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The ampleness of the theta divisor on the compactified jacobian of a proper and integral curve

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0. Introduction

Let X be a proper and integral curve over an algebraically closed field k. If we suppose the curve to be smooth, the set made up by the isomorphism classes of the invertible sheaves of degree 0 over X is the set of rational points of the jacobian, an abelian variety. But, when we allow singularities for X, the jacobian is not proper anymore. Nevertheless, we get a natural compactification by considering the isomorphism classes of rank 1 torsion free sheaves over X. The related compactified functor $\overline{\mathrm{Pic}}(X)$ is representable. Let g be the areithmetic genus of X. Then, we can define the theta divisor on the reduced $g-1$ component, the points of which correspond to the rank 1 torsion free sheaves having non-zero global sections. This work endeavours to prove that the theta divisor on $\overline{\mathrm{Pic}}(X)^{g-1}$ is ample.

In the first chapter, we recall the definition and basic properties of $\overline{\mathrm{Pic}}(X)$. In the second, we define the theta divisor, and we show that two times theta is generated by its sections on the normalization of $\overline{\mathrm{Pic}}(X)$. Then, in the third chapter, the ampleness of theta is proved by demonstrating that no proper curve is included in the complement of its support. Finally, we give some applications.

I. The compactified Picard functor

We suppose that all schemes are locally noetherian. We recall the definition and basic properties of the compactified Picard scheme following [AK1]. For this sake, it is natural to work in the relative situation.

Let $f: X \to S$ be a morphism of finite type, flat and projective, whose geometric fibers are integral curves. Moreover, let us denote by $\mathcal{O}_X(1)$ an invertible and S-ample sheaf on X. In this chapter, T will be an S-scheme.

DEFINITION 1. *A sheaf $\mathcal{M}$ on $X_T = X \times_S T$ is a (relative) quasi-invertible sheaf, if it is coherent, T-flat, and if its restriction to every geometric fiber of f is rank 1 torsion free.*

It is useful to have a definition for the degree of a sheaf on a proper and integral curve, even with singularities. This definition will yield de facto the Riemann-Roch formula.

DEFINITION 2. *Let C be an integral and projective curve on an algebraically closed field k, and $\mathcal{M}$ a quasi-invertible sheaf on C. We define the degree of $\mathcal{M}$ and we write $d(\mathcal{M})$, or simply d, the integer such that:*

$$\chi(\mathcal{M}) = h^0(\mathcal{M}) - h^1(\mathcal{M}) = 1 - g + d$$

where g is the arithmetic genus of X, and $h^i(\mathcal{M})$ denotes $\dim_k H^i(X, \mathcal{M})$.

LEMMA 3. *Let C be a proper and integral curve on an algebraically closed field k, $\mathcal{M}$ a quasi-invertible sheaf, and $\mathcal{L}$ an invertible sheaf corresponding to a Cartier divisor with support in the smooth locus. We then have:*

$$d[\mathcal{M} \otimes \mathcal{L}] = d(\mathcal{M}) + d(\mathcal{L}).$$

Proof. We only need proving this result for $\mathcal{L} = \mathcal{O}_C(x)$ where x is a smooth point. We have the exact sequence:

$$0 \to \mathcal{O}_C(-x) \to \mathcal{O}_C \to k(x) \to 0.$$

This yields the following exact sequence:

$$0 \to \mathcal{M} \to \mathcal{M}(x) \to V \to 0$$

where V is a sheaf in $k(x)$-vectorial spaces of dimension 1, concentrated at the point x.

By the long exact sequence, we get $\chi[\mathcal{M}(x)] - \chi(\mathcal{M}) = 1$. □

Let $\bar{P}'$ be the functor that takes every S-scheme T to the set of all quasi-invertible sheaves on X_T. Moreover, let d be an integer. Then $\bar{P}'^d$ denotes the subfunctor of $\bar{P}'$ made up by the quasi-invertible sheaves of degree d.

DEFINITION 4. *The compactified Picard scheme is the sheaf for the étale topology associated to the functor $\bar{P}'$. We shall denote it by $\overline{\text{Pic}}(X/S)$. In the same way, $\overline{\text{Pic}}^d(X/S)$ is associated to $\bar{P}'^d$.*

$\overline{\mathrm{Pic}}^d(X/S)$ is an open and closed subfunctor of $\overline{\mathrm{Pic}}(X/S)$.

The following theorem has been proved by Altman and Kleiman [AK1 (6.6)], following Grothendieck's sketch [G].

THEOREM 5. *Let d be an integer. The functor* $\overline{\mathrm{Pic}}^d(X/S)$ *is representable by a scheme, that is projective, locally on S.*

We shall denote this scheme again by $\overline{\mathrm{Pic}}^d(X/S)$.

Suppose now that there is a section ε with values in the smooth locus:

$$\varepsilon : S \to X.$$

Consider the functor $\tilde{P}$ of the quasi-invertible rigidified sheaves: for any S-point T, $\tilde{P}(T)$ is the set of isomorphism classes $(\mathscr{M}, \alpha)$ such that $\mathscr{M}$ is quasi-invertible on X_T and α is an isomorphism between $\varepsilon^*(\mathscr{M})$ and $\mathscr{O}_T$.

PROPOSITION 6. *The functor* $\tilde{P}$ *is a sheaf for the étale topology. Moreover, the natural composite map*

$$\tilde{P} \to \bar{P}' \to \overline{\mathrm{Pic}}(X/S)$$

is an isomorphism of étale sheaves.

Proof. First of all, let us show that rigidifying cancels the non-trivial automorphisms.

Indeed, all quasi-invertible sheaves are *simple* i.e. for any S-point T, there is a canonical isomorphism

$$\mathscr{O}_T \cong f_{T*}\,\mathrm{Hom}_{X_T}(\mathscr{M}_T, \mathscr{M}_T)$$

where $\mathscr{M}$ is any quasi-invertible sheaf on X [AK1 (5.2)].

Now, to show that $\tilde{P}$ is a sheaf, let T be an S-point, and suppose we have an exact diagram:

$$T''' \to T'' \to T' \to T$$

where T' covers T, $T'' = T' \times_T T'$, and $T''' = T' \times_T T' \times_T T'$.

We get the following diagram:

$$\tilde{P}(T''') \leftarrow \tilde{P}(T'') \leftarrow \tilde{P}(T') \leftarrow \tilde{P}(T).$$

Take a point in $\tilde{P}(T')$ whose image by both arrows corresponding to the projections coincide, then the cocycle condition is fullfilled in $\tilde{P}(T''')$, because, as we saw, a rigidified sheaf has only trivial isomorphisms. By étale descent of coherent sheaves, we get a T-point of $\tilde{P}$ making commutative the above diagram. □

II. The theta divisor

Now, the base S is the spectrum of an algebraically closed field k. The choice of a smooth point x in X gives us a rigidification ε. We write simply $\overline{\mathrm{Pic}}(X)$ for $\overline{\mathrm{Pic}}(X/k)$.

Let z be a rational point of $X \times_k \overline{\mathrm{Pic}}(X)^{g-1}$. It corresponds to a quasi-invertible sheaf $\mathscr{M}$ on X of degree $g-1$. Thus we have $h^0(\mathscr{M}) = h^1(\mathscr{M})$. We are going to prove that, if we neglect the embedded components of $\overline{\mathrm{Pic}}(X)^{g-1}$, the set of the points z such that $h^0(\mathscr{M}) = h^1(\mathscr{M})$ do not vanish, is the support of a Cartier divisor, namely the theta divisor.

Let $\mathscr{F}$ be the universal sheaf of $\overline{\mathrm{Pic}}(X)^{g-1}$. We consider an affine open set $\mathrm{Spec}(A)$ of $\overline{\mathrm{Pic}}(X)^{g-1}$. There exists a complex (K^j) of locally free A-modules of finite type such that the following holds for any A-module N (see e.g. [H] III 12.2):

$$H^i(\mathscr{F} \otimes_A N) \cong H^i((K^j) \otimes_A N)$$

In particular this complex computes the cohomology of the quasi-invertible sheaves corresponding to the various points of $\mathrm{Spec}(A)$:

$$H^i[\mathscr{F} \otimes_A k(\alpha)] \cong H^i[(K^j) \otimes_A k(\alpha)]$$

where α is a point of $\mathrm{Spec}(A)$, and $k(\alpha)$ denotes its residual field.

Universally there is no cohomology in degree greater than one, thus we can concentrate the complex $K^{\cdot}$ in degree (0, 1). Moreover, as we work on the component of degree $g-1$, K^0 and K^1 have the same rank.

Working for the moment locally, we introduce the determinant $\mathscr{D}$ of the map $K^0 \to K^1$ defined by $\det(K^1) \otimes \det(K^0)^{-1}$. It is an invertible sheaf on $\mathrm{Spec}(A)$. Tensorizing by $\det(K_0)^{-1}$, the map $K^0 \to K^1$ gives rise to a map $\mathcal{O}_A \to \mathscr{D}$. This last map defines in turn a section δ of $\mathscr{D}$ whose set of zeros we denote by Θ. Θ is a closed subscheme locally defined by one equation, but we do not know whether Θ is a Cartier divisor or not.

We do not modify Θ if we replace $K^{\cdot}$ by a quasi-isomorphic complex of the same type. Glueing up, we get Θ globally on $\overline{\mathrm{Pic}}(X)^{g-1}$.

REMARK. Making use of the (quasi)-projectivity of $\overline{\mathrm{Pic}}(X)^{g-1}$, we could have constructed globally on $\overline{\mathrm{Pic}}(X)^{g-1}$ a complex $K^0 \to K^1$ which computes universally the cohomology of $\mathscr{F}$. This is an other way of proving the global existence of Θ.

We denote by U the complement of the support of Θ. It is an open subscheme of $\overline{\text{Pic}}(X)^{g-1}$.

We first notice that there is a natural action of the Picard group scheme $\text{Pic}(X)$ on $\overline{\text{Pic}}(X)$ given by the following formula:

$$(\mathscr{M}, \mathscr{L}) \to \mathscr{M} \otimes \mathscr{L}.$$

In particular, $\text{Pic}^0(X)$ acts on $\overline{\text{Pic}}(X)^{g-1}$.

PROPOSITION 7. *The saturation* $\text{Pic}^0(X) + U$ *of the open set* U *under the action of* $\text{Pic}^0(X)$ *is the whole scheme* $\overline{\text{Pic}}(X)^{g-1}$.

Proof. Let $\mathscr{M}$ be a quasi-invertible sheaf on X of degree $g - 1$. We are going to prove that there exists an invertible sheaf $\mathscr{L}$ on X of degree 0 such that

$$h^0(\mathscr{M} \otimes \mathscr{L}) = 0.$$

To see this, let $\mathscr{N}$ be an invertible sheaf of degree n on X sufficiently ample such that:

$$h^1[\mathscr{M} \otimes \mathscr{N}] = 0.$$

Lemma 3 yields: $d[\mathscr{M} \otimes \mathscr{N}] = d(\mathscr{M}) + n$.

Let us denote now by $\mathscr{M}_0$ the sheaf $\mathscr{M} \otimes \mathscr{N}$. If $\mathscr{M}_0$ has no global section, the problem is solved with $\mathscr{L} = \mathscr{N}$. Otherwise we consider a non-zero global section s of $\mathscr{M}_0$ and a point x_1 of the smooth locus where s does not vanish (i.e. where s generates $\mathscr{M}_0$). Consider $\mathscr{M}_0 \otimes \mathscr{O}_X(-x_1)$; s is not any more a global section of $\mathscr{M}_1 \otimes \mathscr{O}_X(-x_1)$. Thus we have:

$$h^0[\mathscr{M}_1 \otimes \mathscr{O}_X(-x_1)] < h^0(\mathscr{M}_1).$$

On the other hand, we have:

$$h^0[\mathscr{M}_0 \otimes \mathscr{O}_X(-x_1)] - h^1[\mathscr{M}_0 \otimes \mathscr{O}_X(-x_1)] = h^0(\mathscr{M}_0) - 1.$$

Hence the two equalities below hold:

$$h^0[\mathscr{M}_0 \otimes \mathscr{O}_X(-x_1)] = h^0(\mathscr{M}_0) - 1 \quad \text{and} \quad h^1[\mathscr{M}_0 \otimes \mathscr{O}_X(-x_1)] = 0.$$

We set $\mathscr{M}_1 = \mathscr{M}_0 \otimes \mathscr{O}_X(-x_1)$. It is a sheaf of degree $(g - 1) + (n - 1)$. We now go on by descending induction. We denote by x_i the various points appearing in the process, and by $\mathscr{M}_i$ the various sheaves obtained. We find that $\mathscr{M}_n$ belongs to U and arises from $\mathscr{M}$ tensorizing by the sheaf $\mathscr{N} \otimes \mathscr{O}(-\Sigma_{1 \leqslant i \leqslant n} x_i)$ we shall denote by $\mathscr{L}$ which is of degree 0. □

Let us notice that we do not know in general if $\overline{\mathrm{Pic}}(X)$ is reduced or whether it has embedded components. Let $\mathscr{I}$ be the sheaf of those nilpotent ideals of the structural sheaf of $\overline{\mathrm{Pic}}(X)^{g-1}$ generated by the sections whose support are closed sets with empty interior. From now on P^{g-1} will denote the closed subscheme of $\overline{\mathrm{Pic}}(X)^{g-1}$ defined by the sheaf $\mathscr{I}$.

THEOREM AND DEFINITION 8. *The closed subscheme* Θ *induces on* P^{g-1} *a Cartier divisor, namely the theta divisor that we denote again by 0. Moreover we have* $\mathscr{D}_{|P^{g-1}} = \mathscr{O}_{P^{g-1}}(\Theta)$.

Proof. $\mathrm{Pic}^0(X)$ being connected, we find, as a corollary of Proposition 7, that U does not contain any generic point of $\overline{\mathrm{Pic}}(X)^{g-1}$. Thus δ induces a non-zero divisor on $\mathscr{D}_{|P^{g-1}}$. □

We endeavour to show that Θ is an ample divisor. Let P'^{g-1} be the normalization of $(P^{g-1})_{\mathrm{red}}$. Notice that, as $\mathrm{Pic}^0(X)$ is smooth, the action of $\mathrm{Pic}^0(X)$ on P^{g-1} lifts to an action of $\mathrm{Pic}^0(X)$ on P'^{g-1}. We shall denote $\mathrm{Pic}^0(X)$ by J.

The analog of Proposition 7 holds on P'^{g-1}. In fact, if x' is a point of P'^{g-1} over a point x of P^{g-1}, $J + x$ meets U, hence $J + x'$ meets the pullback U' of U in P'^{g-1}. Set also Θ' the pullback of Θ in P'^{g-1}.

PROPOSITION 9. *The invertible sheaf* $\mathscr{O}(2\Theta')$ *on the scheme* P'^{g-1} *is generated by its global sections.*

Proof. Let $\mathscr{P}$ be the Picard scheme of P'^{g-1}. For any point a of J, let T_a be the translation by a operating on P'^{g-1}.

The map: $a \to T_a(\Theta') - \Theta'$, taking divisor classes, defines a morphism of schemes $h\colon J \to \mathscr{P}^0$, where $\mathscr{P}^0$ denotes the neutral component of $\mathscr{P}$.

The next result is a variation on the theorem of the square.

LEMMA 10. *h is a morphism of group schemes.*

Let us prove this lemma. Studying the representability of J we find by ([BLR] 9.2) that, if we start with the jacobian B of the normalization of X (an abelian variety), J arises as an extension of B by a linear group, namely a successive extension of additive groups G_a and multiplicative groups G_m. Thus J is an extension of B by a smooth and connected group H which is a rational variety. On the other hand $\mathscr{P}^0_{\mathrm{red}}$ is an abelian variety as P'^{g-1} is normal ([G] Th. 2.1). Now, every map going from a rational variety to an abelian variety is constant. Hence the map h from J to $\mathscr{P}^0_{\mathrm{red}}$ factorizes through B. We get a morphism of schemes from B to $\mathscr{P}^0_{\mathrm{red}}$ which sends the origin to the origin. By the rigidity lemma (see [M] Ch. 6 Cor. 6.4), this morphism is a morphism of group schemes. Thus h has the same property.

COROLLARY 11. *For any point a in J, $T_a(\Theta') + T_{-a}(\Theta')$ is linearly equivalent to $2\Theta'$.*

End of the Proof of Proposition 9. To show that $2\Theta'$ is generated by its sections, we reduce to proving the following fact: for any point x in P'^{g-1}, there is a point a in J such that x does not lie in $T_a(\Theta') + T_{-a}(\Theta')$. Let us then fix x in J and let us consider the subset V_+ (resp. V_-) of the elements a of J for which $x + a$ (resp. $x - a$) belongs to U'. By Proposition 7, V_+ and V_- are non-empty open sets of J. But, J being irreducible, V_+ meets V_-. We take a to be any element in $V_+ \cap V_-$. □

III. The ampleness of the theta divisor

In this chapter, we work over an algebraically closed field k. We fix a section ε with values in the smooth locus of X and we call x its image. First of all, notice that Θ is ample on P^{g-1} if and only if Θ' is ample on P'^{g-1}.

PROPOSITION 12. *The following properties are equivalent*:

(i) *Θ' is ample*
(ii) *U' is affine*
(iii) *No irreducible complete curve lies in U'.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): well known.
(iii) $\Rightarrow$ (ii): $2\Theta'$ is generated by its sections, so there exists a morphism φ from P'^{g-1} to a projective space $\mathbb{P}^n$ defined by the linear system $|2\Theta'|$. $2\Theta'$ appears as the pullback by φ of a hyperplane section H of $\mathbb{P}^n$. Denote by Ω its complement. Then $U' = \varphi^{-1}(\Omega)$, and the morphism φ' from U' to Ω, restriction of φ to U, is proper. φ' is also quasi-finite, otherwise at least one of its fibers would contain an irreducible curve C. C would be closed in P'^{g-1}, and that would contradict (iii). The morphism φ' is now proper and quasi-finite, hence it is also finite, and in particular affine. We conclude that U' is affine, being the pullback of an affine open set by an affine morphism.

(ii) $\Rightarrow$ (i): As U' is affine, we have the same property for $T_a(U')$, where a is any point in J. P'^{g-1} being separated, $T_a(U') \cap T_{-a}(U')$ is again affine. In other words, this holds also for the complement of the divisor $T_a(\Theta') + T_{-a}(\Theta')$. Moreover, we saw before that these open sets cover P'^{g-1}.

Thus any fiber of φ is contained in an affine open set. From this, we get that any fiber of φ is finite because it is proper. Hence φ is finite and Θ' is ample. □

We are going to prove the amplitude of Θ' using condition (iii) of Proposition 12. So, let us assume that U' contains a proper and integral curve, the normalization of which we denote by C. Then, the composite morphism $C \to U' \to U \to P^{g-1}$ corresponds to a quasi-invertible C-sheaf $\mathcal{M}$ on $X \times C$, rigidified along the section ε. This rigidification corresponds to an isomorphism α from $\mathcal{O}_C$ to $\varepsilon^*(\mathcal{M})$.

In the sequel, S will be the surface $X \times C$, and $p: S \to X$, $q: S \to C$ will denote the two projections. By hypothesis, C is above U, so $q_*(\mathcal{M})$ and $R^1q_*(\mathcal{M})$ vanish; this holds also after base-change, and in particular by restriction to the fibers of q. We are going to show that under these hypotheses the sheaf $\mathcal{M}$ is p-constant i.e. is in the form $p^*(\mathcal{M}_0)$, where $\mathcal{M}_0$ is a quasi-invertible sheaf on X. The map from C to P^{g-1} being non-constant, we will get a contradiction.

The proof runs through six steps.

Step 1. Recall that x denotes the image of the section ε. Let $\mathcal{O}_S(x)$ be the pullback of $\mathcal{O}_X(x)$ by p i.e. the invertible sheaf of degree 1 on X consisting of rational functions having at most a pole of order 1 along x.

Let us write the exact sequence:

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k(x) \to 0.$$

Taking tensor products by $\mathcal{O}_X(x)$, we get:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(x) \to V \to 0$$

where V is a vector space over k of dimension 1 concentrated at x.

Taking the pullback by p of the above exact sequence, we now get an exact sequence on S:

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(x) \to V \otimes_k \mathcal{O}_C \to 0.$$

Next, we take the tensor product by $\mathcal{M}$ of this sequence. As $\mathcal{M}$ is invertible in the neighbourhood of ε, the sequence we obtain on S remains exact:

$$0 \to \mathcal{M} \to \mathcal{M}(x) \to V \otimes_k \varepsilon^*(\mathcal{M}) \to 0.$$

As $\mathcal{M}$ is rigidified along ε, the last term can be identified with $V \otimes_k \mathcal{O}_C$, making use of the isomorphism α. Finally we get the exact sequence:

$$0 \to \mathcal{M} \to \mathcal{M}(x) \to V \otimes_k \mathcal{O}_C \to 0.$$

Step 2. As $q_*(\mathscr{M}) = R^1q_*(\mathscr{M}) = 0$, we get by the cohomological exact sequence for q an isomorphism between $q_*[\mathscr{M}(x)]$ and $V \otimes_k \mathcal{O}_C$. Choose a base e of V. It then corresponds to a global section of $\mathscr{M}(x)$ we shall also denote by e, which in turn induces a non-zero section on the fibers of $\mathscr{M}(x)$ over C, and which generates $\mathscr{M}(x)$ in the neighbourhood of ε. From now on we set $\mathscr{M}' = \mathscr{M}(x)$. Thus the section e defines an injective morphism from $\mathcal{O}_S$ into $\mathscr{M}'$ that remains injective after any base-change. Let $\mathscr{N}'$ be the quotient sheaf $\mathscr{M}'/\mathcal{O}_S$. Its support is closed in S, hence is proper. As it does not meet $[x] \times C$, it is finite over C.

Step 3. Here we want to show that the support of $\mathscr{N}'$ is horizontal i.e. it is the pullback by p of finitely many points of X.

Let C_i be an irreducible component of the support of $\mathscr{N}'$. Projecting C_i by p on X, we get a single closed point of X. Otherwise, the restriction of p to C_i would be a surjective map from C_i onto X, because C_i is proper. In particular, C_i would intersect the curve $[x] \times C$ at one point (x, c) at least. A contradiction.

Step 4. We are now going to show that $q_*(\mathscr{N}')$ is constant.

The section e gives rise to the exact sequence:

$$0 \to \mathcal{O}_S \to \mathscr{M}' \to \mathscr{N}' \to 0.$$

The long exact sequence then yields:

$$0 \to q_*(\mathcal{O}_S) \to q_*(\mathscr{M}') \to q_*(\mathscr{N}') \to R^1q_*(\mathcal{O}_S) \to R^1q_*(\mathscr{M}').$$

Now $q_*(\mathcal{O}_S) \to q_*(\mathscr{M}')$ is an isomorphism by construction. On the other hand, $R^1q_*(\mathscr{M}')$ vanishes. Hence the map from $q_*(\mathscr{N}')$ *to* $R^1q_*(\mathcal{O}_S)$ is an isomorphism. By flat base-change, we get the equality

$$R^1q_*(\mathcal{O}_S) = H^1(X, \mathcal{O}_X) \otimes_k \mathcal{O}_C.$$

We set $E = H^1(X, \mathcal{O}_X)$. Thus we find that $q_*(\mathscr{N}')$ is the sheaf $E \otimes_k \mathcal{O}_C$.

Step 5. We shall now embed $\mathscr{M}'$ into an invertible and p-constant sheaf on S, that is to say a sheaf in the form $p^*(\mathscr{L})$, where $\mathscr{L}$ is invertible on X. Let $\{x_1, \ldots, x_n\}$ be the projection by p of the support of $\mathscr{N}'$. We can find an affine open set W of X containing $x_1, \ldots, x_n$, and a function f on W whose zero-locus, looked upon from the set-theoretical viewpoint, is $\{x_1, \ldots, x_n\}$. We consider the open set $p^{-1}(W)$ and the function $f' = f \otimes_k 1 = p^{-1}(f)$. As the support of $\mathscr{N}'$ is contained in the zero-locus of f', $\mathscr{N}'$ is annihilated by a power of f', say f'^m.

Let $\mathcal{O}_X(D)$ be the invertible sheaf on X, generated on W by f^{-m} and outside W by 1. We just saw that $\mathcal{M}'$ is contained in $p^*[\mathcal{O}_X(D)]$ (that we shall simply denote by $\mathcal{O}_S(D)$). Hence we get the inclusions:

$$\mathcal{O}_S \hookrightarrow \mathcal{M}' \hookrightarrow \mathcal{O}_S(D).$$

Step 6. We now prove that $\mathcal{N}'$ and $\mathcal{M}'$ are constant sheaves. Taking quotients by $\mathcal{O}_S$, the inclusions $\mathcal{O}_S \hookrightarrow \mathcal{M}' \hookrightarrow \mathcal{O}_S(D)$ correspond to an inclusion from $\mathcal{N}' = \mathcal{M}'/\mathcal{O}_S$ into $\mathcal{O}_S(D)/\mathcal{O}_S$. Direct image by q yields:

$$q_*(\mathcal{N}') \hookrightarrow q_*[\mathcal{O}_S(D)/\mathcal{O}_S].$$

By flat base-change, the sheaf $q_*[\mathcal{O}_S(D)/\mathcal{O}_S]$ is isomorphic to the sheaf $H^0[X, \mathcal{O}_X(D)/\mathcal{O}_X] \otimes_k \mathcal{O}_C$.

Set $F = H^0[X, \mathcal{O}_X(D)/\mathcal{O}_X]$, so that $q_*[\mathcal{O}_S(D)/\mathcal{O}_S] = F \otimes_k \mathcal{O}_C$. Thus we have got two constant sheaves over $\mathcal{O}_C$ namely $q_*(\mathcal{N}')$ and $q_*[\mathcal{O}_S(D)/\mathcal{O}_S]$. The curve C being proper, the canonical embedding from $q_*(\mathcal{N}')$ to $q_*[\mathcal{O}_S(D)/\mathcal{O}_S]$ is forced constant and arises from a k-linear injective map from E into F. In other words, the global sections of $\mathcal{N}'$ are constant sections of $\mathcal{O}_S(D)/\mathcal{O}_S$. Fiber to fiber, they generate $\mathcal{N}'$, hence $\mathcal{N}'$ is a p-constant subsheaf of $\mathcal{O}_S(D)/\mathcal{O}_S$, that is to say $\mathcal{N}'$ is the pullback by p of a subsheaf $\mathcal{N}_0$ of $\mathcal{O}_X(D)/\mathcal{O}_X$. We now define $\mathcal{M}_0$ to be the subsheaf $\mathcal{O}_X(D)$ making the diagram below commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X & \to & \mathcal{M}_0 & \to & \mathcal{N}_0 & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \mathcal{O}_X & \to & \mathcal{O}_X(D) & \to & \mathcal{O}_X(D)/\mathcal{O}_X & \to & 0 \end{array} .$$

Pullback by p over S yields the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_S & \to & p^*(\mathcal{M}_0) & \to & \mathcal{N}' & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \mathcal{O}_S & \to & \mathcal{O}_S(D) & \to & \mathcal{O}_S(D)/\mathcal{O}_S & \to & 0 \end{array} .$$

We now see that $\mathcal{M}' = p^*(\mathcal{M}_0)$. Hence $\mathcal{M}'$ is constant, and that ends the proof of the amplitude of the theta divisor. □

IV. Applications and additional remarks

1. The relative case

Suppose we work in the relative situation where $f: X \to S$ is a morphism of finite type, flat and projective, whose geometric fibers are integral curves of genus g. Suppose that f has a section $\varepsilon: S \to X$ with values in the smooth locus.

In a similar way to that of part II, we define the sheaf $\mathscr{D}$ o $\overline{\mathrm{Pic}}^{g-1}(X/S)$ to be the determinant of the relative cohomology of the universal sheaf on $X \times_S \overline{\mathrm{Pic}}^{g-1}(X/S)$.

THEOREM 13. *The sheaf* $\mathscr{D}$ *on* $\overline{\mathrm{Pic}}^{g-1}(X/S)$ *is S-ample.*

Proof. As $\overline{\mathrm{Pic}}^{g-1}(X/S)$ is proper, to prove the amplitude of $\mathscr{D}$ on $\overline{\mathrm{Pic}}^{g-1}(X/S)$, it suffices to show it for the restriction of $\mathscr{D}$ to $\overline{\mathrm{Pic}}^{g-1}(X \otimes_S k(s)/k(s))$ where s is any geometric point of S. So, we reduce to the case worked out in chapter III. □

2. *The case of locally planar curves*

Recall that an integral curve X on a field k is said to be locally planar (for the étale topology), if the Zariski tangent space at any point x of X is of dimension not exceeding 2. This condition is equivalent to saying that the completed local ring at x is the quotient of the ring $k[[u, v]]$ by a reduced non-zero equation. In view of ([AIK], Theorem 9) (see also [R]), $\overline{\mathrm{Pic}}^{g-1}(X)$ is then the schematic closure of $\mathrm{Pic}^{g-1}(X)$ and is a local complete intersection. Thus we get a theta divisor on $\overline{\mathrm{Pic}}(X)^{g-1}$.

COROLLARY 14. *Let X be a planar curve. Then, the theta divisor is an ample positive Cartier divisor on* $\overline{\mathrm{Pic}}^{g-1}(X)$, *the schematic closure of* $\mathrm{Pic}^{g-1}(X)$.

When one accepts curves that are not locally planar, recall (see [R]) that the compactified jacobian is not anymore the closure of the usual jacobian and may have in particular components of dimensions exceeding g.

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