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Rigid subanalytic sets

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Introduction

In this paper, we want to study subanalytic sets in rigid analytic geometry. Let us fix an *ultrametric field* K. With this we mean an algebraically closed field which is complete with respect to a non-trivial non-archimedean rank-one valuation. The norm will be denoted by $|\cdot|$; the valuation ring by R; and the maximal ideal by $\wp$. The standard example is $\mathbb{C}_p$, the completion of the algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers, where p is a (rational) prime.

Real subanalytic sets were first studied by Łojasiewicz in his articles [Łoj 1] and [Łoj 2] and their fundamental properties were discovered by Gabrielov in [Ga 1] and [Ga 2] and by Hironaka in [Hi 1] and [Hi 2]. The p-adic analogue was recently developed by Denef and Van den Dries [DvdD]. Lipshitz [Lip] was the first to develop a theory in the rigid analytic case. Working with a larger class of sets, however, he derived some properties about subanalytic sets as well (see also [LR]).

We present here a theory of a restricted class of subanalytic sets in rigid analytic geometry (called *strongly subanalytic sets*), the restriction being rather natural, since it is a sort of compactness requirement, which occurs also in the real case but is superfluous in the p-adic case, since the space $\mathbb{Z}_p^N$ itself is compact.

We introduce the following function $\mathbf{D}: R^2 \to R$ by

$$\mathbf{D}(a, b) = \begin{cases} a/b & \text{if } |a| \leqslant |b| \text{ and } b \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We will use this function to describe rigid subanalytic sets. Let us give some more precise definitions. Let A be a reduced affinoid algebra and $M = \operatorname{Sp} A$ the corresponding affinoid variety. (See Section 1 for a short overview on rigid analytic geometry.) Consider the algebra $A\langle\langle \mathbf{D} \rangle\rangle$ of (*affinoid*) *strongly*

D-*functions*. This is the smallest K-algebra of functions from M to K, containing A and closed under the following two operations.

(i) If $f, g \in A\langle\langle \mathbf{D} \rangle\rangle$, with $|f|, |g| \leqslant 1$, then $\mathbf{D}(f, g) \in A\langle\langle \mathbf{D} \rangle\rangle$.

(ii) If $f \in A\langle\langle Y \rangle\rangle$ and $g_i \in A\langle\langle \mathbf{D} \rangle\rangle$, with $|g_i| \leqslant 1$, then $f(g_1, \ldots, g_s) \in A\langle\langle \mathbf{D} \rangle\rangle$, where $A\langle\langle Y \rangle\rangle$ denotes the ring of overconvergent power series in $Y = (Y_1, \ldots, Y_s)$ over A, that is, series which converge on $M \times D$, where $D \subset \mathbb{A}^s$ is some disk, depending on the series, of radius strictly bigger than one.

We call now a subset S of M *globally semianalytic* (*globally strongly* **D**-*semianalytic*, respectively) if its is a finite union of sets of the form

$$\{x \in M \mid \forall i \in I \colon |g_i(x)| \Diamond_i |h_i(x)|\},$$

where $g_i, h_i \in A$ ($g_i, h_i \in A\langle\langle \mathbf{D} \rangle\rangle$, respectively) and $\Diamond_i \in \{\leqslant, <\}$, for $i \in I$, and where I is a finite index set. The class of all globally (strongly **D**-) semianalytic subsets of M forms a Boolean algebra on M. We also introduce some more local (for the Grothendieck topology on M) notions (called *semianalytic*, respectively *strongly* **D**-*semianalytic* sets), see (1.3).

We are now interested in the nature of the image of a semianalytic set under a projection map. It turns out that this, in general, is no longer semianalytic; for a counterexample, which is similar to Osgood's example in the real case, see [Sch 1, Chapter III, 4.3.4]. We therefore have to introduce the notion of a subanalytic set as a projection of a semianalytic set. However, we are not able to study this general kind of subanalytic sets. In [Lip] a theory is developed to tackle this kind of sets by introducing a more general kind of sets. We restrict ourselves to the study of (*globally*) *strongly subanalytic* sets, that is projections along the R^m-variables of a (globally) semianalytic set $T \subset M \times R^m$, but with the describing functions of T overconvergent in the R^m-variables.

This overconvergency condition is in a certain sense a relative compactness requirement. Namely, we will prove in a subsequent paper that each strongly subanalytic set is exactly the image of a semianalytic set under a proper map (see [Sch 2, Theorem 2.5]). The compactness requirement also appears in the theory of real subanalytic sets (see for instance [BM] or [Hi 1]). We derive the following structure theorem.

QUANTIFIER ELIMINATION THEOREM. *Let M be an affinoid variety and $S \subset M$. Then the following are equivalent,*

(1) *S is globally strongly subanalytic,*

(2) *S is strongly subanalytic,*

(3) *S is globally strongly* **D**-*semianalytic,*

(4) *S is strongly* **D**-*semianalytic.*

See (4.2) and (5.2).

From this result it follows immediately that the set of stronglysub-analytic subsets of M forms a Boolean algebra. Another consequence of the Elimination Theorem is that the closure in the canonical topology of a strongly subanalytic set is strongly subanalytic again. See (5.4). Here, by the canonical topology on M, we mean the topology induced by the supremum norm on A.

One final remark about the terminology "quantifier elimination". Strictly speaking there is no class of formulas from which we can eliminate arbitrary quantifiers. However, since we do eliminate quantifiers binding overconvergent variables from certain kinds of formulas, we insist on calling it like that, because of the strong resemblance with the analogous Quantifier Elimination Theorems in the real and the p-adic case.

In a second paper we will give some further applications (see [Sch 2]). In a third paper we then prove that strongly subanalytic sets in the plane are semianalytic (see [Sch 3, Theorem 3.2]).

I am highly indebted to my advisor Tom Denef for the many valuable suggestions he gave me. In particular, the idea of using over convergent variables is due to him.

1. Definitions

1.1. DEFINITION. Let K be an ultrametric field. Recall that the free Tate algebra (or *strictly convergent power series*) over K in a finite number of indeterminates $X = (X_1, \ldots, X_n)$ is defined as

$$K\langle X\rangle = \left\{\sum_{i\in\mathbb{N}^n} a_i X^i \in K[[X]] \mid |a_i| \to 0 \text{ for } |i| \to \infty\right\},$$

where we write $|i| \overset{\text{def}}{=} i_1 + i_2 + \cdots + i_n$ for a multi-index i.

An affinoid algebra A is just a quotient of a $K\langle X\rangle$. The set of all maximal ideals of A can be made into a topological space $\operatorname{Sp} A$, called the *affinoid variety* associated to A. Elements of A then can be considered as functions on $\operatorname{Sp} A$. In this paper we will only work with reduced affinoid algebras, so that we tacitly will assume that all appearing affinoid algebras are reduced and all affinoid varieties are reduced. On $\operatorname{Sp} A$ is defined a Grothendieck topology, where the admissible opens are the so called affinoid subdomains of $\operatorname{Sp} A$, see [BGR, Section 7.2] for more details. We want to point out that an affinoid subdomain of a reduced affinoid variety is automatically reduced (see [BGR, 7.3.2, Corollary 10], so that we remain in the reduced case, justifying our restriction to work only on reduced affinoid varieties.

We endow A with an intrinsic norm, the supremum norm (where we consider the elements of A as functions on $\operatorname{Sp} A$, see [BGR, Section 6.2]) and we define similarly as above the ring of strictly convergent power series $A\langle X\rangle$ over A in the variables X. We define the ring of *overconvergent* power series over A by

$$A\langle\langle X\rangle\rangle = \left\{\sum_{i\in\mathbb{N}^n} a_i X^i \in A\langle X\rangle \mid \exists\delta > 1 : |a_i|\delta^{|i|} \to 0 \text{ for } |i| \to \infty\right\}.$$

The only norm on the ring of overconvergent power series we will use, is the supremum norm.

Note that the definition of these rings of strictly and overconvergent power series still makes sense over an arbitrary normed algebra. In particular we could define also the rings $A\langle\langle U\rangle\rangle\langle X\rangle$ and $A\langle\langle U\rangle\rangle\langle X\rangle\langle\langle Y\rangle\rangle$, where X, Y and U are finite sets of variables.

1.2. DEFINITION (**D**-functions). We define the following function $\mathbf{D}: R^2 \to R$ by

$$\mathbf{D}(a, b) = \begin{cases} a/b & \text{if } |a| \leqslant |b| \text{ and } b \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let A be an affinoid algebra and $\mathbb{F}(A)$ the ring of all functions $\operatorname{Sp} A \to K$. Further on we will introduce the notion of a G-basic subset, where G is a K-subalgebra of $\mathbb{F}(A)$. We will always take the supremum norm on each K-subalgebra G of $\mathbb{F}(A)$.

EXAMPLES OF K-SUBALGEBRAS OF $\mathbb{F}(A)$.

(1.2.1) The affinoid algebra A itself is a subalgebra, since we assumed it to be reduced. In this case we will speak of *affinoid functions* on $\operatorname{Sp} A$.

(1.2.2) By $A\langle\langle\mathbf{D}\rangle\rangle$ we will denote the algebra of *strongly* (*affinoid*) **D**-*functions* on $\operatorname{Sp} A$ and define it as the smallest K-subalgebra of $\mathbb{F}(A)$ which contains A and which is closed for the following two operations.

(i) If $f, g \in A\langle\langle\mathbf{D}\rangle\rangle$, with $|f|, |g| \leqslant 1$, then $\mathbf{D}(f, g) \in A\langle\langle\mathbf{D}\rangle\rangle$.

(ii) If $f \in A\langle\langle Y\rangle\rangle$ and $g_i \in A\langle\langle\mathbf{D}\rangle\rangle$, with $|g_i| \leqslant 1$, then $f(g_1, \ldots, g_s) \in A\langle\langle\mathbf{D}\rangle\rangle$, where $Y = (Y_1, \ldots, Y_s)$.

REMARK. If we allow in (ii) of the above definition that f runs over all of $A\langle Y\rangle$, then we get the definition of a general **D**-function.

(1.2.3) By $A\langle\langle\mathbf{D}\rangle\rangle\langle X\rangle\langle\langle Y\rangle\rangle$, where $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ are variables, we mean the K-subalgebra of $\mathbb{F}(A\langle X, Y\rangle)$ consisting of all K-valued functions on $\operatorname{Sp} A \times R^{n+m}$ of the form $G(v)$, where $G \in A\langle\langle U\rangle\rangle\langle X\rangle\langle\langle Y\rangle\rangle$, with

$U = (U_1, \ldots, U_s)$ a set of variables, and where $v = (v_1, \ldots, v_s)$, with each $v_i \in A\langle\langle \mathbf{D} \rangle\rangle$ a $\mathbf{D}$-function.

REMARK. Note that we do not exclude that n or m be zero. It is also important to note that each function in $A\langle\langle \mathbf{D} \rangle\rangle\langle X \rangle\langle\langle Y \rangle\rangle$ has only a finite number of appearances of different $\mathbf{D}$-functions in it, so that a better notation would be $A\langle\langle \mathbf{D} \rangle\rangle_{\mathrm{res}}\langle X \rangle\langle\langle Y \rangle\rangle$, to indicate that we only take restricted power series over $A\langle\langle \mathbf{D} \rangle\rangle$. But to avoid overloaded notation, we will stick to our former notation.

1.3. DEFINITION (SEMIANALYTIC SETS).

(1.3.1. Basic subsets). Let G be a K-subalgebra of $\mathbb{F}(A)$ and $B \subset \operatorname{Sp} A$, then we call B a *G-basic subset* of $\operatorname{Sp} A$, if there exist finitely many functions g_i, $h_i \in G$ and symbols $\lozenge_i \in \{\leqslant, <\}$, for $i = 0, \ldots, s$, such that

$$B = \{x \in \operatorname{Sp} A \mid \forall i\colon |g_i(x)| \lozenge_i |h_i(x)|\}.$$

We call a subset $S \subset \operatorname{Sp} A$ *globally G-semianalytic*, if S is a finite union of G-basic subsets.

REMARKS. (1) Note that in the definition of G-basic subsets also the case of equalities is included. Take namely for h_i the zero element, so that $|g_i(x)| \leqslant 0$ is equivalent to $g_i(x) = 0$.

(2) The intersection of two G-basic subsets is again a G-basic subset. The difference of two G-basic subsets is globally G-semianalytic. Hence the set of globally G-semianalytic subsets forms a Boolean algebra.

(1.3.2) In case G equals A (or $A\langle\langle \mathbf{D} \rangle\rangle$), we call B a *basic subset* (or *strongly* $\mathbf{D}$*-basic subset*, respectively) of $\operatorname{Sp} A$ and we call S *globally semianalytic* (or *globally strongly* $\mathbf{D}$*-semianalytic*, respectively) in $\operatorname{Sp} A$. In the sequel we will define other types of basic subsets and to each of these corresponds the notion of a globally semianalytic subset of that type, but we will not mention this on every occasion.

(1.3.3) We call $B \subset \operatorname{Sp} A \times R^m = \operatorname{Sp} A\langle Y \rangle$, where $Y = (Y_1, \ldots, Y_m)$, a *strongly* $\mathbf{D}$*-basic subset in the* R^m*-direction*, if B is an $A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y \rangle\rangle$-basic subset.

(1.3.4) We call $B \subset \operatorname{Sp} A \times R^m = \operatorname{Sp} A\langle Y \rangle$, where $Y = (Y_1, \ldots, Y_m)$, an *algebraic basic subset in the* R^m*-direction*, if B is an $A[Y]$-basic subset, where clearly $A[Y] \subset \mathbb{F}(A\langle Y \rangle)$. In particular, if $A = K$, then $B \subset R^m$ is just called an *algebraic basic set* and a finite union of these is then called *semialgebraic*. We have the following Quantifier Elimination for semialgebraic sets.

1.3.4.1. THEOREM (Algebraic Quantifier Elimination). *Let $S \subset R^{n+m}$ be semialgebraic. Then $\pi(S)$ is semialgebraic in R^n, where $R^{n+m} \xrightarrow{\pi} R^n$ is the canonical projection map onto the first factor.*

Proof. See [Rob], but for a elaborated proof see also [Wei]. □

We can also give a version for **D**-functions, which we will need to prove our Analytic Quantifier Elimination.

1.3.4.2. PROPOSITION. *Let M be an affinoid variety with affinoid algebra A and let $G = A\langle\langle \mathbf{D} \rangle\rangle\langle\langle X \rangle\rangle[Y]$, where $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ are variables. Let $S \subset M \times R^{n+m}$ be a globally G-semianalytic subset.*

*Then $\pi(S)$ is globally strongly **D**-semianalytic in the R^n-direction, where $M \times R^{n+m} \xrightarrow{\pi} M \times R^n$ is the projection along the Y-variables.*

REMARK. Here we consider G as an algebra of K-valued functions on $M \times R^{n+m}$, where G is just the (full) polynomial ring over $A\langle\langle \mathbf{D} \rangle\rangle\langle\langle X \rangle\rangle$ in the variables Y.

Proof. The proposition follows without any difficulty from the Algebraic Quantifier Elimination (1.3.4.1) in the same way as in [DvdD] or [Lip]. Just replace all the (finitely many) coefficients which appear in all the describing polynomials by new variables, apply (1.3.4.1) and then substitute back all the coefficients in the corresponding variables. □

(1.3.5) Let $W \subset \operatorname{Sp} A \times R^m$ be a strongly rational domain in the R^m-direction. By this we mean that there exist $f_i \in A\langle\langle Y \rangle\rangle$ with $(f_0, \ldots, f_s)A\langle\langle Y \rangle\rangle = (1)$, and where $Y = (Y_1, \ldots, Y_m)$, such that

$$W = \{(x, y) \in \operatorname{Sp} A \times R^m \mid \forall i \colon |f_i(x, y)| \leqslant |f_0(x, y)|\}.$$

Hence, $C = A\langle Y, f/f_0 \rangle$ is the affinoid algebra of W, where we have written $f = (f_1, \ldots, f_s)$ (see [BGR, 6.1.4.]). We denote by $A\langle f/f_0 \rangle\langle\langle Y \rangle\rangle$ the subring of C of all elements which are overconvergent in Y. Explicitly,

$$A\langle f/f_0 \rangle\langle\langle Y \rangle\rangle \stackrel{\text{def}}{=} \frac{A\langle T \rangle\langle\langle Y \rangle\rangle}{(f_0 T_1 - f_1, \ldots, f_0 T_s - f_s)},$$

where $T = (T_1, \ldots, T_s)$. We call a subset $B \subset W$ a *strongly basic subset of W in the R^m-direction*, if B is an $A\langle f/f_0 \rangle\langle\langle Y \rangle\rangle$-basic subset.

(1.3.6) For technical reasons, we introduce in the same way

$$A\langle\langle \mathbf{D} \rangle\rangle\langle f/f_0 \rangle\langle\langle Y \rangle\rangle \stackrel{\text{def}}{=} \frac{A\langle\langle \mathbf{D} \rangle\rangle\langle T \rangle\langle\langle Y \rangle\rangle}{(f_0 T_1 - f_1, \ldots, f_0 T_s - f_s)},$$

and we call a subset $B \subset W$ a *strongly* **D**-*basic subset of* W *in the* R^m-*direction*, if it is an $A\langle\langle \mathbf{D} \rangle\rangle\langle f/f_0 \rangle\langle\langle Y \rangle\rangle$-basic subset. Note that the describing functions of B are overconvergent in Y, but the **D**-symbol is not applied to them. Note also that this definition is an extension of definition (1.3.3) to the case of subsets of an arbitrary rational subdomain $W \subset \operatorname{Sp} A \times R^m$ and that both definitions agree in case $W = \operatorname{Sp} A \times R^m$.

REMARKS. (1) By our previous remark, we know that W, and hence C is reduced. Therefore $(f_0 T_1 - f_1, \ldots, f_0 T_s - f_s)$ is a radical ideal in $A\langle T, Y \rangle$. It is now an exercise to prove that $f_0 T_1 - f_1, \ldots, f_0 T_s - f_s$ also generate a radical ideal in $A\langle T \rangle\langle\langle Y \rangle\rangle$ (respectively in $A\langle\langle \mathbf{D} \rangle\rangle\langle T \rangle\langle\langle Y \rangle\rangle$), hence all of the above defined rings are reduced.

(2) We prefer here the functional definition of these notions, whereas in [DvdD] or [Lip] one makes use of a logical definition, which makes it sometimes easier and more natural to talk about these sets.

(1.3.7. Semianalytic Sets). Let M be an affinoid variety.

(1.3.7.1) Let $S \subset M$. We call S *semianalytic* (*strongly* **D**-*semianalytic*, respectively) in M, if there exists a finite admissible affinoid covering $\mathscr{X} = \{X_i\}_i$ of M, (for a definition, see [BGR, 9.1.4]), such that for each i, $S \cap X_i$ is globally semianalytic (globally strongly **D**-semianalytic, respectively) in X_i.

(1.3.7.2) Let $S \subset M \times R^m$. We call S *strongly semianalytic* (*strongly* **D**-*semianalytic*) *in the* R^m-*direction*, if there exists a strongly rational covering $\mathscr{C} = \{U_i\}_i$ of $M \times R^m$ in the R^m-direction, such that for each i, $S \cap U_i$ is globally strongly semianalytic (globally strongly **D**-semianalytic, respectively) in the R^m-direction in U_i.

REMARK. By a strongly rational covering we mean the following. Let $f_1, \ldots, f_s \in A\langle\langle X \rangle\rangle$ be functions which generate the unit ideal in $A\langle\langle X \rangle\rangle$. Then for each j, this defines a strongly rational subdomain $U_j(\mathfrak{f})$ of A as follows

$$U_j(\mathfrak{f}) \stackrel{\text{def}}{=} \{x \in \operatorname{Sp} A \mid \forall i \colon |f_i(x)| \leqslant |f_j(x)|\},$$

where $\mathfrak{f} = \{f_1, \ldots, f_s\}$. This gives rise to an admissible affinoid covering of $\operatorname{Sp} A$, which we will call the strongly rational covering of $\operatorname{Sp} A$ generated by $\mathfrak{f}$ and which we will denote by

$$\mathscr{C}(\mathfrak{f}) \stackrel{\text{def}}{=} \{U_1(\mathfrak{f}), \ldots, U_s(\mathfrak{f})\}.$$

(1.3.8. Subanalytic Sets). We want to study the behavior of a semianalytic set under projections. As (1.3.4.1) shows, the projection of a semialgebraic set remains semialgebraic. The same statement however for the various semianalytic sets we have defined is false (as in the real or p-adic case). We therefore have to introduce the notion of subanalyticity.

Let M be an affinoid variety and $S \subset M$.

(1.3.8.1) We call S *(rigid) strongly subanalytic (globally strongly subanalytic, respectively)* in M, if there exist an n and a $T \subset M \times R^n$, with T strongly semianalytic in the R^n-direction (globally strongly semianalytic in the R^n-direction, respectively), such that $S = \pi(T)$, where $M \times R^n \xrightarrow{\pi} M$ is the canonical projection on the first factor.

(1.3.8.2) We call S *locally strongly subanalytic* in M, if there exists a finite admissible affinoid covering $\mathscr{X} = \{X_i\}_i$ of M, such that for each i, $S \cap X_i$ is strongly subanalytic in X_i.

(1.3.9) We could also define these notions in an arbitrary rigid analytic variety M by saying that a set $S \subset M$ is of a particular type in M, if for each admissible affinoid $U \subset M$, $S \cap U$ is of that type in U.

1.4. PROPOSITION. *Let M be a rigid analytic variety and S_1, $S_2 \subset M$ strongly subanalytic (globally strongly subanalytic or locally strongly subanalytic, respectively) in M, then $S_1 \cap S_2$ and $S_1 \cup S_2$ are likewise.*

Proof. As in the real or p-adic case. □

REMARK. However, to prove the same for the difference is not trivial at all and we shall need Quantifier Elimination for this.

1.5. PROPOSITION. *Let M be an affinoid variety and $S \subset M$ globally strongly* **D**-*semianalytic in M, then S is globally strongly subanalytic in M.*

More general, if $T \subset M \times R^m$ is globally strongly **D**-*semianalytic in the R^m-direction, then there exists a $V \subset M \times R^{m+n}$ which is globally strongly semianalytic in the R^{m+n}-direction, such that*

$$\theta|_V \colon V \to T$$

is bijective, where $M \times R^{m+n} \xrightarrow{\theta} M \times R^m$ is the projection map.

Proof. As in [DvdD] or [Lip]. □

2. A combinatorial lemma

2.1. We fix in the following $d \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$, and we will make no explicit reference to these two numbers in all what we define below. Let Δ be a non-empty subset of the set of all indices $i \in \mathbb{N}^N$ for which $|i| < d$, where we write $|i| = i_1 + \cdots + i_N$ for $i = (i_1, \ldots, i_N)$. Let $\omega = \operatorname{card}(\Delta) - 1$.

We put the lexicographic ordering on Δ. Suppose that $\Delta = \{\alpha_0, \ldots, \alpha_\omega\}$, with $\alpha_0 < \alpha_1 < \cdots < \alpha_\omega$.

For $s \in \mathbb{N}$, we denote the s-fold Cartesian product of $\{0, 1\}$ by

$$J_s = \{0, 1\}^s,$$

where we agree that $J_0 = \{0\}$. For brevity we will write J for J_ω. We consider each J_s, for $0 \leqslant s \leqslant \omega$, as a subset of J by identifying $(\varepsilon_1, \ldots, \varepsilon_s) \in J_s$ with $(\varepsilon_1, \ldots, \varepsilon_s, 0, \ldots, 0) \in J$.

For a fixed $j = (\varepsilon_1, \ldots, \varepsilon_\omega) \in J$ we define

$$k(j) \stackrel{\text{def}}{=} \alpha_s,$$

if $\varepsilon_s = 1$ and $\varepsilon_{s+1} = \cdots = \varepsilon_\omega = 0$, where we agree that $k(0) = \alpha_0$. For $1 \leqslant s \leqslant \omega$, let

$$\pi_s(j) \stackrel{\text{def}}{=} (\varepsilon_1, \ldots, \varepsilon_{s-1}, 0, \ldots, 0),$$

where we agree that $\pi_1(j) = 0$. So, for $l \in J_s$ we have according to our conventions that $\pi_{s+1}(l) = l$. Finally, we define two symbols; $\diamondsuit_0$ stands for the symbol $>$ and $\diamondsuit_1$ for the symbol $\leqslant$.

Let $A = (A_i)_{i \in \Delta}$ be a set of variables. We will work on the following (Zariski open) subset of $R^{\omega+1}$,

$$\mathbb{D} \stackrel{\text{def}}{=} \{a = (a_i)_{i \in \Delta} \in R^{\omega+1} \mid \text{not all } a_i \text{ vanish}\}.$$

From now on we will also fix $\pi \in \wp$. We set

$$\kappa = (\pi^{d^{N-1}}, \ldots, \pi^d, \pi) \in \wp^N,$$

where we drop reference to π. Note that we have the following estimate for each $i, j \in \Delta$, with $i < j$,

$$|\kappa^{j-i}| \leqslant |\pi|, \tag{1}$$

where we agree that for a vector $\xi = (\xi_1, \ldots, \xi_n) \in R^n$ and for a multi-index $j = (j_1, \ldots, j_n)$, we mean by

$$\xi^j = \xi_1^{j_1} \cdot \xi_2^{j_2} \cdots \xi_n^{j_n}.$$

Define the following functions for $i \in \Delta$ and $j \in J$,

$$H_{ij} \stackrel{\text{def}}{=} \frac{\kappa^{(k(j)-i)|j|} A_i}{A_{k(j)}}.$$

Hence $H_{ij} \in K(A)$. Note that $H_{ij} = 1$ if $i = k(j)$.

We consider the following subsets $V_j^{(t)} \subset \mathbb{D}$, for $j = (\varepsilon_1, \ldots, \varepsilon_\omega) \in J$ and $1 \leqslant t \leqslant \omega$, by the condition that, for $a = (a_i)_{i \in \Delta} \in \mathbb{D}$, we have that $a \in V_j^{(t)}$, if and only if,

$$|a_{k(\tilde{j})} \cdot \kappa^{-(|\tilde{j}|+1)k(\tilde{j})}| \diamondsuit_{\varepsilon_t} |a_{\alpha_t} \cdot \kappa^{-(|\tilde{j}|+1)\alpha_t}|,$$

where $\tilde{j} = \pi_t(j) = (\varepsilon_1, \ldots, \varepsilon_{t-1}, 0, \ldots, 0)$. Note that each $V_j^{(t)}$ is a basic algebraic subset of $R^{\omega+1}$. Finally we define the following (basic algebraic) subsets of $\mathbb{D}$,

$$U_j^{(t)} \stackrel{\text{def}}{=} \bigcap_{1 \leqslant s \leqslant t} V_j^{(s)}.$$

We set $U_j^{(0)} = \mathbb{D}$. We are especially interested in the case $t = \omega$ and therefore we will abbreviate these as

$$U_j \stackrel{\text{def}}{=} U_j^{(\omega)}.$$

We can now prove the following lemma.

2.2. LEMMA. *With the definitions and notations of* (2.1) *above we have the following facts.*

(i) $\{U_j\}_{j \in J}$ *is a partition of* $\mathbb{D}$,

(ii) *for each* $j \in J$ *and each* $a = (a_i)_{i \in \Delta} \in U_j$, *we have that* $a_{k(j)} \neq 0$,

(iii) *for each* $j \in J$ *and each* $a \in U_j$ *and* $i \in \Delta$, *we have that*

$$|H_{ij}(a)| \leqslant \begin{cases} 1 & \textit{for } i \leqslant k(j), \\ |\pi| & \textit{for } i > k(j). \end{cases}$$

REMARK. Note that for $a \in U_j$, $H_{ij}(a)$ is well defined by (ii).

Proof. We will give a proof by induction as follows. Define for each $0 \leqslant t \leqslant \omega$, the following conditions.

(i)$_t$ $\{U_j^{(t)}\}_{j\in J_t}$ is a partition of $\mathbb{D}$,
(ii)$_t$ for each $j\in J_t$ and each $a=(a_i)_{i\in\Delta}\in U_j^{(t)}$, we have that $a_{k(j)}\neq 0$,
(iii)$_t$ for each $j\in J_t$ and each $a\in U_j^{(t)}$ and $i\in\Delta$ with $i\leqslant\alpha_t$, we have that

$$|H_{ij}(a)|\leqslant\begin{cases}1 & \text{for } i\leqslant k(j),\\ |\kappa^{i-k(j)}| & \text{for } k(j)<i\leqslant\alpha_t.\end{cases}$$

Note that (i)$_\omega$ = (i), (ii)$_\omega$ = (ii) and (iii)$_\omega$ = (iii), so if we do induction on t, we have proved the statement. If $t=0$, then $J_0=\{0\}$ and $U_0^{(0)}=\mathbb{D}$ and everything is trivial. So we may assume that $t\neq 0$ and that we have already proved the three conditions for $t-1$.

Now $J_t=J_{t-1}\times\{0,1\}$. Let $j\in J_{t-1}$, then one easily verifies that $\{V_{(j,\varepsilon)}^{(t)}\}_{\varepsilon=0,1}$ is a partition of $\mathbb{D}$. Now, for $\varepsilon=0,1$, we have that

$$U_{(j,\varepsilon)}^{(t)}=U_j^{(t-1)}\cap V_{(j,\varepsilon)}^{(t)},$$

so that by induction we have proved (i)$_t$.

Let $\varepsilon\in\{0,1\}$ and $j\in J_{t-1}$, take $a=(a_i)_{i\in\Delta}\in U_{(j,\varepsilon)}^{(t)}$ and $i\in\Delta$. Then $\pi_t(j,\varepsilon)=j$ and

$$k(j,\varepsilon)=\begin{cases}k(j) & \text{if } \varepsilon=0\\ \alpha_t & \text{if } \varepsilon=1.\end{cases}$$

We split up in two cases. Suppose first of all that $\varepsilon=0$, then we have that $H_{i(j,0)}=H_{ij}$, since $k(j)=k(j,0)$ and $|j|=|(j,0)|$. By induction we also have that $a_{k(j)}\neq 0$ and so, by (iii)$_{t-1}$ we have also (iii)$_t$ for $i<\alpha_t$. But, since $a\in V_{(j,0)}^{(t)}$, we have that

$$|a_{k(j)}\cdot\kappa^{-(|j|+1)k(j)}|>|a_{\alpha_t}\cdot\kappa^{-(|j|+1)\alpha_t}|,$$

hence

$$\begin{aligned}|H_{\alpha_t,(j,0)}|&=\left|\frac{\kappa^{(k(j)-\alpha_t)|j|}a_{\alpha_t}}{a_{k(j)}}\right|\\ &<|\kappa^{(k(j)-\alpha_t)|j|+(|j|+1)(\alpha_t-k(j))}|\\ &=|\kappa^{\alpha_t-k(j)}|.\end{aligned}$$

So, this proves the case $\varepsilon=0$. Suppose therefore now that $\varepsilon=1$. So, now we have that $k(j,1)=\alpha_t$. First of all, from $a\in V_{(j,1)}^{(t)}$ we have that

$$|a_{k(j)}\kappa^{-(|j|+1)k(j)}|\leqslant|a_{\alpha_t}\kappa^{-(|j|+1)\alpha_t}|, \tag{2}$$

hence we must have that $a_{\alpha_t} \neq 0$, since $a_{k(j)} \neq 0$. Now

$$H_{ij} = \frac{\kappa^{(k(j)-i)|j|} A_i}{A_{k(j)}}$$

and

$$H_{i,(j,1)} = \frac{\kappa^{(\alpha_t - i)(|j|+1)} A_i}{A_{\alpha_t}}$$
$$= \frac{\kappa^{(\alpha_t - i)(|j|+1)} A_i}{A_{k(j)}} \cdot \frac{A_{k(j)}}{A_{\alpha_t}}.$$

So that, by putting this all together and using (2), we get that

$$|H_{i,(j,1)}(a)| \leqslant |H_{ij}(a)| \cdot |\kappa^{k(j)-i}|.$$

Take $i < \alpha_t$, then either $i < k(j)$, so that $|H_{ij}(a)| \leqslant 1$ by induction and from (1), we get that $|H_{i,(j,1)}(a)| \leqslant 1$ or either $i > k(j)$ and hence by induction we also have that $|H_{i,(j,1)}(a)| \leqslant 1$. □

REMARK. When we would define $\diamondsuit_0$ as $\geqslant$ instead of $>$, we would get a covering $\{U_j\}_{j \in J}$ instead of a partition with the same properties, where each U_j remains an algebraic subset.

2.3. We show here how we will make use of this combinatorial lemma in order to obtain in the next section Weierstrass Preparation with parameters. With the notations as in (2.1), the A_i will play the role of parameters. More concrete, define the following general polynomial $\Gamma \in \mathbb{Z}[A, Y]$,

$$\Gamma \stackrel{\text{def}}{=} \sum_{i \in \Delta} A_i Y^i,$$

where $Y = (Y_1, \dots, Y_N)$ is set of variables. For a fixed $j \in J$ we define the following rational function over K,

$$\Gamma_j = \frac{\kappa^{k(j)|j|}}{A_{k(j)}} \Gamma(Y/(\kappa^{|j|})). \tag{1}$$

One now easily verifies that

$$\Gamma_j = \sum_{i \in \Delta} H_{ij} Y^i. \tag{2}$$

Let us also introduce for $i \in \Delta$ and $j \in J$, with $i \neq k(j)$, the following functions

$$N_{ij} \overset{\text{def}}{=} \begin{cases} H_{ij} & \text{for } i < k(j) \\ (1/\pi)H_{ij} & \text{for } i > k(j). \end{cases}$$

Hence for $a \in U_j$ we have by (iii) of the Lemma (2.2) that

$$|N_{ij}(a)| \leqslant 1. \tag{3}$$

Let for a fixed $j \in J$, $V_j = (V_{ij})_{\substack{i \in \Delta \\ i \neq k(j)}}$ be another set of variables, set

$$N_j = (N_{ij})_{\substack{i \in \Delta \\ i \neq k(j)}},$$

and define the following polynomials over R,

$$\tilde{\Gamma}_j(Y, V_j) \overset{\text{def}}{=} \sum_{\substack{i \in \Delta \\ i < k(j)}} V_{ij} Y^i + Y^{k(j)} + \sum_{\substack{i \in \Delta \\ i > k(j)}} \pi V_{ij} Y^i.$$

Then from (1) and (2) we finally get for each $a \in U_j$, that

$$\Gamma(a, Y) = a_{k(j)} \kappa^{-k(j)|j|} \tilde{\Gamma}_j(\kappa^{|j|} Y, N_j(a)). \tag{4}$$

The advantage of this formula now is that $\tilde{\Gamma}_j$ has a coefficient one at $Y^{k(j)}$ and all the coefficients at a lexicographically larger exponent are divisible by a fixed element $\pi \in \wp$.

3. Weierstrass preparation with parameters

3.1. THEOREM (Weierstrass Preparation for Overconvergent Power Series). *Let A be an affinoid algebra, then the Weierstrass Preparation Theorem holds for $A\langle\langle X\rangle\rangle$. More explicitly, let $f \in A\langle\langle X\rangle\rangle$ be regular in X_n of degree k, where $X = (X_1, \ldots, X_n)$, and $g \in A\langle\langle X\rangle\rangle$ an arbitrary element. Then there exist unique elements $q \in A\langle\langle X\rangle\rangle$ and $r \in A\langle\langle X'\rangle\rangle[X_n]$, with $\deg_{X_n} r < k$, such that*

$$g = qf + r.$$

Moreover, there exist a unique multiplicative unit $U \in A\langle\langle X\rangle\rangle$ and a unique Weierstrass polynomial $P \in A\langle\langle X'\rangle\rangle[X_n]$ of degree k in X_n, such that

$$f = U \cdot P.$$

REMARK. We call f (ultrametric) *regular* in X_n of degree k, when f can be written as

$$f = \sum_{i=0}^{\infty} a_i(X')X_n^i,$$

where the $a_i \in A\langle X'\rangle$ are such that $|a_i| < |a_k|$ for $i > k$ and a_k is a multiplicative unit of norm equal to $|f|$ and where we mean by $X' = (X_1, \ldots, X_{n-1})$. We call a polynomial $P \in A\langle X'\rangle[X_n]$ a (ultrametric) *Weierstrass polynomial* in X_n, if P is monic and of norm one. In particular, a Weierstrass polynomial is regular.

If G is a K-subalgebra of $\mathbb{F}(A)$, then we call a unit u in G *multiplicative*, if, for each element $a \in G$, we have that $|ua| = |a| \cdot |u|$. This implies that, for each $x \in \operatorname{Sp} A$, we have that $|u(x)| = |u|$.

Proof. See [Sch 1, Chapter III, Theorem 2.2.5]. □

3.2. COROLLARY. *Let A be an affinoid algebra. Then the* (*ultrametric*) *Weierstrass Preparation Theorem holds also for $A\langle\langle \mathbf{D}\rangle\rangle\langle\langle X\rangle\rangle$.*

Proof. Let $f \in A\langle\langle \mathbf{D}\rangle\rangle\langle\langle X\rangle\rangle$ be regular in X_n of degree k. We can find a function $F \in A\langle\langle X, T\rangle\rangle$, regular in X_n of the same degree k, where $T = (T_1, \ldots, T_s)$, and $\mathbf{D}$-functions $v_i \in A\langle\langle \mathbf{D}\rangle\rangle$, such that if we put $v = (v_1, \ldots, v_s)$, we have that

$$f = F(X, v).$$

Applying the Weierstrass Preparation Theorem for overconvergent power series (3.1) to F and then substituting v for T gives us the desired result. □

3.3. LEMMA. *Let A be an affinoid algebra and $f \in A\langle X\rangle$. Write out f as $f = \Sigma_i a_i X^i$, where $X = (X_1, \ldots, X_n)$. Then there exist $d \in \mathbb{N}$ and $b_{ij} \in A$, for $|j| < d$, such that for each i with $|i| \geqslant d$ we have that*

$$a_i = \sum_{|j|<d} b_{ij} a_j,$$

with $|b_{ij}| < 1$ and for fixed j, we have that $|b_{ij}| \to 0$ as $|i| \to \infty$.

Moreover, if $f \in A\langle\langle X\rangle\rangle$, then there exists a $\pi \in \wp$, such that, for fixed $|j| < d$, $|b_{ij}| \cdot |\pi|^{-|i|} \to 0$ as $|i| \to \infty$. In other words, the b_{ij} overconverge.

The same statement holds for A replaced by $A\langle\langle \mathbf{D}\rangle\rangle$.

Proof. Let $\mathfrak{a}$ be the ideal in A generated by all the a_i. Let $d_0 \in \mathbb{N}$ be such that the a_j for $|j| < d_0$ generate $\mathfrak{a}$ (A noetherian), then by [BGR, 5.2.7. Proposition 1], there exists a $\rho > 0$ with the property that, for each $a \in \mathfrak{a}$, we can find $t_j \in A$, such that

$$a = \sum_{|j|<d_0} t_j a_j \quad \text{and} \quad \max_{|j|<d_0} |t_j| \leqslant \rho |a|.$$

Since $|a_i| \to 0$ when $|i| \to \infty$, we can choose $d \geqslant d_0$ such that, for each i with $|i| \geqslant d$, we have that $\rho|a_i| < 1$. So, for $|i| \geqslant d$ and $|j| < d_0$, there exist $b_{ij} \in A$, such that

$$a_i = \sum_{|j| < d_0} b_{ij} a_j, \tag{1}$$

with $|b_{ij}| \leqslant \rho|a_i| < 1$. So, if we set $b_{ij} = 0$ for $d_0 < |j| < |d|$, we get the desired result.

The overconvergency now follows from the estimate $|b_{ij}| \leqslant \rho|a_i|$.

To prove the case over $A\langle\langle \mathbf{D} \rangle\rangle$, just replace the $\mathbf{D}$-functions by overconvergent variables, use the previous case and then substitute the $\mathbf{D}$-functions back. □

REMARK. Let us reformulate this in a more convenient form. Let A be an affinoid algebra and $f \in A\langle X \rangle$ and suppose for simplicity that $|f| \leqslant 1$, then there exists a number $d \in \mathbb{N}$ and, for each j with $|j| < d$, elements $a_j \in A$ of norm less or equal than one and functions $\varphi_j \in A\langle X \rangle$, with $\varphi_j \in (X)^d A\langle X \rangle$ and $|\varphi_j| < 1$, such that

$$f = \sum_{|j| < d} a_j (X^j + \varphi_j).$$

Moreover, if $f \in A\langle\langle X \rangle\rangle$ then also $\varphi_j \in A\langle\langle X \rangle\rangle$. If $f \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle X \rangle\rangle$, then also $\varphi_j \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle X \rangle\rangle$ and $a_j \in A\langle\langle \mathbf{D} \rangle\rangle$.

3.4. LEMMA (Weierstrass Preparation with Parameters). *Let $M = \operatorname{Sp} A$ be an affinoid variety and $f \in A\langle\langle Y \rangle\rangle$, with $|f| \leqslant 1$, where $Y = (Y_1, \ldots, Y_N)$ and take $\delta < 1$. Denote by*

$$D(f) \stackrel{\text{def}}{=} \{x \in M \mid f(x, Y) \textit{ not identically zero}\}.$$

Then we can associate to f and δ the following data

- *a finite partition $\{W_j\}_{j \in J}$ of $D(f)$, with each W_j a basic subset of M,*
- *a Weierstrass automorphism τ of the Y-variables,*
- *elements $\pi_j = (\pi_{j1}, \ldots, \pi_{jN}) \in R^N$ with $\delta \leqslant |\pi_{jk}|$,*
- *Weierstrass polynomials $G_j \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y' \rangle\rangle[Y_N]$ in Y_N,*
- *multiplicative units $U_j \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y \rangle\rangle$,*
- *elements $b_j \in A$, which nowhere vanish on W_j,*

such that for each $j \in J$ and for each $x \in W_j$,

$$f(x, \tau(Y)) = b_j(x) G_j(x, \pi_j Y) U_i(x, \pi_j Y).$$

Moreover, if $f \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y \rangle\rangle$, then for each $j \in J$ we have that W_j is a strongly

D-*basic subset of M and $b_j \in A\langle\langle \mathbf{D} \rangle\rangle$.*

REMARK. By πY we mean $(\pi_1 Y_1, \ldots, \pi_N Y_N)$, for $\pi = (\pi_1, \ldots, \pi_N)$.

Proof. We can find a $d \in \mathbb{N}$, and for all multi-indices $j = (j_1, \ldots, j_N)$ with $|j| < d$, elements $a_j \in A$ ($a_j \in A\langle\langle \mathbf{D} \rangle\rangle$, respectively) of norm less than or equal to one and functions $\varphi_j \in A\langle Y \rangle$ ($\varphi_j \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y \rangle\rangle$, respectively) as given by (3.3), such that

$$f = \sum_{|j|<d} a_j(Y^j + \varphi_j), \tag{1}$$

with $\varphi_j \in (Y)^d$ and $|\varphi_j| < 1$.

We now adopt the notations and terminology of (2.1) and of (2.3) for our d and N of above. We take for Δ the set of all indices i with $|i| < d$. From the given partition $\{U_j\}_{j\in J}$ of $\mathbb{D}$, we can derive a partition of $D(f)$ as follows. Put $a = (a_i)_{i\in\Delta}$. For $j \in J$ define the set

$$W_j \stackrel{\text{def}}{=} \{x \in M \mid a(x) \in U_j\}.$$

Note that by (1), $a(x) \in \mathbb{D}$ if and only if $x \in D(f)$, hence $\{W_j\}_{j\in J}$ does constitute a partition of $D(f)$ and each W_i is a (strongly **D**-) basic subset of M.

Now we have by (1) that

$$f = \Gamma(a, Y) + \sum_{i\in\Delta} a_i \varphi_i. \tag{2}$$

Choose $\pi \in \wp$, such that for $\kappa = (\pi^{d^{N-1}}, \ldots, \pi^d, \pi)$ as defined in (2.1), we have that for each $i \in \Delta$, the functions $\varphi_i(Y/\kappa^{d-1})$ remain overconvergent and of norm strictly less than one, and such that $\delta \leqslant |\pi|^{d^N}$. Hence, if we define for each $i \in \Delta$ and $j \in J$, the functions

$$\Phi_{ij}(Y) \stackrel{\text{def}}{=} \varphi_i(Y/\kappa^{|j|}),$$

then we have that $\Phi_{ij} \in A\langle\langle Y \rangle\rangle$ ($\Phi_{ij} \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y \rangle\rangle$, respectively) and $|\Phi_{ij}| < 1$. Let us call $\pi_j = \kappa^{|j|}$ and define for each $j \in J$,

$$F_j = \tilde{\Gamma}_j(Y, V_j) + \sum_{\substack{i\in\Delta \\ i<k(j)}} \pi_j^i V_{ij}\Phi_{ij} + \pi_j^{k(j)}\Phi_{k(j),j} + \sum_{\substack{i\in\Delta \\ i>k(j)}} \pi\pi_j^i V_{ij}\Phi_{ij},$$

where the $V_j = (V_{ij})_{i\in\Delta}$ are a set of variables and $\tilde{\Gamma}_j$ is defined in (2.3). Then from (4) of (2.3) and (2) of above, we have for each $x \in W_j$, that

$$f\left(x, \frac{Y}{\pi_j}\right) = \pi_j^{-k(j)} a_{k(j)}(x) F_j(Y, N_j(a(x))). \tag{3}$$

Now, since $|\Phi_{ij}| < 1$ we have that

$$F_j \equiv \sum_{\substack{i\in\Delta\\ i<k(j)}} V_{ij}Y^i + Y^{k(j)} \mod \wp.$$

As in [DvdD], we can therefore find a Weierstrass automorphism τ of the Y-variables, such that $\tau(F_j)$ becomes regular in Y_N. More explicitly, let τ be defined by

$$\tau(Y_i) = Y_i + Y_N^{d^{N-i}} \quad \text{for } i = 1,\ldots,N-1,$$
$$\tau(Y_N) = Y_N.$$

We then apply the Weierstrass Preparation Theorem (3.1) ((3.2), respectively) to get

$$\tau(F_j) = u_j \cdot g_j, \tag{4}$$

with u_j a multiplicative unit in $A\langle\langle V_j, Y\rangle\rangle$ (in $A\langle\langle \mathbf{D}\rangle\rangle\langle\langle V_j, Y\rangle\rangle$, respectively) and $g_j \in A\langle\langle V_j, Y'\rangle\rangle[Y_N]$ ($g_j \in A\langle\langle \mathbf{D}\rangle\rangle\langle\langle V_j, Y'\rangle\rangle[Y_N]$, respectively) a Weierstrass polynomial in Y_N. From the definition of the N_{ij}, we are inspired to define the following $\mathbf{D}$-functions for $i \in \Delta$ and $j \in J$,

$$v_{ij} \stackrel{\text{def}}{=} \begin{cases} \mathbf{D}(\pi_j^{k(j)} a_i,\ \pi_j^i a_{k(j)}) & \text{for } i < k(j),\\ \mathbf{D}(\pi_j^{k(j)} a_i,\ \pi\pi_j^i a_{k(j)}) & \text{for } i > k(j),\end{cases}$$

so that by (iii) of (2.2) and the definition of the N_{ij}, we have that for $x \in W_j$,

$$v_{ij}(x) = N_{ij}(a(x)).$$

Put $v_j = (v_{ij})_{i\in\Delta}$ and define, by substituting v_j for V_j,

$$G_j(Y) \stackrel{\text{def}}{=} g_j(Y, v_j),$$
$$U_j(Y) \stackrel{\text{def}}{=} u_j(Y, v_j),$$
$$b_j \stackrel{\text{def}}{=} \pi_j^{-k(j)} a_{k(j)}.$$

So each $U_j, G_j \in A\langle\langle \mathbf{D}\rangle\rangle\langle\langle Y\rangle\rangle$. Moreover, G_j is a Weierstrass polynomial in Y_N and U_j is a multiplicative unit. Note that by (ii) of (2.2), b_j nowhere vanishes on W_j. Now, finally we have, from (3) and (4), that for each $j \in J$ and $x \in W_j$,

$$f(x, \tau(Y)) = b_j(x) \cdot G_j(x, \pi_j Y) \cdot U_j(x, \pi_j Y). \qquad \square$$

REMARK. Note that the Weierstrass automorphism τ does not depend on f, but only on d and N.

4. Locally and globally strongly subanalytic sets

The aim of this section is to prove that strongly subanalytic sets and even locally strongly subanalytic sets are globally strongly subanalytic sets. This result is needed to prove the Quantifier Elimination in the following section.

4.1. LEMMA. *Let M be an affinoid variety and $W \subset M \times R^m$ a strongly rational domain in the R^m-direction. If B is a strongly basic subset of W in the R^m-direction, then there exists $T \subset M \times R^{m+n}$, which is globally strongly semianalytic in the R^{m+n}-direction, such that $\pi(B) = \pi_1(T)$, where $M \times R^m \xrightarrow{\pi} M$ ($M \times R^{m+n} \xrightarrow{\pi_1} M$, respectively) is the projection map.*

Proof. Let $M = \operatorname{Sp} A$ and $Y = (Y_1, \ldots, Y_m)$. By the very definition of a strongly rational domain, there exist $f_0, \ldots, f_s \in A\langle\langle Y\rangle\rangle$, generating the unit ideal in $A\langle\langle Y\rangle\rangle$, such that

$$W = \{(x, y) \in M \times R^m \mid \forall i : |f_i(x, y)| \leqslant |f_0(x, y)|\}.$$

Hence, there exist $g_k, h_k \in A\langle T\rangle\langle\langle Y\rangle\rangle$, where $T = (T_1, \ldots, T_s)$ and symbols $\diamondsuit_k \in \{\leqslant, <\}$, such that

$$B = \left\{(x, y) \in W \mid \forall k : \left| g_k\left(x, y, \frac{f(x, y)}{f_0(x, y)}\right)\right| \diamondsuit_k \left| h_k\left(x, y, \frac{f(x, y)}{f_0(x, y)}\right)\right|\right\}.$$

Therefore, if we define $\tilde{B} \subset M \times R^{m+s}$ by the conditions that an element $(x, y, t) \in M \times R^{m+n}$ belongs to $\tilde{B}$ if and only if

$$\forall k : |g_k(x, y, t)| \diamondsuit_k |h_k(x, y, t)|,$$

$$\forall i : f_i(x, y) = t_i f_0(x, y),$$

then we have that $\kappa(\tilde{B}) = B$. Hence

$$\theta(\tilde{B}) = \pi(B), \tag{1}$$

where $M \times R^{m+s} \xrightarrow{\kappa} M \times R^m$ and $M \times R^{m+s} \xrightarrow{\theta} M$ are the projection maps.

The point now is that $\tilde{B}$ is not strongly basic in the T-variables, since the g_k, h_k are not overconvergent in T. We will overcome this problem by using the $f_i - T_i f_0$, which are overconvergent in Y and T, in order to make also the remaining functions overconvergent (even polynomial) in T, by dividing these functions by Weierstrass polynomials, which we will obtain from (3.4).

Choose $\pi \in \wp$, such that all the $g_k(X, Y/\pi, T)$ and $h_k(X, Y/\pi, T)$ still are in $A\langle T\rangle\langle\langle Y\rangle\rangle$ and all the $f_i(Y/\pi) \in A\langle\langle Y\rangle\rangle$, remaining overconvergent in Y. Note that, since the f_i generate the unit ideal, we must have that, for $(x, y) \in W$, $f_0(x, y) \neq 0$. If we set

$$\gamma_i = f_i - \pi T_i f_0,$$

then $\gamma_i \in A\langle\langle Y, T\rangle\rangle$ and $\pi(W) \subset D(\gamma_i)$. Call $D = \bigcap_i D(\gamma_i)$, then also $\pi(W) \subset D$.

Therefore, we can find for each $i = 1, \ldots, s$ by (3.4) a Weierstrass automorphism τ_i of (Y, T_i) and a partition $\{U_{ij}\}_{j \in J}$ of D by basic subsets of M, where J is a finite set, and, for each $j \in J$, we can find elements $\pi_{ijk}, \theta_{ij} \in R$, with $|\pi_{ijk}| \geqslant |\pi|$ and $|\theta_{ij}| \geqslant |\pi|$, Weierstrass polynomials $H_{ij} \in A\langle\langle \mathbf{D}\rangle\rangle\langle\langle Y\rangle\rangle[T_i]$ in T_i, multiplicative units $v_{ij} \in A\langle\langle \mathbf{D}\rangle\rangle\langle\langle Y, T_i\rangle\rangle$ and elements $a_{ij} \in A$, nowhere vanishing on U_{ij}, such that for each $x \in U_{ij}$, we have that

$$\gamma_i(x, \tau_i(Y), \tau_i(T_i)) = a_{ij}(x) H_{ij}(x, \pi_{ij} Y, \theta_{ij} T_i) v_{ij}(x, \pi_{ij} Y, \theta_{ij} T_i),$$

where $\pi_{ij} = (\pi_{ij1}, \ldots, \pi_{ijN})$.

We extend each τ_i to the other T-variables, by putting $\tau_i(T_j) = T_j$ for $i \neq j$, so that by the definition of the τ_i in the proof of (3.4), we have that all τ_i commute. Define

$$\tau = \tau_1 \circ \cdots \circ \tau_s \quad \text{and} \quad \tau_{\hat{\imath}} = \tau \circ \tau_i^{-1}.$$

Hence $\tilde{H}_{ij} = \tau_{\hat{\imath}}(H_{ij}) \in A\langle\langle \mathbf{D}\rangle\rangle\langle\langle Y, T\rangle\rangle$ remains a Weierstrass polynomial in T_i. So, if we set $\tilde{v}_{ij} = \tau_{\hat{\imath}}(v_{ij})$, then $\tilde{v}_{ij}$ is a multiplicative unit in $A\langle\langle \mathbf{D}\rangle\rangle\langle\langle Y, T\rangle\rangle$. We have then for each $x \in U_{ij}$,

$$\gamma_i(x, \tau(Y), \tau(T)) = a_{ij}(x) \tilde{H}_{ij}(x, \pi_{ij} Y, \theta_{ij} T) \tilde{v}_{ij}(x, \pi_{ij} Y, \theta_{ij} T), \tag{2}$$

where we mean by $\theta_{ij} T = (T_1, \ldots, T_{i-1}, \theta_{ij} T_i, T_{i+1}, \ldots, T_s)$. Note that we have that $\tau(T) = T$.

Now, since τ is a automorphism, we have a $1 - 1$ correspondence between $\tilde{B}$ and the set defined by the conditions

$$\forall k\colon |g_k(x, \tau(y), \tau(t))| \mathbin{\Diamond_k} |h_k(x, \tau(y), \tau(t))|,$$
$$\forall i\colon f_i(x, \tau(y)) = t_i f_0(x, \tau(y)),$$

which both have the same image under the projection map

$$M \times R^{m+s} \xrightarrow{\theta} M,$$

and this image also equals $\pi(B)$ as pointed out by (1). Note that the $\tau(g_k)$ and the $\tau(h_k)$ remain overconvergent in Y since $\tau(T) = T$. Hence, we may as well assume that τ is the identity and therefore we will omit it in the sequel.

Let $\varepsilon_{ij} = (\pi/\theta_{ij})$, then $|\varepsilon_{ij}| \leqslant 1$. Therefore, if $e_{ij} = \deg_{T_i}(\tilde{H}_{ij})$ and if we put

$$\tilde{G}_{ij} \stackrel{\text{def}}{=} \varepsilon_{ij}^{e_{ij}} \cdot \tilde{H}_{ij}(Y, T_i', T_i/\varepsilon_{ij}),$$
$$\tilde{u}_{ij} \stackrel{\text{def}}{=} \varepsilon_{ij}^{-e_{ij}} \cdot \tilde{v}_{ij}(Y, T_i', T_i/\varepsilon_{ij}),$$

where T_i' stands for all the T-variables but the variable T_i, then we obtain that both $\tilde{G}_{ij}, \tilde{u}_{ij} \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y, T\rangle\rangle$ with $\tilde{G}_{ij}$ remaining a Weierstrass polynomial in T_i and $\tilde{u}_{ij}$ a multiplicative unit. So, we get from (2) that for each $x \in U_{ij}$,

$$\begin{aligned} f_i(x, Y) - T_i f_0(x, Y) &= \gamma_i(x, Y, T_i/\pi) \\ &= a_{ij}(x) \cdot \tilde{G}_{ij}(x, \pi_{ij} Y, T) \cdot \tilde{u}_{ij}(x, \pi_{ij} Y, T). \end{aligned} \tag{3}$$

Let $\alpha = (j_1, \ldots, j_s) \in J^s$ be a s-tuple of indices and set

$$V_\alpha = \bigcap_{i=1}^{s} U_{ij_i},$$

then $\{V_\alpha\}_{\alpha \in J^s}$ is a partition of D in basic subsets. Call

$$G_{i\alpha} \stackrel{\text{def}}{=} \tilde{G}_{ij_i}(\pi_{ij_i} Y, T),$$
$$u_{i\alpha} \stackrel{\text{def}}{=} \tilde{u}_{ij_i}(\pi_{ij_i} Y, T),$$

so that we still have that $G_{i\alpha}$, $u_{i\alpha} \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y, T\rangle\rangle$, with $G_{i\alpha}$ a Weierstrass polynomial in T_i and $u_{i\alpha}$ a multiplicative unit. Hence (3) now reads, with $\alpha \in J^s$ and $x \in V_\alpha$ and $1 \leqslant i \leqslant s$,

$$f_i(x, Y) - T_i f_0(x, Y) = a_{ij_i}(x) \cdot G_{i\alpha}(x, \pi_\alpha Y, T) \cdot u_{i\alpha}(x, \pi_\alpha Y, T). \tag{4}$$

We will now, for each $\alpha \in J^s$ and for each k, divide $g_k(Y/\pi_\alpha)$ and $h_k(Y/\pi_\alpha)$ (which are overconvergent in Y, by our choice of π) by all the Weierstrass polynomials $G_{i\alpha}$ to obtain

$$g_k(X, Y/\pi_\alpha, T) = \sum_{i=1}^{s} q_{i\alpha k} G_{i\alpha} + r_{\alpha k},$$

$$h_k(X, Y/\pi_\alpha, T) = \sum_{i=1}^{s} p_{i\alpha k} G_{i\alpha} + s_{\alpha k}, \tag{5}$$

with $q_{i\alpha k}, p_{i\alpha k} \in A\langle\langle \mathbf{D} \rangle\rangle \langle\langle T \rangle\rangle \langle Y \rangle$ and $r_{\alpha k}, s_{\alpha k} \in A\langle\langle \mathbf{D} \rangle\rangle \langle\langle Y \rangle\rangle [T]$.

We can now define, for each $\alpha = (j_1, \ldots, j_s) \in J^s$, a strongly **D**-basic subset $\tilde{B}_\alpha \subset M \times R^{m+s}$ in the R^{m+s}-direction in the following way. Given a point $(x, y, t) \in M \times R^{m+s}$, then $(x, y, t) \in \tilde{B}_\alpha$, if and only if

$$\forall i: G_{i\alpha}(x, \pi_\alpha y, t) = 0$$

and

$$\forall k: |r_{\alpha k}(x, \pi_\alpha y, t)| \diamondsuit_k |s_{\alpha k}(x, \pi_\alpha y, t)|.$$

We now claim that for each $\alpha \in J^s$, we have that

$$B \cap (V_\alpha \times R^m) = \theta(\tilde{B}_\alpha) \cap (V_\alpha \times R^m). \tag{6}$$

Indeed, suppose first of all that $(x, y) \in B \cap (V_\alpha \times R^m)$. Put $t_i = f_i(x, y)/f_0(x, y)$ and $t = (t_1, \ldots, t_s)$, hence $\forall i$: $f_i(x, y) = t_i f_0(x, y)$, so that by (4), since the $a_{i\alpha}$ nowhere vanish on V_α, we have, for each i, that $G_{i\alpha}(x, \pi_\alpha y, t) = 0$, hence by (5), we have for all k,

$$\begin{aligned} g_k(x, y, t) &= r_{\alpha k}(x, \pi_\alpha y, t), \\ h_k(x, y, t) &= s_{\alpha k}(x, \pi_\alpha y, t). \end{aligned} \tag{7}$$

So, $(x, y) \in \theta(\tilde{B}_\alpha)$. This proves one inclusion.

Suppose now that $(x, y) \in \theta(\tilde{B}_\alpha) \cap (V_\alpha \times R^m)$. Hence, there exists a $t \in R^s$, such that $(x, y, t) \in \tilde{B}_\alpha$, therefore all $G_{i\alpha}(x, \pi_\alpha y, t) = 0$. Hence, by (4),

$$t_i = \frac{f_i(x, y)}{f_0(x, y)},$$

and (7) also holds, so that $(x, y) \in B$. So that we have established (6) and we therefore get

$$\pi(B) \cap V_\alpha = \pi\theta(\tilde{B}_\alpha) \cap V_\alpha. \tag{8}$$

Now, since the $\tilde{B}_\alpha$ are globally $A\langle\langle \mathbf{D} \rangle\rangle \langle\langle Y \rangle\rangle [T]$-semianalytic, we get by applying (1.3.4.2) that for each $\alpha \in J^s$, $\theta(\tilde{B}_\alpha) \subset M \times R^m$ is globally strongly **D**-semianalytic in the R^m-direction. Hence by (1.5) there exist $T_\alpha \subset M \times R^{m+n}$, such that

$$\theta_1(T_\alpha) = \theta(\tilde{B}_\alpha), \tag{9}$$

where $M \times R^{m+n} \xrightarrow{\theta_1} M \times R^m$ is the projection map. Therefore, by (8) and (9), we have that

$$\pi_1(T_\alpha) \cap V_\alpha = \pi(B) \cap V_\alpha.$$

So, that if we set

$$T \overset{\text{def}}{=} \bigcup_{\alpha \in J^s} (V_\alpha \times R^{m+n}) \cap T_\alpha,$$

we have that T is globally strongly semianalytic in the R^{m+n}-direction and since $\{V_\alpha\}_\alpha$ is a covering of D and $\pi(B) \subset D$, we have that $\pi_1(T) = \pi(B)$. □

4.2. PROPOSITION. *Let M be a rigid analytic variety and $S \subset M$. Then the following statements are equivalent*

(1) *S is globally strongly subanalytic in M,*

(2) *S is strongly subanalytic in M,*

(3) *S is locally strongly subanalytic in M.*

Proof. It is enough to prove this statement for $M = \operatorname{Sp} A$ affinoid. Obviously (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(2) $\Rightarrow$ (1). So, there exists a $T \subset M \times R^m$ strongly semianalytic in the R^m-direction, such that

$$\pi(T) = S,$$

where $M \times R^m \xrightarrow{\pi} M$ is the projection map. Hence, there exists a strongly rational covering $\mathscr{C} = \{U_i\}_i$ in the R^m-direction of $M \times R^m$, such that for each i, $T \cap U_i$ is globally strongly semianalytic in the R^m-direction in U_i. By the previous Lemma (4.1), there exists therefore for each i, a subset $V_i \subset M \times R^{m+n}$ which is globally strongly semianalytic in the R^{m+n}-direction, such that

$$\pi_1(V_i) = \pi(T \cap U_i),$$

where $M \times R^{m+n} \xrightarrow{\pi_1} M$ is the projection map. So, if we put

$$V = \bigcup_i V_i,$$

then is V globally strongly semianalytic in the R^{m+n}-direction and

$\pi_1(V) = \pi(T) = S.$

(3) $\Rightarrow$ (2). By definition, there exists a finite admissible affinoid covering $\mathscr{X} = \{X_i\}_i$ of M, such that for each i, $S \cap X_i$ is strongly subanalytic in X_i, hence globally strongly subanalytic in X_i, by what we have proved above. So, for each i, there exists $T_i \subset X_i \times R^m$ globally strongly semianalytic in the R^m-direction, such that

$$\pi(T_i) = S \cap X_i.$$

Again by (4.1), since $X_i \times R^m$ is clearly a strongly rational domain of $M \times R^m$ in the R^m-direction, we can find $V_i \subset M \times R^{m+n}$ globally strongly semianalytic in the R^{m+n}-direction, such that $\pi_1(V_i) = \pi(T_i)$ and therefore, if we put

$$V \stackrel{\text{def}}{=} \bigcup_i V_i,$$

then $V \subset M \times R^{m+n}$ is globally strongly semianalytic in the R^{m+n}-direction and $\pi_1(T) = \bigcup_i \pi(T_i) = S$. □

5. Analytic quantifier elimination

5.1. LEMMA. *Let M be an affinoid variety and $B \subset M \times R^m$. Suppose that B is strongly* **D***-basic subset in the R^m-direction, then there exists a subset $\tilde{B} \subset M \times R^{m-1}$ which is globally strongly* **D***-semianalytic in the R^{m-1}-direction, such that*

$$\pi(B) = \tilde{\pi}(\tilde{B}),$$

where $M \times R^m \xrightarrow{\pi} M$ ($M \times R^{m-1} \xrightarrow{\tilde{\pi}} M$, respectively) is the projection map.

Proof. By the definition of a strongly **D**-basic subset, there exist **D**-functions $g_i \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y \rangle\rangle$, for $i = 1, \ldots, 2t$, where $Y = (Y_1, \ldots, Y_m)$ are variables and symbols $\Diamond_i \in \{\leq, <\}$, for $i = 1, \ldots, t$, such that

$$B = \{(x, y) \in M \times R^m \mid \forall i = 1, \ldots, t\colon |g_i(x, y)| \Diamond_i |g_{i+t}(x, y)|\}.$$

For each g_i we can find a d_i as given by (3.3) and let d be the maximum of all the d_i. We can now find by (3.4), for each $i = 1, \ldots, 2t$, a partition $\{U_{ij}\}_{j=1,\ldots,s}$ of $D(g_i)$ consisting out of strongly **D**-basic subsets of M, a Weierstrass automorphism τ (which works for all i by our choice of d and the remark after (3.3)), polynomials $G_{ij} \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y' \rangle\rangle[Y_m]$ in Y_m, multiplicative units $u_{ij} \in A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y \rangle\rangle$ and elements $a_{ij} \in A\langle\langle \mathbf{D} \rangle\rangle$, such that, for each $j = 1, \ldots, s$ and each $x \in U_{ij}$, we have that

$$g_i(x, \tau(Y)) = a_{ij}(x) \cdot G_{ij}(x, Y) \cdot u_{ij}(x, Y). \tag{1}$$

Without any loss of generality, we may assume that each $|u_{ij}| = 1$, so that, since the u_{ij} are multiplicative units, we have, for each $(x, y) \in M \times R^m$, that

$$|u_{ij}(x, y)| = 1. \tag{2}$$

For each i, set $U_{i0} = M \backslash D(g_i)$ and put $a_{i0} = u_{i0} = 1$ and $G_{i0} = 0$, then (1) also holds for $j = 0$ and $\{U_{ij}\}_{j=0,\ldots,s}$ is a partition of M consisting of strongly $\mathbf{D}$-basic subsets of M.

Put $J = \{0, \ldots, s\}^{2t}$ and take $\alpha = (j_1, \ldots, j_{2t}) \in J$, then we define

$$U_\alpha \overset{\text{def}}{=} \bigcap_{i=1}^{2t} U_{ij_i},$$

so that $\{U_\alpha\}_{\alpha \in J}$ is a partition of M in strongly $\mathbf{D}$-basic subsets of M. We therefore have by (1), for all $x \in U_\alpha$ and for each $i = 1, \ldots, 2t$, that

$$g_i(x, \tau(Y)) = a_{ij_i}(x) \cdot G_{ij_i}(x, Y) \cdot u_{ij_i}(x, Y). \tag{3}$$

Let therefore $\alpha = (j_1, \ldots, j_{2t}) \in J$ and define subsets $C_\alpha \subset M \times R^m$, by the following conditions. An element $(x, y) \in M \times R^m$ belongs to C_α, if and only if, $x \in U_\alpha$ and

$$\forall i = 1, \ldots, t : |a_{ij_i}(x) G_{ij_i}(x, y)| \diamondsuit_i |a_{t+i, j_{t+i}}(x) G_{t+i, j_{t+i}}(x, y)|.$$

So, we have, for $x \in M$,

$$(x, y) \in C_\alpha \Leftrightarrow (x, \tau(y)) \in B \cap (U_\alpha \times R^m). \tag{4}$$

Indeed, one verifies this directly using (2) and (3).

Therefore, if we define

$$C \overset{\text{def}}{=} \bigcup_{\alpha \in J} C_\alpha,$$

then C is globally $A\langle\langle \mathbf{D} \rangle\rangle\langle\langle Y' \rangle\rangle[Y_m]$-semianalytic, where $Y' = (Y_1, \ldots, Y_{m-1})$. Put

$$\tilde{B} = \theta(C),$$

where $M \times R^m \xrightarrow{\theta} M \times R^{m-1}$ is the projection map in the Y_m-direction, then we have by (1.3.4.2) that $\tilde{B}$ is globally strongly **D**-semianalytic in the R^{m-1}-direction in $M \times R^{m-1}$. Now, since $\{U_\alpha\}_{\alpha \in J}$ is a partition of M, we get from (4) that $\pi(B) = \pi(C)$ and therefore $\pi(B) = \tilde{\pi}(\tilde{B})$. □

REMARK. This is the analogue of the basic lemma from [DvdD] or from [Lip], from which Quantifier Elimination will easily follow, in the same way as in loc. cit.

5.2. THEOREM (Quantifier Elimination). *Let M be an affinoid variety and $S \subset M$. Then the following are equivalent*

(1) *S is strongly subanalytic,*

(2) *S is globally strongly* **D***-semianalytic.*

Proof. (2) ⇒ (1). This is (1.5).

(1) ⇒ (2). From (4.2) we know that S is globally strongly subanalytic. So, there exists a $T \subset M \times R^m$ globally strongly semianalytic in the R^m-direction, such that

$$\pi(T) = S,$$

where $M \times R^m \xrightarrow{\pi} M$ is the projection map. We will prove the following stronger statement.

ASSERTION. *If T is globally strongly* **D***-semianalytic in the R^m-direction, then $\pi(T)$ is globally strongly* **D***-semianalytic in M.*

We will prove this by induction on m. The case $m = 0$ being trivial, suppose $m > 0$.

Without loss of generality, we may suppose that T is a strongly **D**-basic subset in the R^m-direction. By the previous Lemma (5.1), we can find $\tilde{T} \subset M \times R^{m-1}$ globally strongly **D**-semianalytic in the R^{m-1}-direction, such that

$$\pi(T) = \tilde{\pi}(\tilde{T}),$$

where $M \times R^{m-1} \xrightarrow{\tilde{\pi}} M$ is the projection map and by induction $\tilde{\pi}(\tilde{T})$ is globally strongly **D**-semianalytic in M. □

5.3. COROLLARY. *Let M be a rigid analytic variety, then the set of strongly subanalytic subsets of M forms a Boolean algebra.*

5.4. COROLLARY. *Let M be an affinoid variety and $S \subset M$. If S is strongly subanalytic in M, then also its closure* cl(S) *in the canonical topology is strongly subanalytic.*

REMARK. With the canonical topology on M, we mean the topology induced by the supremum norm on A, where $M = \operatorname{Sp} A$. This is the same as the topology induced on M by any closed immersion into R^n, the latter with the topology induced by the norm

$$|(x_1,\ldots,x_n)| \overset{\text{def}}{=} \max_i |x_i|,$$

see [BGR, 7.2].

Proof. As pointed out by the remark, we can restrict ourselves to the case of $M = R^n$. Now we have the following equivalence for $x \in R^n$, which is used for instance in [DvdD] to prove the same statement in the p-adic case,

$$x \in \mathrm{cl}(S) \Leftrightarrow \forall u \in R\colon (u \neq 0 \Rightarrow \exists z \in S\colon |z - x| \leqslant |u|). \tag{1}$$

However, the formula on the right hand of (1) is not a formula from which we can eliminate the quantifiers (whereas in [DvdD], this caused no problems), since the condition $\exists z \in S$, need not to be overconvergent in the z-variable. The reason for our problem is the fact that the projection of a strongly subanalytic set need not to be strongly subanalytic anymore (though subanalytic). Therefore we have to refine (1) in order to obtain a good formula. Namely, let $\pi \in \wp$, then we have for $x \in R^n$,

$$\begin{aligned} x \in \mathrm{cl}(S) &\Leftrightarrow \forall u \in R\colon (u \neq 0 \Rightarrow \exists z \in S\colon |z - x| \leqslant |\pi^2 u|) \\ &\Leftrightarrow \neg(\exists u \in R\colon u \neq 0 \wedge \neg(\exists t \in R^n\colon x + \pi^2 u t \in S)). \end{aligned}$$

From this it is now an exercise in elimination of quantifiers to prove that cl(S) is strongly subanalytic. □

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