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An elliptic analogue of the multiple Dedekind sums

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1. Introduction

The multiple Dedekind sum investigated by Carlitz ([Ca1], [Ca2]), Zagier ([Za1], [Za2], [H-Z]), and Berndt ([Ber]) is a natural generalization of the classical Dedekind sum. In this paper we introduce its elliptic analogue. Our method and results are quite similar to [Za1] except the use of an elliptic function in place of the cotangent function which appeared there. In fact we show the reciprocity law and explicit calculation for our Dedekind sums in some special cases. By a limiting procedure we can recover the corresponding results on multiple Dedekind (cotangent) sums.

Elliptic analogues of the classical Dedekind sum were already introduced by Sczech and investigated in detail ([Scz], [It1], [It2]). We could not clarify the relation between their results and ours since they use nonanalytic doubly periodic function and their main interest seems to be in CM case.

In the above cited works Zagier investigated relations between topology and multiple Dedekind sums. Though "formal" topological invariants appear also in our case, any real relation between our Dedekind sums and topology is not known.

2. Review from the theory of elliptic functions

We quote from ([HBJ], Appendix 1 by Skoruppa) some results on elliptic functions and modular forms required later. Let τ be an element of the upper half plane and L_τ(resp. L'_τ) denotes the lattice $2\pi i(\mathbf{Z}\tau + \mathbf{Z})$ (resp. $2\pi i(2\mathbf{Z}\tau + \mathbf{Z})$) of the complex plane. We denote as usual

$$\begin{aligned}
\wp(\tau, z) &= \frac{1}{z^2} + \sum{}' \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \\
g_2(\tau) &= 60 \sum{}' \frac{1}{\omega^4}, \\
g_3(\tau) &= 140 \sum{}' \frac{1}{\omega^6},
\end{aligned}$$

$$e_1(\tau) = \wp(\tau, \pi i),$$

where $\sum'$ denotes the summation over all $\omega \in L_\tau$ except the origin. It is well known that $g_2(\tau)$ and $g_3(\tau)$ are modular forms for $SL_2(\mathbf{Z})$, while $e_1(\tau)$ is a modular form for

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}); c \equiv 0 (\bmod 2) \right\}.$$

Now we introduce the function $\varphi(\tau, z)$ determined by

$$\varphi(\tau, z) = \sqrt{\wp(\tau, z) - e_1(\tau)},$$
$$\varphi(\tau, z) = \frac{1}{z} + O(1), \quad (z \to 0),$$

which plays the principal role throughout this paper. The function $\varphi(\tau, z)$ is a meromorphic Jacobi form for $\Gamma_0(2)$ of weight 1 and index 0 by the periodicity for L'_τ and

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)(c\tau + d)^{-1} = \varphi(\tau, z),$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$(see [E-Z] for Jacobi forms). Furthermore $\varphi(\tau, z)$ has the following q-expansion:

$$\varphi(\tau, z) = \frac{1}{2}\frac{\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} \prod_{n=1}^{\infty} \frac{(1 + q^n\zeta)(1 + q^n\zeta^{-1})(1 - q^n)^2}{(1 - q^n\zeta)(1 - q^n\zeta^{-1})(1 + q^n)^2}, \tag{1}$$

where $\zeta = e^z$ and $q = e^{\pi i \tau}$.

From the definition of $\varphi(\tau, z)$ it is easily seen that

$$\varphi(\tau, z) = \frac{1}{z}(1 + H_2 z^2 + H_4 z^4 + H_6 z^6 + \cdots), \tag{2}$$

where H_{2n} is a polynomial in e_1, g_2, g_3 with rational coefficients and a modular form for $\Gamma_0(2)$ of weight $2n$, whose first few terms are

$$H_2 = -\tfrac{1}{2}e_1,$$
$$H_4 = \tfrac{1}{40}g_2 - \tfrac{1}{8}e_1^2,$$
$$H_6 = \tfrac{1}{56}g_3 + \tfrac{1}{80}g_2 e_1 - \tfrac{1}{16}e_1^3.$$
$$\vdots$$

Let $t_0, \ldots, t_n$ be $n + 1$ indeterminates and consider the formal expansion

$$z^{n+1} \prod_{k=0}^{n} \varphi(\tau, t_k z) = 1 + H_2(t_0^2 + \cdots + t_n^2)z^2 + \cdots. \tag{3}$$

We define the polynomial $M_{n,\tau}(t_0,\dots,t_n)$ by the coefficient of z^n in the above power series. Note that the coefficients of $M_{n,\tau}$ are modular forms for $\Gamma_0(2)$ of weight n and vanish identically for odd n. Since $M_{n,\tau}$ is a symmetric polynomial in $t_0,\dots,t_n$ it can be written in a polynomial of the elementary symmetric polynomials $p_1,\dots,p_n$ of t_k's i.e. $M_{n,\tau}(t_0,\dots,t_n) = K_{n,\tau}(p_1,\dots,p_n)$. Polynomial $K_{n,\tau}$ appeared in "elliptic genus theory" in topology ([HBJ, Ch1]).

3. Definition and the reciprocity law

Let p be a natural number and $a_1,\dots,a_r$ integers coprime to p such that $p + a_1 + \cdots + a_r$ is even. For $\Im(\tau) > 0$ we define the *multiple elliptic Dedekind sum* by

$$D_\tau(p; a_1,\dots,a_n) = \sum_{\substack{m,n=0\\(m,n)\neq(0,0)}}^{p-1} (-1)^m \prod_{k=1}^{r} \varphi\left(\tau, \frac{2\pi i a_k(m\tau+n)}{p}\right). \tag{4}$$

It is easily seen that $D_\tau(p;\dots)$ is a modular form of level p and weight r ([E-Z, p.10 Theorem 1.3]). Then we obtain the following *reciprocity law*:

THEOREM 1. *Let $a_0, a_1,\dots,a_r$ be pairwise coprime natural numbers such that $a_0 + \cdots + a_r$ is even. Then*

$$\sum_{k=0}^{r} \frac{1}{a_k} D_\tau(a_k; a_0,\dots,a_{k-1},a_{k+1},\dots,a_r) = -M_{r,\tau}(a_0,\dots,a_r). \tag{5}$$

Proof. Set $\Phi(\tau,z) = \prod_{k=0}^{r}\varphi(\tau,a_k z)$. From the periodicity of $\varphi(\tau,z)$ with respect to the lattice L'_τ, $\Phi(\tau,z)$ is an elliptic function for the same lattice. But moreover by the assumption that $a_0 + \cdots + a_r$ is even and the fact $\varphi(\tau,z+\tau) = -\varphi(\tau,z)$ it is an elliptic function for the larger lattice L_τ. Thus the sum of residues of $\Phi(\tau,z)$ over a fundamental domain $\mathbf{C}/L_\tau$ must be zero.

Now we consider the fundamental domain $2\pi i(x\tau + y), 0 \leqslant x,y < 1$, and calculate the residues at each poles. At first the residue at the origin is nothing but $M_{r,\tau}(a_0,\dots,a_r)$. As for the other poles, since a_k's are pairwise coprime they all are simple and can be represented uniquely in the form

$$a(k,m,n) = \frac{2\pi i(m\tau+n)}{a_k}, \quad 0 \leqslant m,n < a_k, (m,n) \neq (0,0).$$

Since

$$\operatorname*{Res}_{z=a(k,m,n)} \varphi(\tau,a_k z) = \frac{1}{a_k}\operatorname*{Res}_{z=2\pi i(m\tau+n)} \varphi(\tau,z) = \frac{(-1)^m}{a_k},$$

the residue of $\Phi(\tau,z)$ at $z = a(k,m,n)$ is

$$\frac{(-1)^m}{a_k}\prod_{j=0,j\neq k}^{r} \varphi\left(\tau, \frac{2\pi i a_j(m\tau+n)}{a_k}\right),$$

which proves the theorem. □

Since $D_\tau(1; a_1, \ldots, a_r) = 0$ we can calculate the sum of the form $D_\tau(n; 1, \ldots, 1)$ as polynomials of n with coefficients in $\mathbf{Q}[e_1, g_2, g_3]$. For example

$$D_\tau(n; 1, 1) = \tfrac{1}{2} e_1(n^2 + 2) \tag{6}$$

$$D_\tau(n; 1, 1, 1, 1) = -\left(\left(\frac{g_2}{40} - \frac{1}{8}e_1^2\right) n^4 + e_1^2 n^2 + \left(\frac{g_2}{10} + e_1^2\right)\right). \tag{7}$$

4. The relation to cotangent sums

Let p be a natural number and $a_1, \ldots, a_r$ be integers coprime to p. Zagier [Za1] considered the following multiple Dedekind sum:

$$d(p; a_1, \ldots, a_r) = (-1)^{\frac{r}{2}} \sum_{k=1}^{p-1} \cot\left(\frac{\pi k a_1}{p}\right) \cdots \cot\left(\frac{\pi k a_r}{p}\right). \tag{8}$$

Note that the above sum vanishes identically when r is odd. The relation to our sum is the following:

THEOREM 2. *If $p + a_1 + \cdots + a_r$ is even then*

$$\lim_{\Im(\tau)\to\infty} D_\tau(p; a_1, \ldots, a_r) = \frac{1}{2^r}\left(d(p; a_1, \ldots, a_r) + (-1)^r p \sum_{\nu=1}^{p-1} (-1)^{\nu + [\frac{a_1 \nu}{p}] + \cdots + [\frac{a_r \nu}{p}]}\right).$$

Proof. Consider the q-expansion (1) of $\varphi(\tau, z)$. When $0 \leqslant x < 1$ in $z = 2\pi i(x\tau + y)$, the infinite product tends to 1 as $\Im(\tau) \to \infty$. Furthermore since

$$\frac{1}{2}\frac{\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} \to \begin{cases} -\frac{1}{2}, & x > 0 \\ \frac{1}{2i}\cot(\pi y), & x = 0 \end{cases}$$

we have

$$\lim_{\Im(\tau)\to\infty} \varphi(\tau, 2\pi i(x\tau + y)) = \begin{cases} \frac{1}{2}(-1)^{1+[y]}, & x \notin \mathbf{Z} \\ \frac{1}{2i}\cot(\pi y), & x \in \mathbf{Z}. \end{cases}$$

Thus from the definition (4) we have

$$\lim_{\Im(\tau)\to\infty} D_\tau(p; a_1, \ldots, a_r) = \left(\frac{1}{2i}\right)^r \sum_{m=1}^{p-1} \cot\left(\frac{\pi a_1 m}{p}\right) \cdots \cot\left(\frac{\pi a_r m}{p}\right) + \frac{1}{2^r} \sum_{n=1}^{p-1} (-1)^n \sum_{m=0}^{p-1} \prod_{j=1}^{r} (-1)^{1 + [\frac{a_j n}{p}]},$$

which shows the theorem. □

Taking limit $\Im(\tau) \to \infty$ in (6) and (7) we have

$$d(n;1,1) = n - \tfrac{1}{3}(n^2+2) \tag{9}$$
$$d(n;1,1,1,1) = n + \tfrac{1}{45}(n^4 - 20n^2 - 26), \tag{10}$$

for even n, which are consistent to [Za1, p.166].

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