollary).

### 1. Certain homomorphism $\delta^{(n)}$ of $U_n^{(1)}$ to $\mathcal{O}_K/l^n \mathcal{O}_K$ and the calculation of $\delta^{(n)}(J_m^{(a)}(\mathfrak{a}))$

Let  $l$  be an odd prime number<sup>†</sup> and let  $n$  be a positive integer. Let  $\mathbb{Z}_l$  and  $\mathbb{Q}_l$  denote the ring of  $l$ -adic integers and the field of  $l$ -adic numbers respectively. We fix an algebraic closure  $\overline{\mathbb{Q}}_l$  of  $\mathbb{Q}_l$  once for all, and we consider that all algebraic extensions of  $\mathbb{Q}_l$  and all elements which are algebraic over  $\mathbb{Q}_l$  are

<sup>†</sup>Though almost all parts of the present paper are valid for  $l = 2$  with slight modification, we will discuss in the case  $l = 2$  elsewhere.

contained in  $\bar{\mathbb{Q}}_l$ . All congruences in the present paper are those in  $\bar{\mathbb{Q}}_l$ . Fix a sequence  $\zeta_l, \zeta_{l^2}, \dots, \zeta_{l^i}, \zeta_{l^{i+1}}, \dots$  of a primitive  $l^i$ th root of unity such that  $\zeta_{l^{i+1}} = \zeta_{l^i}$  for  $i = 1, 2, 3, \dots$  and put  $\pi_i = 1 - \zeta_{l^i}$ . Fix any finite unramified extension  $K$  of  $\mathbb{Q}_l$  and let  $\mathcal{O}_K$  be the ring of integers of  $K$ . Put  $K_n = K(\zeta_{l^n})$  and  $K_\infty = \bigcup_{i=1}^\infty K_i$ . Then  $\text{Gal}(K_\infty/K) \cong \mathbb{Z}_l^\times$  (the group of units in  $\mathbb{Z}_l$ ) by  $\sigma_a \leftrightarrow a$ , where  $\sigma_a \in \text{Gal}(K_\infty/K)$  is such that  $\zeta_{l^i}^{\sigma_a} = \zeta_{l^i}^a$  for all  $i \geq 1$ . Put

$$\delta^{(n)}(\alpha) = \frac{-1}{l^{n-1}(l-1)} \text{Tr}_{K_n/K} \left( \zeta_{l^n} \alpha^{-1} \frac{d\alpha}{d\pi_n} \right) \text{ for } \alpha \in U_n^{(1)},$$

where  $U_n^{(1)}$  is the group of principal units in  $K_n$ :

$$U_n^{(1)} = \{x \in \mathcal{O}_{K_n}^\times \mid x \equiv 1 \pmod{\pi_n}\},$$

$\text{Tr}_{K_n/K}$  is the trace from  $K_n$  to  $K$ , and  $d\alpha/d\pi_n = f'(\pi_n)$ . Here  $f(T)$  is a formal power series in  $T$  with coefficients in  $\mathcal{O}_K$  satisfying  $\alpha = f(\pi_n)$ , and  $f'(T)$  is the formal derivative of  $f(T)$  with respect to  $T$ . Let  $[\alpha, \beta]_n \in \mathbb{Z}/l^n\mathbb{Z}$  be such that  $(\alpha, \beta)_n = \zeta_{l^n}^{[\alpha, \beta]_n}$  for  $\alpha, \beta \in \mathbb{Q}_l(\zeta_{l^n})^\times$ , where  $(\alpha, \beta)_n$  is the Hilbert norm residue symbol in  $\mathbb{Q}_l(\zeta_{l^n})$  for the power  $l^n$  defined by

$$(\alpha, \beta)_n = (\sqrt[l^n]{\beta})^{\rho(\alpha)-1}.$$

Here  $\rho: \mathbb{Q}_l(\zeta_{l^n})^\times \rightarrow \text{Gal}(\mathbb{Q}_l(\zeta_{l^n})^{ab}/\mathbb{Q}_l(\zeta_{l^n}))$  is the Artin map in local class field theory and  $\mathbb{Q}_l(\zeta_{l^n})^{ab}$  is the maximum abelian extension of  $\mathbb{Q}_l(\zeta_{l^n})$ . Then the following Lemma 1 is a direct consequence of Iwasawa [8] (though he assumes  $K = \mathbb{Q}_l$ , the proof is the same for general  $K$ ).

LEMMA 1. *Let the notation and assumptions be as above. Then  $\delta^{(n)}$  is a well-defined homomorphism of  $U_n^{(1)}$  to  $\mathcal{O}_K/l^n\mathcal{O}_K$  satisfying the following properties (i) ~ (v) for  $\alpha \in U_n^{(1)}$ :*

- (i)  $\delta^{(n)}(\alpha^{\sigma_a}) \equiv a\delta^{(n)}(\alpha) \pmod{l^n\mathcal{O}_K}$  for  $a \in \mathbb{Z}_l^\times$ .
- (ii)  $\delta^{(n)}(\zeta_{l^n}) \equiv 1 \pmod{l^n\mathcal{O}_K}$ .
- (iii)  $\delta^{(n)}(\alpha) \equiv -c [1 + l, \alpha]_n \pmod{l^n\mathcal{O}_K}$  if  $\alpha \in U_n^{(1)} \cap \mathbb{Q}_l(\zeta_{l^n})$ , where  $c = ((1 - 1/l) \log(1 + l))^{-1} \in \mathbb{Z}_l^\times$  and  $\log$  is the  $l$ -adic logarithm.
- (iv)  $\delta^{(n)}(\alpha) \equiv 0 \pmod{l^n\mathcal{O}_K}$  if  $\alpha \equiv 1 \pmod{\pi_1^2}$  and  $\alpha \in \mathbb{Q}_l(\zeta_{l^n})$ .
- (v)  $\delta^{(n+1)}(\alpha') \equiv \delta^{(n)}(N_{n+1,n}(\alpha')) \pmod{l^n\mathcal{O}_K}$  if  $\alpha' \in U_{n+1}^{(1)}$ , where  $N_{n+1,n}$  is the norm map of  $K_{n+1}$  to  $K_n$ .

REMARK. (i) In [16], we used the Coates-Wiles homomorphism [3] to prove the existence of a homomorphism satisfying the above properties (i) ~ (v) of Lemma 1, but here we adopt a more direct method using Iwasawa [8]. For the

details of the relation between  $\delta^{(n)}$  and the Coates-Wiles homomorphism, we will discuss elsewhere.

(ii) Conversely, if  $K = \mathbb{Q}_l$ , then we can define  $\delta^{(n)}$  by  $\delta^{(n)}(\alpha) = -c[1 + l, \alpha]_n$  for  $\alpha \in U_n^{(1)}$ . Then the property (i) of Lemma 1 is a well-known property of the norm residue symbol, and the property (ii) of Lemma 1 is one of Artin-Hasse's explicit formulas for the norm residue symbol [2]. Once we determine the value of  $[1 + l, J_n^{(a)}(\alpha)]_n$ , the homomorphism  $\delta^{(n)}$  only for  $K = \mathbb{Q}_l$  and only the properties (i) and (ii) of Lemma 1 are sufficient for our proof of Theorem 3, but it is crucial for our calculation of  $[1 + l, J_n^{(a)}(\alpha)]_n$  to define  $\delta^{(n)}$  for any finite unramified extension  $K$  of  $\mathbb{Q}_l$  (see the proof of Theorem 1).

(iii) We do not use the property (v) in Lemma 1 in the present paper.

Let  $\bar{\mathbb{Q}}$  be the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ . We consider that all algebraic extensions of  $\mathbb{Q}$  and all elements algebraic over  $\mathbb{Q}$  are contained in  $\bar{\mathbb{Q}}$ . By a fixed imbedding  $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$ , we consider  $\bar{\mathbb{Q}}$  as a subfield of  $\bar{\mathbb{Q}}_l$ .

For any positive integer  $m$  and any  $a \in \mathbb{Z}$ , put

$$g_m(\mathfrak{p}, a) = - \sum_{x \in \mathbb{Z}[\zeta_m]/\mathfrak{p}} \chi_p^a(x) \psi_p(x) \in \mathbb{Z}[\zeta_{mp}],$$

where  $\psi_p(x) = \zeta_p^{T(x)}$  ( $p$  is a prime number such that  $p \in \mathfrak{p}$  and  $T$  is the trace of  $\mathbb{Z}[\zeta_m]/\mathfrak{p}$  to  $\mathbb{Z}/p\mathbb{Z}$ ), and put

$$g_m(\mathfrak{a}, a) = \prod_{\mathfrak{p}} g_m(\mathfrak{p}, a)^{e_{\mathfrak{p}}} \quad \text{and} \quad g_m(\mathfrak{a}) = g_m(\mathfrak{a}, 1),$$

where  $\mathfrak{a} = \prod \mathfrak{p}^{e_{\mathfrak{p}}}$  is the prime ideal decomposition of any fractional ideal  $\mathfrak{a}$  of  $\mathbb{Q}(\zeta_m)$  which is prime to  $m$ . This is called the *Gauss sum*. Clearly  $g_m(\mathfrak{a}\mathfrak{b}, a) = g_m(\mathfrak{a}, a)g_m(\mathfrak{b}, a)$ . It is well known that if  $a = (a_1, \dots, a_r) \not\equiv (0, \dots, 0) \pmod{m}$ , then

$$J_m^{(a)}(\mathfrak{a}) = N\mathfrak{a}^{-1} \cdot \prod_{i=1}^r g_m(\mathfrak{a}, a_i), \tag{1}$$

where

$$a_0 = - \sum_{i=1}^r a_i.$$

Now assume  $m = l^n$  and the following condition (\*) on  $K$  and  $\mathfrak{a}$ :

$$K \ni \zeta_p \text{ for any prime number } p \text{ contained in any prime ideal dividing } \mathfrak{a}. \tag{*}$$

Then  $g_{l^n}(\mathfrak{a}, a) \in K_n$ . By the following Lemma 2, we can see the action of the Galois group  $\text{Gal}(K_n/\mathbb{Q}_l) = \text{Gal}(K_n/K) \times \text{Gal}(K_n/\mathbb{Q}_l(\zeta_{l^n}))$  (direct product) on  $g_{l^n}(\mathfrak{a}, a)$ .

LEMMA 2. Under the above assumption (\*), we have the following:

- (i)  $g_m(\mathfrak{a}, a)^{\sigma^c} = g_m(\mathfrak{a}, ac)$  for  $c \in \mathbb{Z}_l^\times$ .
- (ii)  $g_m(\mathfrak{a}, a)^\tau = \zeta_m^{-\langle l, \mathfrak{a} \rangle a} g_m(\mathfrak{a}, a)$ , where  $\tau \in \text{Gal}(K_n/\mathbb{Q}_l(\zeta_m))$  is the Frobenius automorphism, and  $\langle x, \mathfrak{a} \rangle \in \mathbb{Z}/l^n\mathbb{Z}$  is defined by  $(x/\mathfrak{a})_m = \zeta_m^{\langle x, \mathfrak{a} \rangle}$ .

*Proof.* It suffices to prove for  $\mathfrak{a} = \mathfrak{p}$ . Since (i) is trivial, we prove (ii). Since  $\tau$  acts trivially on  $\chi_{\mathfrak{p}}^a(x)$ ,

$$g_m(\mathfrak{p}, a)^\tau = - \sum_{x \in \mathbb{Z}[\zeta_m]/\mathfrak{p}} \chi_{\mathfrak{p}}^a(x) \psi_{\mathfrak{p}}(x)^\tau,$$

so

$$\begin{aligned} g_m(\mathfrak{p}, a)^\tau &= - \sum_x \chi_{\mathfrak{p}}^a(x) \psi_{\mathfrak{p}}(xl) \\ &= \chi_{\mathfrak{p}}(l)^{-a} g_m(\mathfrak{p}, a), \end{aligned}$$

hence we have the assertion.

For  $a \in \mathbb{Z}$ , we write  $a = a' a''$ , where  $a'$  is the power of  $l$  and  $a'' \in \mathbb{Z}$  is prime to  $l$ . If  $a = 0$ , then put  $a' = 0$  and  $a'' = 1$ . Under this notation, we have the following congruence (mod  $l$ ) for the Gauss sum  $g_m(\mathfrak{a}, a)$  and the Jacobi sum  $J_m^{(a)}(\mathfrak{a})$ :

LEMMA 3. Under the assumption (\*) before Lemma 2, we have the following congruences:

- (i)  $g_m(\mathfrak{a}, a^{l^j}) \equiv \zeta_m^{\langle l^j, \mathfrak{a} \rangle a^{l^j}} g_m(\mathfrak{a}, a)^{l^j} \pmod{l}$  for  $a, j \in \mathbb{Z}, j \geq 1$ .
- (ii)  $g_m(\mathfrak{a}, a) \equiv \zeta_m^{\langle l, \mathfrak{a} \rangle (\text{ord}_l(a) \cdot a)} g_m(\mathfrak{a})^{a' \sigma_{a''}} \pmod{l}$  for  $a \in \mathbb{Z}$ , where  $\text{ord}_l$  is the normalized additive valuation of  $\mathbb{Q}_l$ , and  $\text{ord}_l(0) \cdot 0 = \infty \cdot 0 = 0$ .
- (iii)  $J_m^{(a)}(\mathfrak{a}) \equiv N \mathfrak{a}^{-1} \cdot \zeta_m^{\langle l, \mathfrak{a} \rangle g} g_m(\mathfrak{a})^\omega \pmod{l}$  if  $a = (a_1, \dots, a_r) \not\equiv (0, \dots, 0) \pmod{l^n}$ , where  $a_0 = -\sum_{i=1}^r a_i$ ,  $g = \sum_{i=0}^r \text{ord}_l(a_i) \cdot a_i$ ,  $\omega = \sum_{i=0}^r a_i' \sigma_{a_i''} \in \mathbb{Z}[\text{Gal}(K_n/K)]$  (the group ring of  $\text{Gal}(K_n/K)$  over  $\mathbb{Z}$ ).

*Proof.* It is sufficient to prove for  $\mathfrak{a} = \mathfrak{p}$ .

- (i) Put  $a_1 = a^{l^j}$ . Then

$$\begin{aligned} g_m(\mathfrak{p}, a)^{l^j} &= \left( - \sum_x \chi_{\mathfrak{p}}^a(x) \psi_{\mathfrak{p}}(x) \right)^{l^j} \\ &\equiv - \sum_x \chi_{\mathfrak{p}}^{a_1}(x) \psi_{\mathfrak{p}}(l^j x) \pmod{l} \\ &\equiv \chi_{\mathfrak{p}}^{-a_1}(l^j) g_m(\mathfrak{p}, a_1) \pmod{l} \\ &\equiv \zeta_m^{-\langle l^j, \mathfrak{p} \rangle a_1} g_m(\mathfrak{p}, a_1) \pmod{l} \end{aligned}$$

by the equality  $\psi_p(x)^{l^j} = \psi_p(l^j x)$  and the definition of  $\langle l^j, p \rangle$ . Thus we obtain the desired congruence.

(ii) Since the case  $a = 0$  is trivial, we may assume  $a \neq 0$ . We can write  $a = a' a''$ , where  $a' = l^j, j = \text{ord}_l(a)$  and  $a'' \in \mathbb{Z}$  is prime to  $l$ . Then the congruence (ii) is a direct consequence of (i), since  $\langle l^j, p \rangle \equiv j \langle l, p \rangle \pmod{l^n}$  and  $g_m(p)^{\sigma_{a'}} = g_m(p, a'')$  by (i) of Lemma 2.

(iii) This follows immediately from the congruence (ii) and the equality (1).

By Lemmas 1, 2, and 3, we will determine the value of  $\delta^{(n)}(J_m^{(a)}(\mathfrak{a}))$ , i.e., that of  $[1 + l, J_m^{(a)}(\mathfrak{a})]_n$ :

**THEOREM 1.** *If  $\mathfrak{a} = (a_1, \dots, a_r) \not\equiv (0, \dots, 0) \pmod{l^n}$ , then*

$$\delta^{(n)}(J_m^{(a)}(\mathfrak{a})) \equiv \langle l, \mathfrak{a} \rangle g \pmod{l^n},$$

i.e.

$$[1 + l, J_m^{(a)}(\mathfrak{a})]_n \equiv -\left(1 - \frac{1}{l}\right) \log(1 + l) \cdot \langle l, \mathfrak{a} \rangle g \pmod{l^n},$$

where  $g = \sum_{i=1}^r \text{ord}_l(a_i) \cdot a_i$ , and  $\langle l, \mathfrak{a} \rangle$  is as in Lemma 2.

*Proof.* Take  $K = \mathbb{Q}_l(\zeta_p \mid p \in P)$ , where  $P$  is the set of all prime numbers contained in any prime ideal dividing  $\mathfrak{a}$ . Since

$$(g_m(\mathfrak{a})^\omega)^\tau = g_m(\mathfrak{a})^\omega$$

by using (ii) of Lemma 2 and the equality  $\sum_{i=1}^r a_i = 0$ , we have  $g_m(\mathfrak{a})^\omega \in \mathbb{Q}_l(\zeta_m)$ , hence the congruence (iii) of Lemma 3 implies that

$$J_m^{(a)}(\mathfrak{a}) = N \mathfrak{a}^{-1} \cdot \zeta_m^{\langle l, \mathfrak{a} \rangle g} g_m(\mathfrak{a})^\omega \cdot \xi \tag{2}$$

with  $\xi \in \mathbb{Q}_l(\zeta_m)$ ,  $\xi \equiv 1 \pmod{l}$ . Clearly  $N \mathfrak{a} \equiv 1 \pmod{l^n}$  and  $g_m(\mathfrak{a}) \in U_n^{(1)}$ . Taking  $\delta^{(n)}$  of both members of (2), we have immediately the assertion, since

$$\delta^{(n)}(N \mathfrak{a}^{-1}) \equiv \delta^{(n)}(\xi) \equiv 0 \pmod{l^n}$$

by (iv) of Lemma 1,

$$\delta^{(n)}(\zeta_m^{\langle l, \mathfrak{a} \rangle g}) \equiv \langle l, \mathfrak{a} \rangle g \pmod{l^n}$$

by (ii) of Lemma 1, and

$$\delta^{(n)}(g_{l^n}(a)^\omega) = \left( \sum_{i=0}^r a'_i a''_i \right) \delta^{(n)}(g_{l^n}(a)) = 0$$

by (i) of Lemma 1 and the equality  $\sum_{i=0}^r a_i = 0$ .

## 2. Calculation of a certain sum $S_l^{(a)}$

For  $a \in \mathbb{Z}$ , put

$$W_n(a) = \sum_{\substack{0 < t < l^n \\ (t, l) = 1}} \left( \left\{ \frac{at}{l^n} \right\} - a \left\{ \frac{t}{l^n} \right\} \right) (-t)^{-1} \in \mathbb{Z}_l,$$

where  $\{x\}$  is the fractional part of  $x \in \mathbb{Q}$ , i.e.  $0 \leq \{x\} < 1$  such that  $\{x\} \equiv x \pmod{\mathbb{Z}}$ .

In this section, we will calculate  $W_n(a)$  (see Theorem 2 below), and as its corollary, we will get the value of a certain sum  $S_l^{(a)}$ , which we need for our proof of Theorem 3.

If  $(a, l) = 1$ , then the calculation of  $W_n(a)$  was made by Iwasawa (see his formula in the line 2, p. 82 of [10]; replace  $(1 + q_0)$  and  $a$  in the formula by  $a$  and  $t$  in our notation respectively):

LEMMA 4. *If  $a \in \mathbb{Z}$ ,  $(a, l) = 1$ , then*

$$W_n(a) \equiv \left( 1 - \frac{1}{l} \right) \log \langle a \rangle^a \pmod{l^n},$$

where  $\log$  is the  $l$ -adic logarithm and  $\langle a \rangle$  is a unique element in  $\mathbb{Z}_l^\times$  such that  $\langle a \rangle \equiv 1 \pmod{l}$  and  $a/\langle a \rangle$  is an  $(l - 1)$ th root of unity.

REMARK. By Iwasawa's construction of the  $l$ -adic  $L$ -function [9],

$$g_a((1 + l)^s - 1) = (\omega(a)\langle a \rangle^s - a)L_l(s, 1),$$

where  $\omega(a) = a/\langle a \rangle$  and  $g_a(T)$  is the unique power series in  $T$  with coefficients in  $\mathbb{Z}_l$  satisfying

$$g_a(T) \equiv \sum_{\substack{0 < t < l^n \\ (t, l) = 1}} \left( a \left\{ \frac{t}{l^n} \right\} - \left\{ \frac{at}{l^n} \right\} \right) \omega^{-1}(t)(1 + T)^{-it} \pmod{(1 + T)^{l^n} - 1}$$



for all  $n \geq 1$ . Here  $i(t) = \log\langle t \rangle / \log(1 + l)$ . Hence

$$g_a(l) \equiv W_n(a) \pmod{l^n} \quad \text{and} \quad W_n(a) \equiv \lim_{s \rightarrow 1} (\omega(a)\langle a \rangle^s - a)L_1(s, 1) \pmod{l^n}.$$

Since  $L_1(s, 1)$  has a pole of order 1 with residue  $(1 - 1/l)$  at  $s = 1$ , this gives another proof of Lemma 4. This is a method used in [16], but here we adopt a more elementary and direct calculation of Iwasawa (see pp. 81–82 of [10]).

We need the following Lemmas 5, 6 and 7 to generalize Lemma 4 for arbitrary  $a \in \mathbb{Z}$ .

LEMMA 5. For  $c \in \mathbb{Z}$ , we have the following (i) and (ii):

(i) If  $c \geq 1$ , then

$$\sum_{j=0}^{l^n} j^c \equiv \sum_{j=1}^{l^n-1} j^c \equiv \begin{cases} 0 & \pmod{l^n} \quad \text{if } (l-1) \nmid c \\ -l^{n-1} & \pmod{l^n} \quad \text{if } (l-1) \mid c. \end{cases}$$

(ii)

$$\sum_{\substack{0 < j < l^n \\ (j, l) = 1}} j^c \equiv \begin{cases} 0 & \pmod{l^n} \quad \text{if } (l-1) \nmid c \\ -l^{n-1} & \pmod{l^n} \quad \text{if } (l-1) \mid c. \end{cases}$$

*Proof.* (i) First, suppose  $c$  is odd. Then  $(l^n - j)^c \equiv -j^c \pmod{l^n}$ , so, by pairing  $j^c$  and  $(l^n - j)^c$  for  $j \in \mathbb{Z}$ ,  $0 \leq j < l^n/2$  in the sum, we get the desired congruence. Next, assume  $c$  is even. Since  $lB_i \in \mathbb{Z}_l$  for all  $i \geq 0$  by the von Staudt-Clausen (cf. [22], Theorem 5.10), using a well known identity (cf. [22], Proposition 4.1)

$$B_c = \frac{1}{l^n} \sum_{j=1}^{l^n} (l^n)^c B_c \left( \frac{j}{l^n} \right),$$

we have easily a congruence

$$\sum_{j=1}^{l^n} j^c \equiv l^n B_c \pmod{l^n}, \tag{1}$$

where  $B_c(X) = \sum_{i=0}^c \binom{c}{i} B_i X^{c-i}$  and  $B_i$  is the  $i$ th Bernoulli number. Again, by the von Staudt-Clausen theorem, we have

$$B_c \equiv \begin{cases} 0 & \pmod{\mathbb{Z}_l} \quad \text{if } (l-1) \nmid c, \\ -\frac{1}{l} & \pmod{\mathbb{Z}_l} \quad \text{if } (l-1) \mid c \end{cases} \tag{2}$$

By (1) and (2), we have the desired congruence.

(ii) Put  $c' = c + l^s(l-1)$  with sufficiently large  $s \geq n$ , then  $c' \geq 1$  and

$$j^{c'} \equiv \begin{cases} j^c \pmod{l^n} & \text{if } l \nmid j, \\ 0 \pmod{l^n} & \text{if } l \mid j, \end{cases}$$

since  $j^{l^s(l-1)} \equiv 1 \pmod{l^n}$  if  $l \nmid j$ . Hence

$$\sum_{\substack{0 < j < l^n \\ (j,l)=1}} j^c \equiv \sum_{j=1}^{l^n} j^{c'} \pmod{l^n}.$$

By this and (i), we get (ii).

LEMMA 6. For  $0 \leq m \leq n-1$ , put

$$A = \sum_{\substack{0 < t < l^n \\ (t,l)=1}} \left( \left\{ \frac{t}{l^{n-m}} \right\} - l^m \left\{ \frac{t}{l^n} \right\} \right) (-t)^{-1} \in \mathbb{Z}_l.$$

Then

$$A \equiv \begin{cases} -l^{n-1} \pmod{l^n} & \text{if } l=3 \text{ and } m=n-1 \geq 1, \\ 0 \pmod{l^n} & \text{otherwise.} \end{cases}$$

*Proof.* If  $m=0$ , then it is trivial, so we may assume  $m \geq 1$ . Put

$$B = \sum_{\substack{0 < t < l^n \\ (t,l)=1}} \left\{ \frac{t}{l^{n-m}} \right\} (-t)^{-1}$$

and

$$C = l^m \sum_{\substack{0 < t < l^n \\ (t,l)=1}} \left\{ \frac{t}{l^n} \right\} (-t)^{-1}.$$

Then  $A = B - C$ . We can write

$$t = t_1 + t_2 l^{n-m} \quad \text{with } 0 \leq t_1 < l^{n-m}, 0 \leq t_2 < l^m.$$

Then  $(t, l) = 1$  implies  $(t_1, l) = 1$ . Since

$$\left\{ \frac{t}{l^{n-m}} \right\} = \frac{t_1}{l^{n-m}},$$

we have

$$\begin{aligned}
B &= - \sum_{t_1, t_2} \frac{t_1}{l^{n-m}} (t_1 + t_2 l^{n-m})^{-1} \\
&= - \frac{1}{l^{n-m}} \sum_{t_1, t_2} \left( 1 + \frac{t_2}{t_1} l^{n-m} \right)^{-1} \\
&= - \frac{1}{l^{n-m}} \sum_{t_1, t_2} \sum_{i=0}^{\infty} (-1)^i \left( \frac{t_2}{t_1} \right)^i (l^{n-m})^i \\
&= - \frac{1}{l^{n-m}} \sum_{i=0}^{\infty} (-1)^i \left( \sum_{t_1} t_1^{-i} \right) \left( \sum_{t_2} t_2^i \right) l^{(n-m)i} \\
&= -l^m \left( 1 - \frac{1}{l} \right) - \sum_{i=1}^{\infty} (-1)^i \left( \sum_{t_1} t_1^{-i} \right) \left( \sum_{t_2} t_2^i \right) l^{(n-m)(i-1)}.
\end{aligned}$$

Since  $C = -l^m(1 - 1/l)$ , this implies

$$A = - \sum_{i=1}^{\infty} (-1)^i \left( \sum_{t_1} t_1^{-i} \right) \left( \sum_{t_2} t_2^i \right) l^{(n-m)(i-1)}. \quad (*)$$

Since

$$\sum_{t_1} t_1^{-i} \equiv 0 \pmod{l^{n-m-1}}$$

and

$$\sum_{t_2} t_2^i \equiv 0 \pmod{l^{m-1}}$$

by Lemma 5, the  $i$ th term of (\*) is congruent to 0 modulo  $l^{n-m-1} \cdot l^{m-1} \cdot l^{2(n-m)}$ , hence, mod  $l^n$  for  $i \geq 3$ . In the same way, the first term of (\*) is congruent to 0 modulo  $l^n$ . Thus we have

$$A \equiv - \left( \sum_{t_1} t_1^{-2} \right) \left( \sum_{t_2} t_2^2 \right) l^{n-m} \pmod{l^n}.$$

By this and Lemma 5, we have the desired congruence.

By Lemma 6, we will prove the following Lemma 7, which enables us to reduce the computation of  $W_n(a)$  for arbitrary  $a \in \mathbb{Z}$  to Lemma 4.

**LEMMA 7.** *Let  $W_n(a)$  be as in the beginning of Section 2, and let  $a \in \mathbb{Z}$  be of the*

form  $a = a'l^m$  with  $a', m \in \mathbb{Z}$ ,  $(a', l) = 1$  and  $0 \leq m \leq n - 1$ . Then

$$W_n(a) \equiv \begin{cases} l^m W_n(a') - a \pmod{l^n} & \text{if } l = 3 \text{ and } m = n - 1 \geq 1, \\ l^m W_n(a') \pmod{l^n} & \text{otherwise.} \end{cases}$$

*Proof.* If  $m = 0$ , then it is trivial, so we may assume  $m \geq 1$ . Put

$$D = W_n(a) - l^m W_n(a') \in \mathbb{Z}_l.$$

Then

$$\begin{aligned} D &= \sum_{\substack{0 < t < l^n \\ (t, l) = 1}} \left( \left\{ \frac{at}{l^n} \right\} - l^m \left\{ \frac{a't}{l^n} \right\} \right) (-t)^{-1} \\ &\equiv \sum_{t \in (\mathbb{Z}/l^n\mathbb{Z})^\times} \left( \left\{ \frac{at}{l^n} \right\} - l^m \left\{ \frac{a't}{l^n} \right\} \right) (-t)^{-1} \pmod{l^n}, \end{aligned}$$

since  $\{at/l^n\}$  and  $\{a't/l^n\}$  are determined by  $t \pmod{l^n}$  and since  $\{at/l^n\} - l^m\{a't/l^n\} \in \mathbb{Z}$ . Putting  $t' = a't$ , we have

$$D \equiv a' \sum_{t' \in (\mathbb{Z}/l^n\mathbb{Z})^\times} \left( \left\{ \frac{l^m t'}{l^n} \right\} - l^m \left\{ \frac{t'}{l^n} \right\} \right) (-t')^{-1} \pmod{l^n},$$

i.e.,

$$D \equiv a'A \pmod{l^n},$$

where  $A$  is as in Lemma 6. Thus the assertion follows from Lemma 6.

By Lemmas 4 and 7, we have the following:

**THEOREM 2.** Let  $W_n(a)$  be as in the beginning of Section 2. Then for any  $a \in \mathbb{Z}$ , we have

$$W_n(a) \equiv \begin{cases} \left(1 - \frac{1}{l}\right) \log \langle a \rangle^a - a \pmod{l^n} & \text{if } l = 3 \text{ and } \text{ord}_l(a) = n - 1 \geq 1, \\ \left(1 - \frac{1}{l}\right) \log \langle a \rangle^a - \frac{a}{l} \pmod{l^n} & \text{if } \text{ord}_l(a) \geq n, \\ \left(1 - \frac{1}{l}\right) \log \langle a \rangle^a \pmod{l^n} & \text{otherwise,} \end{cases}$$

where we define  $\langle a \rangle$  by  $\langle a \rangle = \langle a' \rangle$  for  $a = a'l^m$  with  $m \geq 1$ ,  $a' \in \mathbb{Z}_l^\times$  and  $\langle 0 \rangle = 1$ .

*Proof.* Assume  $\text{ord}_l(a) \geq n$ . Then we have easily

$$W_n(a) \equiv -\frac{a}{l} \pmod{l^n}$$

and

$$\log \langle a \rangle^a \equiv 0 \pmod{l^{n+1}}.$$

Hence we have the assertion in the case  $\text{ord}_l(a) \geq n$ . The proof in the other cases follows from Lemmas 4 and 7.

**COROLLARY.** For  $a = (a_1, \dots, a_r) \in \mathbb{Z}^r$ , put

$$S_l^{(a)} = \sum_{\substack{0 < t < l^n \\ (t, l) = 1}} \left( \sum_{i=0}^r \left\{ \frac{a_i t}{l^n} \right\} \right) (-t)^{-1} \in \mathbb{Z}_l,$$

where

$$a_0 = -\sum_{i=1}^r a_i.$$

Then

$$S_l^{(a)} \equiv \left(1 - \frac{1}{l}\right) \log \left( \prod_{i=0}^r \langle a_i \rangle^{a_i} \right) - T_1 - T_2 \pmod{l^n},$$

where

$$T_1 = \frac{1}{l} \sum_{\substack{\text{ord}_l(a_i) = n \\ 0 \leq i \leq r}} a_i,$$

and

$$T_2 = \begin{cases} \sum_{\substack{\text{ord}_l(a_i) = n-1 \\ 0 \leq i \leq r}} & \text{if } l = 3 \text{ and } n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Since  $\sum_{i=0}^r a_i = 0$ , we have

$$S_n^{(a)} = \sum_{i=0}^r W_n(a_i).$$

Hence the assertion follows directly from Theorem 2.

### 3. Purely number theoretic proof of Coleman-McCallum's theorems

By Lemma 1, Theorem 1, and Corollary to Theorem 2, we will give another proof of Coleman-McCallum's Theorem 3 below ([5], Theorems 5.3, 7.1 and 7.2 when  $m = p^n$  and  $x \equiv 1 \pmod{\pi_n}$  in their notation), which gives the precise value of the Jacobi sum  $J_n^{(a)}(\mathfrak{a})$  at any principal ideal  $\mathfrak{a} = (\alpha)$  with  $\alpha \in \mathbb{Q}(\zeta_{l^n})$ ,  $\alpha \equiv 1 \pmod{\pi_n}$  in terms of the Hilbert symbol. Note that when  $l = 3$ , they give the formula only for  $r = 2$ , but it is easy to derive our following formula for general  $r$  even if  $l = 3$  from the case  $r = 2$  in the same way as in the proof of Theorem 7.1 of [5], and note also that our formulation is slightly different from theirs, but they are essentially the same.

By Stickelberger's theorem on the prime ideal decomposition of Gauss sums, we have

$$J_n^{(a)}((\alpha)) = \zeta_{l^n}^{i(\alpha)} \alpha^{\omega_n(a)} \quad \text{with } i(\alpha) = i_n^{(a)}(\alpha) \in \mathbb{Z}/l^n\mathbb{Z}, \tag{*}$$

for any  $\alpha \in \mathbb{Q}(\zeta_{l^n})$  such that  $\alpha \equiv 1 \pmod{\pi_n}$ , where

$$\omega_n(a) = \sum_{\substack{0 < t < l^n \\ (t, l) = 1}} \left( \sum_{i=0}^r \left\{ \frac{a_i t}{l^n} \right\} \right) \sigma_t^{-1} - \sum_{\substack{0 < t < l^n \\ (t, l) = 1}} \sigma_t \in \mathbb{Z}[G_n]$$

(cf. Weil [23]). Here  $a_0 = -\sum_{i=1}^r a_i$ ,  $G_n = \text{Gal}(\mathbb{Q}_l(\zeta_{l^n})/\mathbb{Q}_l)$  and  $\sigma_t \in G_n$  is such that  $\zeta_{l^n}^{\sigma_t} = \zeta_{l^n}^t$ .

**THEOREM 3.** *Let the notation and assumptions be as above. Assume that  $a = (a_1, \dots, a_r) \not\equiv (0, \dots, 0) \pmod{l^n}$ . Then*

$$\begin{aligned} i_n^{(a)}(\alpha) &\equiv g[l, \alpha]_n + h[1 + l, \alpha]_n \pmod{l^n} \\ &\equiv \left[ \left( \prod_{i=0}^r a_i^{a_i} \right) (1 + l)^{T_1 + T_2}, \alpha \right]_n \pmod{l^n} \\ &\equiv \left[ \left( \prod_{i=0}^r a_i^{a_i} \right) (1 + (T_1 + T_2)l), \alpha \right]_n \pmod{l^n} \end{aligned}$$

for  $\alpha \in \mathbb{Q}(\zeta_m)$ ,  $\alpha \equiv 1 \pmod{\pi_n}$ , where

$$g = \sum_{i=0}^r \text{ord}_l(a_i) \cdot a_i,$$

$$h = cS_l^{(a)} \left( \equiv \left[ \log \left( \prod_{i=0}^r a_i^{a_i} \right) \right] / \log(1+l) + T_1 + T_2 \pmod{l^n} \right),$$

$$c = \left( \left( 1 - \frac{1}{l} \right) \log(1+l) \right)^{-1},$$

$$S_l^{(a)} = \sum_{\substack{0 < i < l^n \\ (t,l)=1}} \left( \sum_{i=0}^r \left\{ \frac{a_i t}{l^n} \right\} \right) (-t)^{-1} (\in \mathbb{Z}_l),$$

$$T_1 = \frac{1}{l} \sum_{\substack{\text{ord}_l(a_i) = n \\ 0 \leq i \leq r}} a_i,$$

and

$$T_2 = \begin{cases} \sum_{\substack{\text{ord}_l(a_i) = n-1 \\ 0 \leq i \leq r}} a_i & \text{if } l = 3 \text{ and } n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Here  $\text{ord}_l(0) \cdot 0 = \infty \cdot 0 = 0$  and  $0^0 = 1$ .

*Proof.* Taking  $\delta^{(n)}$  for  $K = \mathbb{Q}_l$  (or for any  $K$ ) of both members of the above (\*) and using the properties (i) and (ii) in Lemma 1, we have

$$\delta^{(n)}(J_l^{(a)}(\alpha)) \equiv i_l^{(a)}(\alpha) + S_l^{(a)} \delta^{(n)}(\alpha) \pmod{l^n}.$$

Hence by (iii) of Lemma 1 and Theorem 1, we have

$$g \langle l, \alpha \rangle \equiv i_l^{(a)}(\alpha) - cS_l^{(a)}[1+l, \alpha]_n \pmod{l^n}.$$

Since  $\langle l, \alpha \rangle = [l, \alpha]_n$  by class field theory, we have the first congruence. (Note that we use only Lemma 1 and Theorem 1 to obtain the first congruence). Now we use Corollary to Theorem 2 to transform the first congruence to the second one. By Corollary to Theorem 2 and the first congruence,

$$\begin{aligned} i_l^{(a)}(\alpha) &\equiv [l^g, \alpha]_n + c \left( \left( 1 - \frac{1}{l} \right) \log \left( \prod_{i=0}^r \langle a_i \rangle^{a_i} \right) - T_1 - T_2 \right) [1+l, \alpha]_n \pmod{l^n} \\ &\equiv [l^g, \alpha]_n + [(1+l)^h, \alpha]_n - c(T_1 + T_2)[1+l, \alpha]_n \pmod{l^n}, \end{aligned}$$

where

$$h' = \log \left( \prod_{i=0}^r \langle a_i \rangle^{a_i} \right) / \log(1 + l).$$

Since

$$(1 + l)^{h'} = \prod_{i=0}^r \langle a_i \rangle^{a_i},$$

$$c \equiv -1 \pmod{l},$$

and

$$T_1 + T_2 \equiv 0 \pmod{l^{n-1}},$$

we have

$$i_n^{(a)}(\alpha) \equiv \left[ \prod_{i=0}^r (l^{\text{ord}_l(a_i)} \langle a_i \rangle)^{a_i}, \alpha \right]_n + (T_1 + T_2)[1 + l, \alpha]_n \pmod{l^n}.$$

Since we can write

$$a_i^{a_i} = (\omega(a_i) \langle a_i \rangle^{l^{\text{ord}_l(a_i)}})^{a_i} \quad \text{for } 0 \leq i \leq r$$

( $\omega(a_i)^{l-1} = 1$ ,  $\omega(0) = \langle 0 \rangle = 1$ , and  $l^\infty = 0$ ) and since

$$\left[ \prod_{i=0}^r \omega(a_i)^{a_i}, \alpha \right]_n \equiv 0 \pmod{l^n},$$

we have the second congruence. The last one follows directly from the second one, since  $(1 + l)^{T_i} \equiv 1 + T_i l \pmod{l^{n+1}}$  and since  $x \equiv 1 \pmod{l^{n+1}}$  implies  $x \in (1 + l\mathbb{Z}_l)^{l^n}$  for  $x \in \mathbb{Z}_l$ .

If  $c$  is the minimum integer  $c \geq 0$  such that

$$J_n^{(a)}(\alpha) = \alpha^{\omega_n(a)} \tag{*}$$

for all  $\alpha \in \mathbb{Q}(\zeta_{l^n})$ ,  $\alpha \equiv 1 \pmod{\pi_n^c}$ , then we call the ideal  $(\pi_n^c)$  the *conductor* of the Jacobi sum Hecke character  $J_n^{(a)}$ , which we denote by  $C_n^{(a)}$ . Note that  $c = 0$  if and only if the above (\*) holds for all  $\alpha \in \mathbb{Q}(\zeta_{l^n})$  which are prime to  $l$ .

By the above Theorem 3, Lemma 8 below, and Coleman-McCallum's determination of the conductor  $f_n(g, h)$  of the character  $\alpha \mapsto [\alpha, l^g(1 + l)^h]_n$  with



$g \in \mathbb{Z}$ ,  $h \in \mathbb{Z}_l$  ([5], Theorem 6.1) (note that we can also determine  $f_n(g, h)$  by developing a certain computation in Iwasawa [8] (see Miki [18]), though Coleman-McCallum used Coleman's formula on the Hilbert norm residue symbol), we can get the precise conductor  $C_{l^n}^{(a)}$  as follows:

## COROLLARY

$$C_{l^n}^{(a)} = \begin{cases} (\pi_j \pi_{j+1}) & \text{if } 1 \leq j \leq n-1 \quad \text{and } \text{ord}_l(g+hl) > j, \\ & \text{otherwise:} \\ (\pi_j^2) & \text{if } 1 \leq j \leq n-1 \quad \text{or } n = \text{ord}_l(h) + 1 \leq \text{ord}_l(g), \\ (\pi_n) & \text{if } j \geq n+1 \quad \text{or } n = \text{ord}_l(g) \leq \text{ord}_l(h) \\ & \text{and if } r_1 \text{ is odd,} \\ (1) & \text{if } j \geq n+1 \quad \text{or } n = \text{ord}_l(g) \leq \text{ord}_l(h) \\ & \text{and if } r_1 \text{ is even,} \end{cases}$$

where  $j = \min(\text{ord}_l(g), \text{ord}_l(h) + 1)$  and  $r_1$  is the number of  $i$  such that  $l^n \nmid a_i$  for  $0 \leq i \leq r$ .

Note that we have always  $l \mid g$ , since each term in  $g$  is 0 or 0 mod  $l$  according as  $l \nmid a_i$  or not.

LEMMA 8. *Let the notation and assumptions be as in the above corollary. Furthermore, assume that*

$$J_{l^n}^{(a)}((\alpha)) = \alpha^{\omega_n(a)} \quad (*)$$

for all  $\alpha \in \mathbb{Q}(\zeta_{l^n})$  such that  $\alpha \equiv 1 \pmod{\pi_n}$ . Then  $C_{l^n}^{(a)} = (\pi_n)$  or (1) according as  $r_1$  is odd or even.

*Proof.* For all  $\alpha \in \mathbb{Q}(\zeta_{l^n})$  such that  $(\alpha, l) = 1$ , we can write

$$J_{l^n}^{(a)}((\alpha)) = \mathcal{E}(\alpha) \alpha^{\omega(a)}, \quad \mathcal{E}(\alpha)^{2l^n} = 1 \quad (1)$$

(cf. Weil [23]). Since we can write

we have  $\mathcal{E}(\alpha) = \alpha = \alpha_0 \alpha_1$ ,  $\alpha_0 \in \mathbb{Z}$ ,  $1 \leq \alpha_0 \leq l-1$ ,  $\alpha_1 \in \mathbb{Q}(\zeta_{l^n})^{\times}$ ,  $\alpha_1 \equiv 1 \pmod{\pi_n}$ ,  $\mathcal{E}(\alpha_1) = \mathcal{E}(\alpha_0)$  by the above (\*). Since  $\alpha_0^{l-1} \equiv 1 \pmod{l}$ , by (\*) we have  $\mathcal{E}(\alpha_0^{l-1}) = \mathcal{E}(\alpha_0)^{l-1} = 1$ . On the other hand, by (1) we have  $\mathcal{E}(\alpha_0)^{2l^n} = 1$ . Hence  $\mathcal{E}(\alpha_0) = \pm 1$ . Since  $J_{l^n}^{(a)}((\alpha_0)) \equiv 1 \pmod{\pi_n}$ , by (1) we have

$$\mathcal{E}(\alpha_0) \equiv \alpha_0^{-\omega_n(a)} \pmod{\pi_n}. \quad (2)$$

Since  $\alpha_0 \in \mathbb{Z}$ , by the definition of  $\omega_n(a)$  we have

$$\alpha_0^{-\omega_n(a)} = \alpha_0^{-S + l^{n-1}(l-1)}, \tag{3}$$

where

$$S = \sum_{t \in (\mathbb{Z}/l^n\mathbb{Z})^*} \left( \sum_{i=0}^r \left\{ \frac{a_i t}{l^n} \right\} \right).$$

Now, if necessary, we change the numbers of  $a_i$  so that  $l^n \nmid a_i$  for  $0 \leq i < r_1$ , and  $l^n \mid a_i$  for  $r_1 \leq i \leq r$ . Then

$$\begin{aligned} S &= \sum_{t \in (\mathbb{Z}/l^n\mathbb{Z})^*} \sum_{i=0}^{r_1-1} \left\{ \frac{a_i t}{l^n} \right\} \\ &= \sum_t \sum_{i=0}^{r_1-1} \left\{ \frac{-a_i t}{l^n} \right\}. \end{aligned}$$

Hence

$$\begin{aligned} 2S &= \sum_t \sum_{i=0}^{r_1-1} \left( \left\{ \frac{a_i t}{l^n} \right\} + \left\{ \frac{-a_i t}{l^n} \right\} \right) \\ &= l^{n-1}(l-1)r_1, \end{aligned}$$

since  $\{x\} + \{-x\} = 1$  if  $x \in \mathbb{Q} - \mathbb{Z}$ . Hence

$$S = r_1 l^{n-1} \cdot \frac{l-1}{2}. \tag{4}$$

By (2), (3), and (4), we have

$$\mathcal{E}(\alpha_0) \equiv \alpha_0^{-r_1 \cdot (l-1)/2} \pmod{\pi_n}, \tag{5}$$

since  $\alpha_0^{l^{n-1}} \equiv \alpha_0 \pmod{l}$ . Since  $\mathcal{E}(\alpha_0) = \pm 1$ ,  $\mathcal{E}(\alpha_0) = 1$  if and only if  $\mathcal{E}(\alpha_0) \equiv 1 \pmod{\pi_n}$ . Hence by (5) we see that  $\mathcal{E}(\alpha_0) = 1$  for all  $\alpha_0 \in \mathbb{Z}$  such that  $1 \leq \alpha_0 \leq l-1$  if and only if  $r_1 \equiv 0 \pmod{2}$ .

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