COMPOSITIO MATHEMATICA

ROBERT BRAUN
GUNNAR FLØYSTAD

A bound for the degree of smooth surfaces in P^4 not of general type

Compositio Mathematica, tome 93, nº 2 (1994), p. 211-229
<http://www.numdam.org/item?id=CM_1994__93_2_211_0>

Compositio Mathematica **93**: 211–229, 1994.

A bound for the degree of smooth surfaces in $\mathbf{P}^4$ not of general type

ROBERT BRAUN[1] and GUNNAR FLØYSTAD[2]

[1] *Mathematisches Institut, Universität Bayreuth, D-8580 Bayreuth, Germany*
[2] *Matematisk Institutt, Allegaten 55, 5007 Bergen, Norway*

Received 30 June 1993; accepted in final form 17 September 1993

0. Introduction

In [3], G. Ellingsrud and C. Peskine proved that there are only a finite number of components in the Hilbert scheme of surfaces in $\mathbf{P}^4$ parametrizing smooth surfaces not of general type. This has inspired research trying to classify these surfaces.

Let S be a smooth surface in $\mathbf{P}^4$. We prove in this paper that if S is not of general type then $\deg(S) \leqslant 105$. This is an improvement compared to the previous upper bound deducible from [3], which was of order several thousands. It is conjectured that the upper bound is 15.

Let $d = \deg(S)$ and let $s = \min\{k \mid h^0\mathscr{I}_S(k) \neq 0\}$. Then we derive a lower bound for $\chi\mathcal{O}_S$ in terms of d and s which is a polynomial of degree 3 in d. Note that Ellingsrud and Peskine in [3] found a lower bound for $\chi\mathcal{O}_S$ which was a polynomial of degree 6 in $\sqrt{d}$. The initial terms are the same, $d^3/6s^2$ but [3] has a negative coefficient of $d^{5/2}$ so our lower bound is an improvement. There is however nothing canonical with the polynomial we find here. Using this lower bound we easily deduce our upper bound for the degree of surfaces not of general type.

We work over an algebraically closed field of characteristic 0.

Note. In a talk in Oberwolfach in September 1991 Peskine also gave a proof that χ may be bounded below by a third degree polynomial with initial term $d^3/6s^2$.

1. Deriving the upper bound

(1.1) Let S be a smooth surface in $\mathbf{P}^4$. We have the following invariants.

$d =$ the degree of S.

$\pi =$ the genus of a general hyperplane section.

$\chi = \chi\mathcal{O}_S = h^0\mathcal{O}_S - h^1\mathcal{O}_S + h^2\mathcal{O}_S$.

$s = \min\{k \mid h^0\mathcal{I}_S(k) \neq 0\}$.

$K =$ the canonical divisor on S.

We have the following facts.

(a)

$$\pi \leqslant d^2/2s + (s-4)d/2 + 1 - r(s-r)(s-1)/2s$$

for $d > (s-1)^2 + 1$ where $d \equiv r \pmod{s}$ with $0 \leqslant r < s$. This is due to [5]. We also deduce it in (4.2).

(b) With r as above let

$$\gamma = d^2/2s + (s-4)d/2 + 1 - r(s-r)(s-1)/2s - \pi.$$

Then from [3, Lemma 1] we have

$$\gamma \leqslant d(s-1)^2/2s.$$

(c)

$$d^2 - 5d - 10(\pi - 1) + 2(6\chi - K^2) = 0.$$

This is the double-point formula [6, Appendix A, Ex. 4.1.3].

(d) If S is not of general type then $K^2 \leqslant 9$ and $6\chi \geqslant K^2$, except for the case in which S is a rational surface with $K^2 > 6$. In this case $d \leqslant 4$. We exclude this case in the considerations in (1.2) and (1.3) below. These facts are from [3, Corollaire p. 2 and Appendice B].

(e)

$$\chi\mathcal{O}_S \geqslant d^3/6s^2 + d^2(s-5)/4s + d(3s^2 - 30s + 71)/24$$
$$- \tfrac{1}{48}(s^4 - 5s^3 - s^2 + 5s) - \gamma^2/2 - \gamma(d/s + s - 5/2).$$

We deduce this in Section 5.

(1.2) Suppose now that S is not of general type. Then $6\chi \geqslant K^2$ which by the double-point formula gives $d^2 - 5d - 10(\pi - 1) \leqslant 0$. Inserting the upper bound for π we easily deduce that this gives $d \leqslant 90$ or $s \leqslant 5$. This argument is by [3, Proposition 1].

(1.3) Also, in the case of S not of general type, we have by (c) and (d) that

$$18 \geqslant 2K^2 = d^2 - 5d - 10(\pi - 1) + 12\chi.$$

Now let $s = 5$. The lower bound for $\chi\mathcal{O}_S$ then becomes

$$d^3/150 - d/6 - \gamma^2/2 - \gamma(d/5 + 5/2).$$

In the special case $s = 5$, we will see that the double point formula gives a better bound for γ than the bound given in (b). From the double point formula and the definition of γ, we deduce, by eliminating π, that

$$\gamma = d - r(5 - r)4/10 - (6\chi - K^2)/5 \leqslant d - r(5 - r)2/5.$$

We then get the following

$$\begin{aligned} 18 &\geqslant d^2 - 5d - 10(\pi - 1) + 12\chi \\ &\geqslant -10d + 10\gamma + 12(d^3/150 - d/6 - \gamma^2/2 - \gamma(d/5 + 5/2)) \\ &= 4d^3/50 - 12d - 6\gamma^2 + 10\gamma - 12\gamma d/5 - 30\gamma \end{aligned}$$

If $d \equiv 0 \pmod 5$ then $\gamma \leqslant d$ and the above is

$$\geqslant 4d^3/50 - 42d^2/5 - 32d.$$

This implies $d \leqslant 108$. But since $d \equiv 0 \pmod 5$ we get $d \leqslant 105$. For other cases of r we only get better bounds.

If now $s < 5$ then we have $\gamma \leqslant d(s - 1)^2/2s$ by (b). Using an argument analogous to the above we deduce

$$s = 4 \Rightarrow d \leqslant 97$$

$$s = 3 \Rightarrow d \leqslant 36$$

$$s = 2 \Rightarrow d \leqslant 14.$$

Note that if $s = 3$ then it is known that $\gamma \leqslant 1$, [7], and if $s = 2$ then $\gamma = 0$. Using this we may deduce

$$s = 3 \Rightarrow d \leqslant 12$$

$$s = 2 \Rightarrow d \leqslant 7.$$

For $s = 2$ the bound is sharp for surfaces not of general type. For $s = 3$ the bound is known to be 8 (using [7]).

(1.4) By (1.2) and (1.3) we may then conclude that if S is a surface in $\mathbf{P}^4$ not of general type, then $d \leqslant 105$.

To complete the picture we show how this implies that there is only a finite number of Hilbert scheme components of surfaces in $\mathbf{P}^4$ containing surfaces not of general type as proven in [3].

(1.5) Now if S is not of general type we have established that there is only a finite number of possible values for d. Then there is only a finite number of values of s and by (1.1) (a) also for π. By (1.1) (d) and the first inequality in (1.3) this gives that there is only a finite number of possible values for χ.

So there is only a finite number of possible invariants (d, π, χ) for S a smooth surface not of general type. This implies that there is only a finite number of Hilbert polynomials and hence a finite number of Hilbert scheme components containing smooth surfaces not of general type.

2. Initial ideals

In the following we introduce the concept of initial ideal and state some of its properties. Much in this section is written by inspiration from [4] and conversations with Green. Note that in this section we use d to denote the degree of a monomial.

2.1. Basic properties

(2.1.1) Let V be a vector space over $\mathbf{C}$ with basis $x_0, x_1, \ldots, x_n$, and let $S = \oplus_k S^k V$ denote the symmetric algebra on V. We will use multi-index notation, so that $X^I = x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$, where $I = (i_0, i_1, \ldots, i_n)$. We let $|I| = \Sigma_{j=0}^n i_j$. We order the variables by

$$x_0 > x_i > \cdots > x_n.$$

We now order the monomials of the same degree by reverse lexicographic order so that $x^I > x^J$ if $i_k < j_k$, where $k = \max(\{m \mid i_m \neq j_m\})$. For $n = 2$, we have for example that

$$x_0^2 > x_0 x_1 > x_1^2 > x_0 x_2 > x, x_2 > x_2^2.$$

If $f \in S^k V$ is a homogeneous polynomial, write $f = \Sigma_I f_I x^I$. Let

$$I_m = \max\{I \mid f_I \neq 0\}.$$

The initial monomial of f is

$$\mathrm{in}(f) = x^{I_m}.$$

We note the formula

$$\mathrm{in}(fg) = \mathrm{in}(f)\mathrm{in}(g).$$

If $I \subseteq S$ is a homogeneous ideal, then the *initial ideal* of I is the ideal $\mathrm{in}(I)$ generated by

$$\{\mathrm{in}(f) \mid f \in I\}.$$

By abuse of notation we will write $J \in \mathrm{in}(I)$ if $x^J \in \mathrm{in}(I)$. We note that I and $\mathrm{in}(I)$ have the same Hilbert function, i.e. the same dimension in every degree. Indeed, by Gaussian elimination, there exists a basis $\{f_J\}_{J \in \mathrm{in}(I)_d}$ for I_d so that $\mathrm{in}(f_J) = x^J$ for all $J \in \mathrm{in}(I)_d$.

(2.1.2) Now let $I \subseteq S$ be a monomial ideal, i.e. an ideal generated by monomials. For $k \in [0, n-1]$ define $e_k(x^J) = x^K$ where

$$K = (j_0, \ldots, j_{k-1}, j_k + 1, j_{k+1} - 1, j_{k+2}, \ldots, j_n).$$

By definition we set $x^K = 0$ if some $k_m < 0$. A monomial ideal is *Borel-fixed* if, for all $k \in [0, n-1]$ and $x^J \in I$ we have $e_k(x^J) \in I$.

We have the standard action of $GL(V)$ on $S^k V$. If $g \in GL(V)$ and $f \in S^k V$, we denote the product by $g(f)$. If $I \subseteq S$ is a homogeneous ideal we let

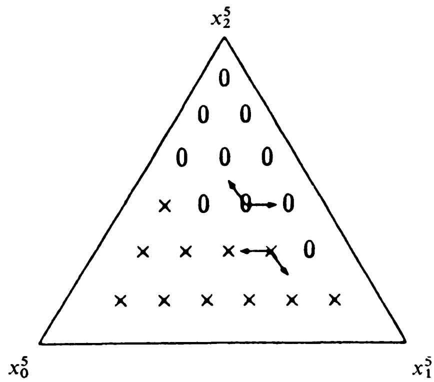
$$g(I) = \{g(f) \mid f \in I\}.$$

THEOREM (Galligo). *For any homogeneous* $I \subseteq S$ *there is a Zariski-open subset* $U \subseteq GL(V)$ *s.t.* $\mathrm{in}(g(I))$ *is constant and Borel-fixed for* $g \in U$.

For a proof see [1, Prop. 1]. For generic g we call $\mathrm{in}(g(I))$ the *generic initial ideal* of I.

(2.1.3) We now introduce the *diagram* of a monomial ideal. We may envision the monomials of degree d in $(n+1)$ variables as an n-simplex whose vertices correspond to $x_0^d, \ldots, x_n^d$, and where x^J corresponds to the point with barycentric coordinates $(j_0/d, \ldots, j_n/d)$. We insert a 0 in a point if the corresponding monomial does not belong to the monomial ideal and a $\times$ if the corresponding monomial does belong to the ideal.

We illustrate this with a diagram. Let $I_0 \subseteq k[x_0, x_1, x_2]$ be a monomial ideal. The monomials of degree five in I_0 may for example be represented by a diagram



These monomials of degree five satisfy the Borel-fixedness property, since this means that if one starts at a 0 and goes in the directions indicated by arrows above, one will only encounter 0'es. Also, if one starts at a × and goes in the directions indicated above, one will only encounter ×'es.

In the rest of Section 2.1 we assume that a homogeneous ideal $I \subseteq S$ is $g(I')$ for some homogeneous $I' \subseteq S$ and generic $g \in GL(V)$. Hence in(I) will be Borel-fixed.

We now have:

(2.1.4) PROPOSITION. *Suppose that $I \subseteq S$ is saturated. Then* $\text{in}(I) : x_n = \text{in}(I)$.

Proof. [2, 2.2(b)]. □

(2.1.5) COROLLARY. $\text{in}(I)_d : x_n = \text{in}(I)_{d-1}$.

Hence $(j_0, \ldots, j_n)$ is in $\text{in}(I)_d$ if and only if $(j_0, \ldots, j_{n-1}, 0)$ is in $\text{in}(I)_{d-j_n}$. In terms of the diagram of in(I) the corollary means that we get the diagram for $\text{in}(I)_{d-1}$ by disregarding the face of the n-simplex whose vertices are $x_0^d, \ldots, x_{n-1}^d$.

Referring to the example in (2.1.3), supposing that $I_0 = \text{in}(I)$, we see that we get the diagram for $I_{0,4}$ by disregarding the last line and the diagram for $I_{0,3}$ by disregarding the last two lines and so on.

(2.1.6) For $h \in V$ let H be the corresponding hyperplane and I_H the restriction of I to H. Let $V_H = V/(h)$. If h is general, the coefficient of x_n is non-zero, so that we may identify $S^k V_H$ with $S^k \bar{V}$ in a natural way, where $\bar{V}$ is the vector space spanned by $x_0, \ldots, x_{n-1}$. We consider I_H as a graded ideal in the symmetric algebra on $\bar{V}$.

(2.1.7) PROPOSITION. *For H general we have* $\text{in}(I_H) = \text{in}(I)_{x_n}$.

Proof. [2, 2.2(a)]. □

In terms of the diagram of in(I) this means that the diagram of $\text{in}(I_H)_d$ is the face of the diagram of $\text{in}(I)_d$ spanned by the vertices $x_0^d, \ldots, x_{n-1}^d$.

2.2. Decomposition of the set of monomials

(2.2.1) For $n \geqslant -1$ and $d \geqslant 0$ let

$$M(n) = \{j_0, \ldots, j_n) \mid \forall i \in [0, n]: j_i \in \mathbf{Z}, \quad j_i \geqslant 0\}$$

$$M(n, d) = \left\{(j_0, \ldots, j_n) \in M(n) \mid \sum_{i=0}^{n} j_i = d\right\}.$$

Note that $M(-1) = \{()\}$ and $M(-1, d) = \varnothing$. Let $I \subseteq S$ be any monomial ideal. For $i \in [-1, n-1]$ we define functions

$$f_{i,I}: M(n-2-i) \to \mathbf{N}_0 \cup \{\infty\}$$

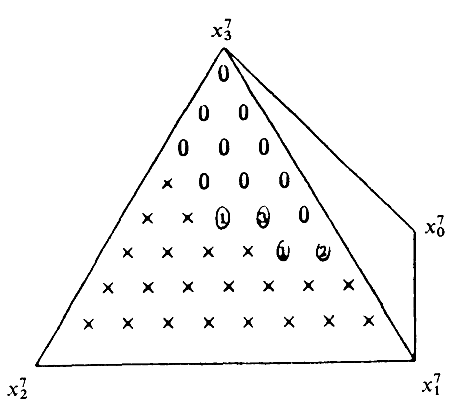
by

$$f_{-1,I}(j_0, \ldots, j_{n-1}) = \begin{cases} \infty & \text{if } (j_0, \ldots, j_{n-1}) \text{ is not in } I, \\ 0 & \text{if } (j_0, \ldots, j_{n-1}) \text{ is in } I, \end{cases}$$

$$f_{i,I}(j_0, \ldots, j_{n-2-i}) = \min\{j_{n-1-i} \mid f_{i-1,I}(j_0, \ldots, j_{n-1-i}) < \infty\}.$$

If it is clear to which ideal these functions are associated we drop the ideal as an index.

(2.2.2) We illustrate again by a diagram. Suppose I is the initial ideal of the homogeneous saturated ideal of a curve C in $\mathbf{P}^3$, so I is a monomial ideal in $k[x_0, x_1, x_2, x_3]$. If I is Borel-fixed we may for example have a diagram of the monomials of degree 7 displayed in the following way



A 0 (resp. ×) on the front face in location $(j_0, j_1, 0, d - j_0 - j_1)$ (we have $d = 7$ in the above diagram) indicates that we have 0'es (resp. ×'es) in the locations $(j_0, j_1, j_2, d - j_0 - j_1 - j_2)$ for all $j_2 \in [0, d - j_0 - j_1]$ (for every d). Such a 0 on the front face is called an ordinary 0.

A 0 with a number n in it at location $(j_0, j_1, 0, d - j_0 - j_1)$ (with $d = 7$ in the above diagram) indicates that we have 0'es at locations $(j_0, j_1, j_2, d - j_0 - j_1 - j_2)$ where $j_2 \in [0, \min(n-1, d - j_0 - j_1)]$ and ×'es for $j_2 \in [n, d - j_0 - j_1]$.

We may as a way of example verify the following

$$f_0(1,3) = 3,\ f_0(1,1) = \infty,\ f_1(2) = 2,\ f_1(1) = 3,\ f_1(0) = 5,\ f_2(()) = 0.$$

Note furthermore that if $\Lambda = C \cap H$ for a general plane H in $\mathbf{P}^3$, then $I_\Lambda = I_H^{\text{sat}}$ so $I_{\Lambda,d} = I_{H,d}$ for $d \gg 0$. Hence in $(I_\Lambda)_d = \text{in}(I_H)_d = \text{in}(I)_{x_n,d}$ for $d \gg 0$, so the initial ideal of the plane section Λ of C in degree five is just the diagram given in (2.1.3).

(2.2.3) LEMMA. *Let* $i \geqslant 0$ *and suppose* $f_i(j_0, \ldots, j_{n-2-i}) = \infty$. *Then* $f_{i-1}(j_0, \ldots, j_{n-1-i}) = \infty$ *for all* $j_{n-1-i} \geqslant 0$.

Proof. This follows immediately from the definition. □

Hence $f_i(j_0, \ldots, j_{n-2-i}) = \infty$ if and only if $(j_0, \ldots, j_{n-2-i}, j_{n-1-i}, \ldots, j_n) \notin I$ for all $j_{n-1-i}, \ldots, j_n \geqslant 0$. In example (2.2.2) we have $f_{-1}(j_0, j_1, j_2) = \infty$ if the location (j_0, j_1, j_2, j_3) is represented by 0 in the 3-simplex of degree $d = \Sigma_{i=0}^3 j_i$, for every $j_3 \geqslant 0$. We have $f_0(j_0, j_1) = \infty$ if the location $(j_0, j_1, 0, j_3)$ is represented by an ordinary 0 on the front face, where $d = j_0 + j_1 + j_3$, for every $j_3 \geqslant 0$.

Suppose now that I is Borel-fixed.

(2.2.4) PROPOSITION.
(a) *Let* $0 \leqslant k < n - 2 - i$ *and* $J = (j_0, \ldots, j_{n-2-i})$. *If* $e_k(x^J) \neq 0$ *then*

$$f_i(J) \geqslant f_i(e_k(J)).$$

(*b*) *If* $f_i(j_0, \ldots, j_{n-2-i}) > 0$ *then*

$$f_i(j_0, \ldots, j_{n-2-i} + 1) \leqslant f_i(j_0, \ldots, j_{n-2-i}) - 1.$$

(c) *If* $f_i(j_0, \ldots, j_{n-2-i}) = 0$ *then* $f_i(j_0, \ldots, j_{n-2-i} + 1) = 0$.

Proof. We use induction on i.

(a) This is clear for $i = -1$. It is furthermore clear if $f_i(J) = \infty$. Suppose that $i \geqslant 0$ and $f_i(J) = r < \infty$. Then $f_{i-1}(J, r) < \infty$. By induction $f_{i-1}(e_k(J), r) < \infty$ and $f_i(e_k(J)) \leqslant r$.

(b), (c) It is clear if $i = -1$. Suppose now that $f_i(j_0, \ldots, j_{n-2-i} + 1) = r > 0$ then $f_{i-1}(j_0, \ldots, j_{n-2-i} + 1, r-1) = \infty$. Hence $f_{i-1}(j_0, \ldots, j_{n-2-i}, r) = \infty$. By induction we will have $f_{i-1}(j_0, \ldots, j_{n-2-i}, t) = \infty$ for all $t \leqslant r$. This gives $f_i(j_0, \ldots, j_{n-2-i}) \geqslant r + 1$. □

(2.2.5) COROLLARY. *Suppose* $f_i(j_0, \ldots, j_{n-2-i}) = r$. *Then*

$$f_{i-1}(j_0, \ldots, j_{n-2-i}, s) = \infty$$

for $s \in [0, \ldots, r-1]$, *and* $f_{i-1}(j_0, \ldots, j_{n-2-i}, s)$ *is a strictly decreasing sequence of non-negative integers for* $s \geqslant r$ *until it becomes* 0. *Then it remains* 0.

(2.2.6) We now define for $i \geqslant 0$

$$\Lambda_i(I) = \{(j_0, \ldots, j_{n-2-i}) \in M(n-2-i) \mid 0 < f_i(j_0, \ldots, j_{n-2-i}) < \infty\}$$

$$\Gamma_i^d(I) = \{(j_0, \ldots, j_n) \in M(n, d) \mid (j_0, \ldots, j_{n-2-i}) \in \Lambda_i(I), \\ f_{i-1}(j_0, \ldots, j_{n-1-i}) = \infty\}$$

In example (2.2.2) $\Lambda_0(I)$ is the set of pairs (j_0, j_1) such that the corresponding location on the front face has a 0 with a number in, so $\Lambda_0(I) = \{(2, 2), (1, 3), (1, 4), (0, 5)\}$. $\Gamma_0^d(I)$ is the set of locations with 0 in the diagram "inwards" from 0'es with number in, so for d big $\Gamma_0^d(I)$ has $1 + 3 + 1 + 2 = 7$ elements.

$\Lambda_1(I)$ is the set of j_0 with a sequence of ordinary 0'es "diagonally under" it. In the example $\Lambda_1(I) = \{0, 1, 2\}$, and $\Gamma_1^d(I)$ is the set of 0'es in the diagram lying "inwards" from the ordinary 0'es. One sees that $\#\Gamma_1^d(I)$ for d big will be a linear polynomial in d with the number of ordinary 0'es 10 as the coefficient of d.

(2.2.7) PROPOSITION. $M(n, d) - I_d = \amalg_{i \geqslant 0} \Gamma_i^d(I)$.

Proof. That the $\Gamma_i^d(I)$ are disjoint for $i \geqslant 0$ follows easily by the definition of $\Lambda_i(I)$ and (2.2.3). Also that $\Gamma_i^d(I) \subseteq M(n, d) - I_d$ follows easily by (2.2.3).

Suppose then that $J \in M(n, d) - I_d$. Let k be minimal such that

$$f_k(j_0, \ldots, j_{n-2-k}) < \infty.$$

If $M(n, d) \neq I_d$ such a k exists. If

$$f_k(j_0,\dots,j_{n-2-k}) = 0$$

then $f_{k-1}(j_0,\dots,j_{n-2-k},0) < \infty$. By (2.2.4) we then have

$$f_{k-1}(j_0,\dots,j_{n-2-k},j_{n-1-k}) < \infty$$

which is impossible. Hence $f_k(j_0,\dots,j_{n-2-k}) \in (0,\infty)$ and $(j_0,\dots,j_{n-2-k}) \in \Lambda_k(I)$. So $J \in \Gamma_i^k(I)$. □

(2.2.8) PROPOSITION. *The $\Lambda_i(I)$ are finite sets.*

Proof. Suppose $\Lambda_i(I)$ is not finite. Let $k-1$ be the maximum of all r such that there exists infinitely many $(j_0,\dots,j_{n-2-r}) \in M(n-2-r)$ which may be completed to an

$$(j_0,\dots,j_{n-2-r},\dots,j_{n-2-i}) \in \Lambda_i(I).$$

Then there exists a $(j_0,\dots,j_{n-2-k})$ which may be completed to

$$(j_0,\dots,j_{n-2-k},j_{n-1-k},\dots,j_{n-2-i}) \in \Lambda_i(I)$$

with j_{n-1-k} arbitrarily large.

If $f_k(j_0,\dots,j_{n-2-k}) = \infty$ then $f_i(j_0,\dots,j_{n-2-i}) = \infty$ by (2.2.3), and this is not so. Hence $f_k(j_0,\dots,j_{n-2-k}) < \infty$ and there exists $r_{n-1-k},\dots,r_n$ such that

$$(j_0,\dots,j_{n-2-k},r_{n-1-k},\dots,r_n) \in I.$$

Now we have

$$(j_0,\dots,j_{n-2-k},j_{n-1-k},\dots,j_{n-2-i},0,\dots,0) \notin I$$

where j_{n-1-k} can be arbitrarily large. By Borel-fixedness we can, by applying inverse elementary transformations to this, get an

$$(j_0,\dots,j_{n-2-k},j'_{n-1-k},\dots,j'_n) \notin I$$

where $j'_{n-1-t} \geq r_{n-1-t}$ for $t \in [-1,k]$. This is a contradiction. □

(2.2.9) For $J \in \Lambda_i(I)$ let

$$T_J = \{(j_0,\dots,j_n) \in M(n) \mid (j_0,\dots,j_{n-2-i}) = J,\ f_{i-1}(j_0,\dots,j_{n-1-i}) = \infty\}.$$

We then easily see that $\Gamma_i^d(I) = {}_{J\in\Lambda_i(I)} T_J$.

(2.2.10) LEMMA.

$$\#T_J = \binom{d - |J| + i + 1}{i + 1} - \binom{d - |J| - f_i(J) + i + 1}{i + 1}$$

for $d \geqslant |J| + f_i(J) - 1$.

Proof. $T_J = \amalg_{k=0}^{f_i(J)-1} T_{J,k}$ where

$$T_{J,k} = \{(j_0, \ldots, j_n) \in T_J \,|\, j_{n-i} + \cdots + j_n = d - |J| - k\}.$$

Since

$$\#T_{J,k} = \binom{d - |J| + i - k}{i}$$

we get the lemma. □

(2.2.11) COROLLARY. *For* $d \gg 0$

$$\#\Gamma_i^d(I) = \sum_{J \in \Lambda_i(I)} \left(\binom{d - |J| + i + 1}{i + 1} - \binom{d - |J| - f_i(J) + i + 1}{i + 1} \right)$$

Hence $\#\Gamma_i^d(I)$ is a polynomial $p_{i,I}(d)$ of degree i for $d \gg 0$. We will write $p_{i,I}(d) = \Sigma_{j=0}^{i} p_{ij,I} d^j$. We may drop the index I if it is clear to which ideal we are referring to.

(2.2.12) LEMMA.

$$\#\Gamma_i^d(I) = \sum_{t=0}^{d} \sum_{\substack{(j_0, \ldots, j_{n-1}, 0) \\ \in \Gamma_i^t(I)}} 1$$

Proof. By (2.1.4)

$$(j_0, \ldots, j_n) \in \Gamma_i^d(I) \Leftrightarrow (j_0, \ldots, j_{n-1}, 0) \in \Gamma_i^{d-j_n}(I)$$ □

2.3. Hyperplane sections

(2.3.1) Now let $X \subseteq \mathbf{P}^n$ be a subscheme. Let I_X be its homogeneous ideal. We will always suppose that X is in generic coordinates so $\mathrm{in}(I_X)$ is the generic initial ideal of X. The homogeneous ideal of X and $\mathrm{in}(I_X)$ have the same Hilbert function. Hence for $d \gg 0$

$$\begin{aligned}\chi\mathcal{O}_X(d) &= h^0\mathcal{O}_{\mathbf{P}^n}(d) - h^0\mathcal{I}_X(d)\\ &= \#M(n,d) - \#\mathrm{in}(I_X)_d\\ &= \#(M(n,d) - \mathrm{in}(I_X)_d)\\ &= \sum_{i\geqslant 0} \#\Gamma_i^d(\mathrm{in}(I_X)) = \sum_{i\geqslant 0} p_i(d)\end{aligned}$$

Let H be a general hyperplane and let $\Lambda = X \cap H$.

(2.3.2) PROPOSITION. $\Lambda_{i-1}(\mathrm{in}(I_\Lambda)) = \Lambda_i(\mathrm{in}(I_X))$ *for* $i \geqslant 1$, *and*

$$f_{i,\mathrm{in}(I_X)}(j_0,\ldots,j_{n-2-i}) = f_{i-1,\mathrm{in}(I_\Lambda)}(j_0,\ldots,j_{n-2-i}) \quad \textit{for } i \geqslant 1.$$

Proof. The first follows from the latter. For the latter it is enough to show that

$$f_{-1,\mathrm{in}(I_\Lambda)}(j_0,\ldots,j_{n-2}) = \begin{cases}\infty & \text{if } f_{0,\mathrm{in}(I_X)}(j_0,\ldots,j_{n-2}) = \infty\\ 0 & \text{if } f_{0,\mathrm{in}(I_X)}(j_0,\ldots,j_{n-2}) < \infty\end{cases}$$

Let I_Λ be the homogeneous ideal of Λ. We have $I_\Lambda = (I_{X,H})^{\mathrm{sat}}$. Hence for $m \gg 0$

$$\begin{aligned}(I_\Lambda)_m &= (I_{X,H})_m\\ \mathrm{in}(I_\Lambda)_m &= \mathrm{in}(I_{X,H})_m\\ &= \mathrm{in}(I_X)_{x_n,m}\end{aligned}$$

Suppose that $f_{0,\mathrm{in}(I_X)}(j_0,\ldots,j_{n-2}) = \infty$. Then $(j_0,\ldots,j_{n-1},0)\notin \mathrm{in}(I_X)$ for all $j_{n-1} \geqslant 0$. Hence $(j_0,\ldots,j_{n-1})\notin \mathrm{in}(I_\Lambda)_m$ if $m = \Sigma j_k$ and $m \gg 0$. This gives $f_{-1,\mathrm{in}(I_\Lambda)}(j_0,\ldots,j_{n-2}) = \infty$.

Suppose that $f_{0,\mathrm{in}(I_X)}(j_0,\ldots,j_{n-2}) < \infty$. Then $(j_0,\ldots,j_{n-1},0)\in \mathrm{in}(I_X)$ for all sufficiently big j_{n-1}. Hence $(j_0,\ldots,j_{n-1})\in\mathrm{in}(I_\Lambda)_m$ for $j_{n-1} \gg 0$ and $m = \Sigma j_k$. By (2.1.4) this gives $(j_0,\ldots,j_{n-2},0)\in\mathrm{in}(I_\Lambda)$ and $f_{-1,\mathrm{in}(I_\Lambda)}(j_0,\ldots,j_{n-2}) = 0$. □

(2.3.3) LEMMA. *Let* $i \geqslant 1$. *Then*

$$\#\Gamma_i^d(\mathrm{in}(I_X)) = \sum_{t=0}^{d} \sum_{\substack{(j_0,\ldots,j_{n-2},0)\\ \in\Gamma_{i-1}^t(\mathrm{in}(I_\Lambda))}} (d-t+1)$$

Proof. For $i \geqslant 1$ we have

$$\Gamma_i^d(\mathrm{in}(I_X)) = \{J\in M(n,d) \mid (j_0,\ldots,j_{n-2-i})\in\Lambda_i(\mathrm{in}(I_X)),\; f_{i-1,\mathrm{in}(I_X)}(j_0,\ldots,j_{n-1-i}) = \infty\}$$

$$= \{J \in M(n, d) \mid (j_0, \ldots, j_{n-2-i}) \in \Lambda_{i-1}(\mathrm{in}(I_\Lambda)),$$
$$f_{i-2,\mathrm{in}(I_\Lambda)}(j_0, \ldots, j_{n-1-i}) = \infty\}$$
$$= \{J \in M(n, d) \mid \exists t \leqslant d\colon (j_0, \ldots, j_{n-1}) \in \Gamma^t_{i-1}(\mathrm{in}(I_\Lambda))\}$$

But $(j_0, \ldots, j_{n-1}) \in \Gamma^t_{i-1}(\mathrm{in}(I_\Lambda))$ if and only if $(j_0, \ldots, j_{n-2}, 0) \in \Gamma^{t-j_{n-1}}_{i-1}(\mathrm{in}(I_\Lambda))$. Hence

$$\Gamma^d_i(\mathrm{in}(I_X)) = \amalg^d_{t=0}\{J \in M(n, d) | (j_0, \ldots, j_{n-2}, 0) \in \Gamma^t_{i-1}(\mathrm{in}(I_\Lambda))\}. \qquad \square$$

2.4. *The invariants*

(2.4.1) Now let $\lambda_i = f_{n-2}(i)$. If X has codimension 2, then $\Lambda_{n-2}(\mathrm{in}(I_X)) \neq \emptyset$. We then have a sequence of positive integers $\lambda_0, \ldots, \lambda_r$ where $r = \max \Lambda_{n-2}(\mathrm{in}(I_X))$, and call this sequence the *invariants* of X. Since $\mathrm{in}(I_X)$ is Borel-fixed we have by Proposition (2.2.4) that $\lambda_{i+1} \leqslant \lambda_i - 1$. Let $\Lambda = X \cap P$ where P is a general plane in $\mathbf{P}^n$. Then we see by (2.3.2) that giving the invariants is equivalent to giving $\Lambda_0(\mathrm{in}(I_\Lambda))$ and the values $f_{0,\mathrm{in}(I_\Lambda)}(i)$ for $i \in \Lambda_0(\mathrm{in}(I_\Lambda))$, which again is equivalent to giving the initial ideal of Λ. Note that $\Sigma \lambda_j = \deg \Lambda = \deg X$. Note also that in example (2.2.2) we have $\lambda_0 = 5$, $\lambda_1 = 3$ and $\lambda_2 = 2$.

The invariants of a set of points in the plane are related to the numerical character $(n_0, \ldots, n_{\sigma-1})$ by $r = \sigma - 1$ and $n_j = \lambda_j + j$. This is easily worked out by the fact that giving either of the numerical character or the invariants for a set of points in the projective plane is equivalent to giving the Hilbert function of the set of points.

(2.4.2) PROPOSITION. *Suppose X is integral of dimension $\geqslant 1$. Then $\lambda_{i+1} \geqslant \lambda_i - 2$.*

Proof. This is just the fact that the numerical character is connected [5, Corollaire 2.2]. $\square$

We say that a tuple of non-negative integers $(\lambda_0, \ldots, \lambda_r)$ is *connected* if it satisfies $\lambda_i - 1 \geqslant \lambda_{i+1} \geqslant \lambda_i - 2$ for $i = 0, \ldots, r-1$. Its *degree* is $\Sigma \lambda_j$.

2.5. *Laudals lemma and generalizations*

Suppose now that X is an integral non-degenerate variety in $\mathbf{P}^n$ of positive dimension. From [8] we have the following, essentially due to Re.

(2.5.1) THEOREM. *Let $\Lambda = X \cap H$ be a generic hyperplane section. Let σ be an integer such that* $\deg(X) > \sigma(\sigma + 1)$. *Then the map $I_{X,\sigma} \to I_{\Lambda,\sigma}$ is surjective.*

Note that if X is a curve (this case is Laudals lemma), we can replace the assumption $\deg(X) > \sigma(\sigma + 1)$ by $\deg(X) > \sigma^2 + 1$ by [8]. The implication of this for initial ideals is as follows.

(2.5.2) PROPOSITION. *Let t be an integer such that* $\deg(X) > t(t - 1)$ (*resp.* $(t - 1)^2 + 1$ *if* X *is a curve*). *Then* $\Sigma_{k=0}^{n-2} j_k \geqslant t$ *for all* $(j_0, \ldots, j_{n-2}) \in \Lambda_0(\mathrm{in}(I_X))$.

Proof. We assume that X is in generic coordinates. Let $\sigma < t$. Then $\deg(X) > \sigma(\sigma + 1)$. We have a diagram

$$\begin{array}{ccc} (I_X)_\sigma & \xrightarrow{\beta} & (I_\Lambda)_\sigma \\ & \searrow \quad \nearrow_{\alpha} & \\ & (I_{X,H})_\sigma & \end{array}$$

Since β is a surjection and α is always injective, α must be an isomorphism. Then

$$\mathrm{in}(I_{X,H})_\sigma = \mathrm{in}(I_\Lambda)_\sigma$$
$$\mathrm{in}(I_X)_{x_n,\sigma} = \mathrm{in}(I_\Lambda)_\sigma.$$

Let $(j_0, \ldots, j_{n-2}) \in \Lambda_0(\mathrm{in}(I_X))$ and suppose $\Sigma_{k=0}^{n-2} j_k = \sigma$. Then

$$(j_0, \ldots, j_{n-1}, 0) \in \mathrm{in}(I_X)$$

for j_{n-1} sufficiently big. Hence $(j_0, \ldots, j_{n-1}) \in \mathrm{in}(I_X)_{x_n,m} = \mathrm{in}(I_\Lambda)_m$ for $j_{n-1} \gg 0$ and $(j_0, \ldots, j_{n-2}, 0) \in \mathrm{in}(I_\Lambda)_\sigma = \mathrm{in}(I_X)_{x_n,\sigma}$. But then $(j_0, \ldots, j_{n-2}, 0, 0) \in \mathrm{in}(I_X)$ and $f_{-1,\mathrm{in}(I_X)}(j_0, \ldots, j_{n-2}, 0) = 0$ which does not square with

$$(j_0, \ldots, j_{n-2}) \in \Lambda_0(\mathrm{in}(I_X)). \qquad \square$$

(2.5.3) COROLLARY. *Let* $i \in [0, \dim(X) - 1]$. *Then for all*

$$(j_0, \ldots, j_{n-2-i}) \in \Lambda_i(\mathrm{in}(I_X))$$

we have $\Sigma_{k=0}^{n-2-i} j_k \geqslant t$.

Proof. Let $X_i = X \cap H_1 \cdots \cap H_i$ be an intersection with i general hyperplanes. The result follows from the fact that $\Lambda_i(\mathrm{in}(I_X)) = \Lambda_0(\mathrm{in}(I_{X_i})$. $\square$

Now suppose that X has codimension 2.

(2.5.4) COROLLARY. *If* $h^0 \mathscr{I}_X(t - 1) = 0$ *then* $\lambda_{t-1} > 0$.

Proof. Note that $n \geqslant 3$. Suppose $f_{n-2}(t - 1) = 0$, then

$$f_{n-3}(t-1,0) < \infty.$$

If $f_{n-3}(t-1,0) > 0$ then $(t-1,0) \in \Lambda_{n-3}$ which is not possible by (2.5.3). Hence $f_{n-3}(t-1,0) = 0$. We now continue. Either there is some $k \leqslant n-2$ such that $f_{n-2-k}(t-1,0,\ldots,0) \in (0,\infty)$ which gives

$$(t-1,0,\ldots,0) \in \Lambda_{n-2-k}.$$

This is impossible. Else will have $f_{-1}(t-1,0,\ldots,0) = 0$ and hence $\mathrm{in}(I_X)_{t-1} \neq 0$ which implies $h^0\mathscr{I}_X(t-1) \neq 0$. □

(2.5.5) LEMMA. *If $h^0\mathscr{I}_X(t) \neq 0$ then $\lambda_t = 0$.*

Proof. Suppose $\lambda_t \neq 0$. Then Borel-fixedness implies $\lambda_j \geqslant t-j+1$. Let $(j_0,\ldots,j_n) \in \mathrm{in}(I_X)_t$. Then $j_0 + j_1 \leqslant t$. Since $f_{n-2}(j_0) \geqslant t - j_0 + 1 > j_1$ we get $f_{-1}(j_0,\ldots,j_{n-1}) = \infty$ which is impossible. □

3. Some numerical lemmas

(3.1) LEMMA. *Let $\lambda_0,\ldots,\lambda_{t-1}$ be a connected sequence of invariants of degree d. Then*

$$\sum_{i=0}^{t-1}\left(\binom{\lambda_i+i-1}{2}-\binom{i-1}{2}\right) \leqslant d^2/2t + (t-4)d/2 - r(t-r)(t-1)/2t$$

with r the residue of d (mod *t*).

Proof. [5, Theoreme 2.7]. □

(3.2) LEMMA. *Let $\lambda_0,\ldots,\lambda_{t-1}$ be a connected sequence of invariants of degree d. Then $\lambda_0 \leqslant d/t + t - 1$.*

Proof. We have $\lambda_i \geqslant \lambda_0 - 2i$. Hence

$$d \geqslant \sum_{i=0}^{t-1}(\lambda_0 - 2i) = t\lambda_0 - t(t-1)$$

which implies the result. □

Now let $F(x) = \binom{x}{3}$. Let $m \geqslant 2(k-1)$ where $k \geqslant 2$ and let $m = \Sigma_{i=1}^{i=k} m_i$, where the $m_i \geqslant 0$.

(3.3) LEMMA.

$$kF(m/k) \leqslant \sum_{i=1}^{k} F(m_i)$$

This follows by induction from the following

(3.4) LEMMA. *Let* $0 \leqslant x \leqslant y$ and $m \geqslant 2(k-1)$ where $k \geqslant 2$. Suppose that $x + (k-1)y = m$. Then

$$F(x) + (k-1)F(y) \geqslant kF(m/k)$$

Proof. This is of an elementary nature. □

4. Curves in $\mathbf{P}^3$

Let C be a curve in $\mathbf{P}^3$. We need the following for later use. (Note that this is quite obvious when thinking of the diagram as displayed in example (2.2.2)).

(4.1) LEMMA. *Let* $\lambda_0 = f_1(0)$. *Suppose that* $J = (j_0, j_1, j_2, 0) \in \Gamma_0^t(\mathrm{in}(I_C))$ *where* $t > \lambda_0$. *Then there exists* $J' = (j_0', j_1', j_2', 0) \in \Gamma_0^{t-1}(\mathrm{in}(I_C))$.

Proof. (i) Suppose $j_2 \neq 0$. Then Borel-fixedness gives $(j_0, j_1, j_2 - 1, 1) \notin \mathrm{in}(I_C)$ and hence $(j_0, j_1, j_2 - 1, 0) \notin \mathrm{in}(I_C)$. Since $(j_0, j_1) \in \Lambda_0$ we get $(j_0, j_1, j_2 - 1, 0) \in \Gamma_0^{t-1}(\mathrm{in}(I_C))$.

(ii) Suppose $j_2 = 0$ and $j_1 \neq 0$. Again by Borel-fixedness

$$(j_0, j_1 - 1, 0, 0) \notin \mathrm{in}(I_C).$$

Furthermore we have $\lambda_j \leqslant \lambda_0 - j$ and since $j_0 + j_1 - 1 \geqslant \lambda_0$ we get $f_1(j_0) \leqslant j_1 - 1$. Hence $f_0(j_0, j_1 - 1) < \infty$ and $(j_0, j_1 - 1, 0, 0) \in \Gamma_0^{t-1}(\mathrm{in}(I_C))$.

(iii) Suppose $j_2 = j_1 = 0$ so $j_0 \neq 0$. Again by Borel-fixedness $(j_0 - 1, 0, 0, 0) \notin \mathrm{in}(I_C)$. Since $f_1(j_0 - 1) = 0$ we have $f_0(j_0 - 1, 0) \in (0, \infty)$ and hence $(j_0 - 1, 0, 0, 0) \in \Gamma_0^{t-1}(\mathrm{in}(I_C))$. □

(4.2) Let C have degree d and genus g. The Hilbert polynomial of C is

$$dm + 1 - g = \chi\mathcal{O}_C(m) = p_1(m) + p_0(m),$$

which gives $1 - g = p_{10} + p_{00}$.

By (2.2.11) we have

$$p_1(m) = \sum_{i \in \Lambda_1} \left(\binom{m - i + 2}{2} - \binom{m - i + 2 - \lambda_i}{2} \right).$$

This gives

$$p_{10} = -\sum_{i=0}^{r}\left(\binom{\lambda_i + i - 1}{2} - \binom{i-1}{2}\right), \quad r = \max \Lambda_1.$$

We have

$$g = 1 - p_{00} - p_{10} \leqslant 1 - p_{10}.$$

Suppose now that C is an integral curve with $h^0\mathscr{I}_C(t-1) = 0$ and $d > (t-1)^2 + 1$. Then the invariants of C are connected and $\lambda_{t-1} > 0$ by (2.5.4). By the above and (3.1) we get that

$$g \leqslant d^2/2t + (t-4)d/2 + 1 - r(t-r)(t-1)/2t$$

where r is the residue of d (mod t), which is a well known bound from [5]. The authors learned this argument from Mark Green. By (3.1) we also get

$$\begin{aligned} p_{00} = 1 - g - p_{10} \leqslant d^2/2t + (t-4)d/2 \\ -r(t-r)(t-1)/2t + 1 - g. \end{aligned} \tag{1}$$

5. Lower bound for the Euler-Poincare characteristic for surfaces in $\mathbf{P}^4$

(5.1) Let S be a smooth surface in $\mathbf{P}^4$ with invariants as given in (1.1). Let $C = S \cap H$ be a general hyperplane section. Suppose that $d > s(s-1)$.

By (2.3.1) we have

$$\chi\mathcal{O}_S(m) = p_2(m) + p_1(m) + p_0(m).$$

The constant terms here are

$$\chi\mathcal{O}_S = p_{20} + p_{10} + p_{00}.$$

We proceed to calculate the $p_i(m)$ and find lower bounds for the p_{i0}.

(5.2) By (2.5.4) we have $\lambda_{s-1} > 0$ and by (2.5.5) we have $\lambda_s = 0$. Hence

$$\begin{aligned} p_2(m) &= \sum_{t\in\Lambda_2}\left(\binom{m-t+3}{3} - \binom{m-t+3-\lambda_t}{3}\right) \\ &= \sum_{t=0}^{s-1}\left(\binom{m-t+3}{3} - \binom{m-t+3-\lambda_t}{3}\right), \end{aligned}$$

which gives

$$p_{20} = \sum_{t=0}^{s-1} \left(\binom{\lambda_t + t - 1}{3} - \binom{t-1}{3} \right).$$

We have

$$\sum_{t=0}^{s-1} (\lambda_t + t - 1) = \sum_{t=0}^{s-1} \lambda_t + \sum_{t=0}^{s-1} (t-1)$$
$$= d + s(s-3)/2.$$

Since $s \geqslant 2$ and $d > (s-1)^2 + 1$ gives $d + s(s-3)/2 \geqslant 2(s-1)$, we get by (3.3) that

$$p_{20} \geqslant sF(d/s + (s-3)/2) + 1 - \binom{s-1}{4}$$
$$= \frac{s}{6}(d/s + (s-3)/2)(d/s + (s-5)/2)(d/s + (s-7)/2) + 1 - \binom{s-1}{4}$$
$$= d^3/6s^2 + d^2(s-5)/4s + d(3s^2 - 30s + 71)/24 - \frac{1}{48}(s^4 - 5s^3 - s^2 + 5s)$$

(5.3) Let

$$\gamma = d^2/2s + (s-4)d/2 + 1 - r(s-r)(s-1)/2s - \pi$$

where $d \equiv r(\operatorname{mod} s)$ and $0 \leqslant r < s$. From (1) in (4.2) we have

$$p_{00,\mathrm{in}(I_C)} \leqslant \gamma$$

(5.4) From (2.3.3) we have

$$p_{10} = \sum_{t=0}^{m} \sum_{\substack{(j_0, j_1, j_2, 0) \\ \in \Gamma_0^t(\mathrm{in}(I_C))}} -(t-1).$$

for $m \gg 0$. From (2.2.12) we have

$$\sum_{t=0}^{m} \sum_{\substack{(j_0, j_1, j_2, 0) \\ \in \Gamma_0^t(\mathrm{in}(I_C))}} 1 = p_{00,\mathrm{in}(I_C)} \leqslant \gamma.$$

for $m \gg 0$. From (3.2) and (4.1) we then easily get the following